Homework 3 for Math 530

1. Show the function

$$f(z) = \sum_{n=1}^{\infty} \frac{z^2}{n^2 z^2 + 8}$$

is defined and continuous for all real values of z. Determine the region of the complex plane in which this function is analytic.

2. Let $f(z) = \frac{1}{(z-1)(z-2)}$. Find the Laurent series of $f(z)$ when (a) $1 < |z| < 2$ and when (b) $|z| > 2$.

3. Evaluate

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$$

(Hint: Consider the integral of $\frac{1-e^{2ix}}{x^2}$.)

4. Evaluate $\int_0^{2\pi} \frac{1}{2+\sin\theta}$.

5. Suppose that f is a holomorphic function on the unit disc D such that $|f(z) - i| < 1$ for all $z \in D$. Show that $\int_\gamma f(z)dz = 0$ for every closed path in D.

6. Let $P(z) = z^8 - 5z^3 + z - 2$. How many zeros of P are in the annulus $1 < |z| < 2$.

7. Find the first three terms of the Laurent series expansion of the function

$$f(z) = \frac{e^z}{(z-1)^2(z+1)}$$

that converges in $0 < |z| < 1$.

8. Find the residue of the function $\frac{z^2}{(1+z^2)^3}$ at the points i and $-i$.

9. If f is analytic in the annulus $1 \le |z| \le 2$ and $|f(z)| \le 3$ on $|z| = 1$ and $|f(z)| \le 12$ on $|z| = 2$, then prove that $|f(z)| \le 3|z|^2$ for $1 \le |z| \le 2$
(Hint: Consider $f(z)/3z^2$)