Theorem. If f is a connected open set in Ω and $\{f_n\}_{n=1}^{\infty}$ a sequence of injective holomorphic functions on Ω that converges uniformly on every compact subset K, then either f is injective or a constant.

Proof: Suppose g is not constant and there exist two points z_1 and z_2 so that $f(z_1) = f(z_2)$. By setting

$g(z) = f(z) - f(z_1)$ and

$g_n(z) = f_n(z) - f_n(z_1)$,

then g_n converges uniformly on compact subsets to

$g(z) = f(z) - f(z_1)$. If f is not a constant, then g has an isolated zero at z_1. Hence there is a small circle γ_1 about z_1 and a positive number m_1 so that

$$m_1 = \frac{1}{2\pi i} \int_{\gamma_1} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta .$$

Since $g_n \to g$ on γ_1, we have

$$\frac{1}{2\pi i}\int \frac{g_n'(z)\,dz}{g_n(z)} \to \int_\gamma \frac{g'(z)}{g(z)}\,dz = m_1$$

Thus, if n is large, then

g_n has m_1 zeros inside γ_1.

Following the same logic, since

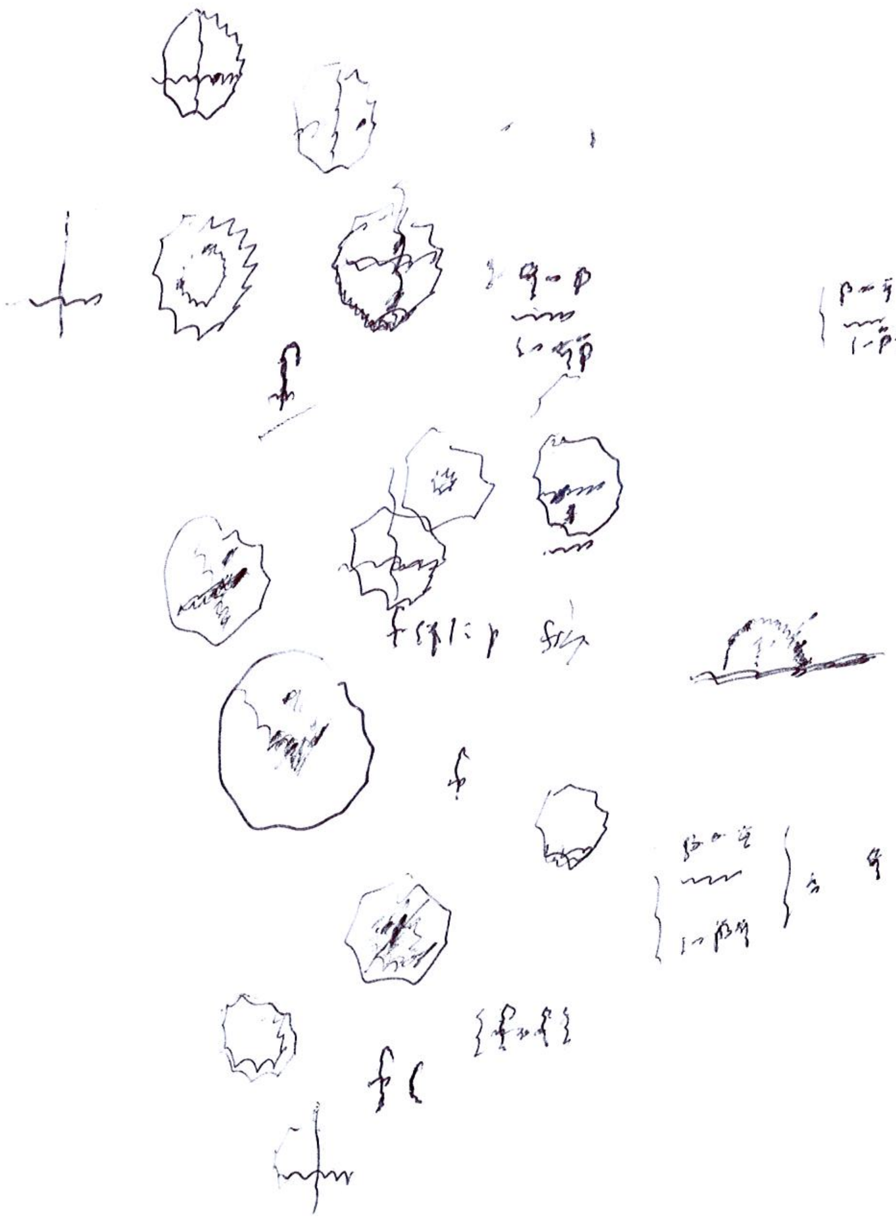
g has m_2 zeros inside a circle γ_2

about z_2, as does g_n if n is

large. It follows that

g_n has at least 2 distinct

zeros, which is a contradiction.



We are now ready to prove the

Riemann Mapping Theorem

Step 1. Recall that $\exists\, \alpha \notin \Omega$.

Since Ω is simply-connected,

there is a function $f(z) = \log(z-\alpha)$

$\Rightarrow$ $e^{f(z)} = z - \alpha$, which proves

that f is injective.

Now pick $w \in \Omega$, and

observe that

$$f(z) \neq f(w) + 2\pi i \quad , \quad \text{for all } z \text{ in } \Omega$$

For else, we get

$$e^{f(z)} = e^{f(w)}$$

or $\quad z = w. \quad \to f(z) = f(w).$

Contradiction

$$\Rightarrow f(z) \neq f(w) + 2\pi i \quad \text{for all } z \in \Omega.$$

In fact, there is a disc

centered at $f(w) + 2\pi i$

that contains no points of $f(\Omega)$

Otherwise, $\exists$ a sequence z_n in Ω

such that $f(z_n) \to f(w) + 2\pi i$.

This implies $e^{f(z_n)} \to e^{f(w)}$

$\Rightarrow z_n \to w \Rightarrow f(z_n) \to f(w)$

as $n \to \infty$

Contradiction

Step 2

Now let

$$F(z) = \frac{1}{f(z) - (f(w) + 2\pi i)}$$

Then $|F(z)| \leq M$ for all $z \in \Omega$.

If $F(z_0) = w_0$ for some $z_0 \in \Omega$,

then $$F_1(z) = \frac{F(z) - w_0}{2M}$$

satisfies $|F_1(z)| < \underset{1}{\cancel{M}}$, for $z \in \Omega$

and $0 \in$ Range of F_1

If $\Omega_1 = \{ F_1(z) ;\ z \in \Omega \}$,

then we want to find $\tilde{f} : \Omega_1 \to \mathbb{D}$

such that $\tilde{f}$

is an injective map of Ω_1 onto $\mathbb{D}$,

which would prove the Riemann

Mapping Theorem. We define

$$\mathcal{F} = \left\{ f : \Omega_1 \to \mathbb{D} \text{ holomorphic injective and } f(0) = 0 \right\}$$

Note that $\mathcal{F}$ is non-empty since it contains the identity.

Also, note that $\mathcal{F}$ is uniformly bounded by construction, since all functions map into $\mathbb{D}$.

We want to define a function that maximizes $|f'(0)|$.

Since Ω_1 is an open set,

there is $r > 0$ so that $\overline{D}_r(0) \subset \Omega_1$

In addition, $|f(z)| < 1$ for all

$z \in \overline{D}_r$. Hence Cauchy's

Inequalities imply $|f'(0)| \leq \frac{1}{r}$

for all $f \in \mathcal{F}$.

Next, we let

$$s = \sup_{f \in \mathcal{F}} |f'(0)| = s$$

We choose a sequence $\{f_n\} \subset \mathcal{F}$

such that $|f_n'(0)| \to S$ as $n \to \infty$.

By Montel's theorem, there

is a subsequence that

converges uniformly on compact

sets to a function f on Ω_1.

Since $S \geq 1$ (because $z \mapsto z$

belongs to $\mathcal{F}$) f is nonconstant hence injective.

Also, by continuity, we have

$|f(z)| \leq 1$ for all $z \in \Omega$, and from

the maximum modulus prinsiple,

we see that $|f(z)| < 1$. Since

we clearly have $f(0) = 0$. we

conclude that $|f'(0)| = s$

and $f \in \mathfrak{F}$.

Step 3.

We will show that $f: \Omega_1 \to \mathbb{D}$ is onto. Suppose f is not surjective. This implies that we could choose a function with $|f'(0)| > s$. In fact, if $f(z) \neq \alpha$, consider the automorphism $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ that interchanges 0 and α.

Since Ω is simply-connected,

so is $U = (\psi_\alpha \circ f)(\Omega)$,

so U does not contain the origin.

We can thus define a square root

on U by $g(w) = e^{\frac{1}{2}\log w}$

Next consider the function

$$F = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f.$$

We claim that $F \in \mathcal{F}$. Clearly

F is holomorphic and it maps 0 to 0

Also F maps into the unit disc

since that is true for all of

the functions in the composition.

Finally F is injective, this

is true for ψ_α and $\psi_{g(\alpha)}$.

It's also true for the square

root g and the function f,

since the latter is injective

by assumption. If h is the

square fcn. $h(w) = w^2$, then

we have

$$f = \psi_\alpha^{-1} \circ h \circ \psi_{g(\alpha)}^{-1} \circ F = \Phi \circ F$$

But Φ maps $\mathbb{D}$ into $\mathbb{D}$

with $\Phi(0) = 0$ and is not injective

because F is and h is not.

By the last part of the Schwarz lemma, we conclude that $|\Phi'(0)| < 1$.

The proof is complete once we observe that

$$f'(0) = \Phi'(0)\,F'(0)$$

and thus $|f'(0)| < |F'(0)|$

which contradicts the

maximality of $|f'(0)|$ in $\mathfrak{F}$.

Finally we multiply f by a complex number of absolute value $=1$ so that $f'(0)>0$ which ends the proof.