Runge Approximation Thm

Let Ω be an open set in $\mathbb{C}$ and

K a compact subset of Ω.

The following are equivalent:

(a) Every function which is

holomorphic in a neighborhood

of K can be approximated

uniformly on K by functions

in $A(\Omega)$.

(b) The open set $\Omega \setminus K$

$= \Omega \cap \complement K$ has no component

which is relatively compact in Ω

(c) For every z in $\Omega \setminus K$, there

is a function $f \in A(\Omega)$ that

satisfies $|f(z)| > \sup_K |f|$

Corollary.

Every function which is holomorphic in a neighborhood of the compact set K can be approximated by polynomials uniformly on K if and only if and only if $\complement K$ is connected, or equivalently, for every

$z \in \complement K$, there is a polynomial f such that

$$|f(z)| > \sup_K |f| \quad \text{is valid.}$$

Proof. We first prove that (c) $\Rightarrow$ (b) and that (a) $\Rightarrow$ (b).

So, assume that (b) is not valid, that is, that $\Omega \setminus K$ has a component O such that

$\bar{O}$ is compact and $\bar{O} \subset \Omega$.

Then the boundary of O is a subset of K. Then the Maximum Principle implies that

$$\sup_{O} |f| \leq \sup_{K} |f| , \quad \text{all } f \in K.$$

which contradicts (c).

If (a) were valid, then for every f which is ~~not~~ holomorphic in a neighborhood of K, we could choose $f_n \in A(\Omega)$ that uniformly $\to f$ on K. It follows from (2) that $f_n - f_m \to 0$ so that f_n converges uniformly in $\bar{O}$ to a limit F. We have $F = f$ on the boundary of O and F is holomorphic in O.

In particular, we can choose

$f(z) = \dfrac{1}{z - \zeta}$ if $\zeta \in O$, and

then we have $(z-\zeta) F(z) = 1$

on the boundary of O,

hence $(z-\zeta) F(z) = 1$ in O

This gives a contradiction

when $z = \zeta$.