MA 30300
Chapter 10 practice

NAME ______________________ INSTRUCTOR ______________________

1. Instructor's names: Chen

2. Course number: MA30300.

3. **TEST/QUIZ NUMBER is:**

01 if this sheet is **yellow**

02 if this sheet is **blue**

03 if this sheet is **white**

4. Sign the scantron sheet.

5. Laplace transform table and a formula sheet on Fourier series and some PDEs are provided at the end of this booklet.

6. There are 20 questions, each worth 5 points. Do all your work on the question sheets. Turn in both the scantron form and the question sheets when you are done.

7. Show your work on the question sheets. Although no partial credit will be given, any disputes about grades or grading will be settled by examining your written work on the question sheets.

8. NO CALCULATORS, BOOKS, OR PAPERS ARE ALLOWED. Use the back of the test pages for scrap paper.

1. Find all $T(t)$ such that $u(x,t) = \cos(2x)T(t)$ is a solution of

$$4u_{xx} = u_{tt}.$$

A. $T(t) = c_1 e^{4t} + c_2 e^{-4t}$.

B. $T(t) = c_1 e^{2t} + c_2 e^{-2t}$.

C. $\boxed{T(t) = c_1 \cos(4t) + c_2 \sin(4t).}$

D. $T(t) = c_1 \cos(2t) + c_2 \sin(2t)$.

E. $T(t) = c_1 \cos(t) + c_2 \sin(t)$.

2. Find the equation for $Y(y)$ such that $u(x,y) = e^{-\lambda x}Y(y)$ is a solution of

$$u_x + yu_y = 0.$$

A. $Y' - n^2 Y = 0$

B. $\boxed{yY' - \lambda Y = 0}$

C. $yY' - n^2 Y = 0$

D. $yY'' - \lambda Y' = 0$

E. $yY' + \lambda Y = 0$

3. Determine whether the method of separation of variables can be used to replace the equation $u_{xx} - xu_{tt} = 0$ by a pair of ordinary differential equations.

A. The equation can be replaced by $X'' - \lambda x X = 0$, $T'' + \lambda T = 0$, where λ is some constant.

B. The equation cannot be replaced by a pair of ordinary differential equations using this method.

C. The equation can be replaced by $X'' + \lambda X = 0$, $T'' + \lambda x T = 0$, where λ is some constant.

D. The equation can be replaced by $X'' - \lambda X = 0$, $T'' + \lambda T = 0$, where λ is some constant.

E. $\boxed{\text{The equation can be replaced by } X'' + \lambda x X = 0,\ T'' + \lambda T = 0, \text{ where } \lambda \text{ is some constant.}}$

4. Which is the solution for

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le 1,\ y \ge 0$$
$$u(0,y) = 0, \qquad u(1,y) = 0$$
$$u(x,0) = 2\sin(3\pi x), \qquad \lim_{y\to\infty} u(x,y) = 0?$$

A. $2e^{-3\pi y}\sinh(3\pi x)$

B. $\boxed{2e^{-3\pi y}\sin(3\pi x)}$

C. $2\sinh(3\pi y)\sin(3\pi x)$

D. $2\sinh(3\pi y)\cos(3\pi x)$

E. none of the above.

5. Find the eigenvalues and eigenfunctions of the given boundary value problem:

$$u'' - 2u' + \lambda u = 0, \quad u(0) = u(\pi) = 0$$

(Assume $\lambda > 1$)

A. $\lambda_n = n^2\pi^2,\ \ u_n(x) = \sin(n + \frac{1}{2})\pi x,\ \ n = 1, 2, \ldots$

B. $\lambda_n = n^2\pi^2,\ \ u_n(x) = \sin(n\pi x),\ \ n = 1, 2, \ldots$

C. $\lambda_n = n^2 + 1,\ u_n(x) = e^{-x}\sin(nx),\quad n = 1, 2, \ldots$

D. $\boxed{\lambda_n = n^2 + 1,\ u_n(x) = e^{x}\sin(nx),\quad n = 1, 2, \ldots}$

E. $\lambda_n = n^2 + 1,\ u_n(x) = e^{x}\cos(nx),\quad n = 1, 2, \ldots$

6. Solve the boundary value problem if possible:

$$y'' + 4y = \cos x, \quad y(0) = 0, \quad y(\pi) = 0.$$

A. $y(x) = \cos x$

B. $y(x) = \sin x + c_1 \cos x$

C. $y(x) = \cos 2x + (\cos x)/3$

D. $y(x) = \cos 2x + c_1 \sin x + (\cos x)/3$

E. $\boxed{\text{No solution}}$

7. Find a, such that the functions 1 and $2x - ax^2$ are orthogonal on the interval $(0, 3)$

A. $\frac{1}{3}$

B. $-\frac{1}{3}$

C. 2

D. $\boxed{1}$

E. 3

8. Given that the Fourier series of period 4 for the function $f(x) = 2 - |x|$ defined on $(-2, 2)$ is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{2}).$$

What is the value of a_3?

A. $\dfrac{\pi}{3}$

B. $\dfrac{8}{\pi^2}$

C. $\boxed{\dfrac{8}{9\pi^2}}$

D. $\dfrac{\pi^2+1}{9}$

E. $3\pi^2$

9. Let $g(x)$ be the Fourier **sine** series of period 4 for the function $f(x)$ defined on $(0, 2)$ by

$$f(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & 1 \leq x < 2. \end{cases}$$

Determine the values of $g(0)$, $g(.5)$ and $g(5)$. **Hint:** Use the Fourier Convergence Theorem rather than finding coefficients in the series.

A. $-1, \quad 0, \quad -1$

B. $\boxed{0, \quad 1, \quad 0.5}$

C. $0, \quad 1, \quad -1$

D. $1, \quad 1, \quad 0.5$

E. $1, \quad 0, \quad -1$

10. The Fourier **sine** series of period 4 for the function $f(x)$ defined on $(0,2)$ by

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 \le x < 2. \end{cases}$$

is

A. $\boxed{\sum_{n=1}^{\infty}(1-\cos(\frac{n\pi}{2}))\frac{2}{n\pi}\sin(\frac{n\pi x}{2})}$

B. $\sum_{n=1}^{\infty}(1-\sin(\frac{n\pi}{2}))\frac{2}{n\pi}\sin(\frac{n\pi x}{2})$

C. $\sum_{n=1}^{\infty}(1-(-1)^n)\frac{2}{n\pi}\sin(\frac{n\pi x}{2})$

D. $\sum_{n=1}^{\infty}(1-(-1)^n)\frac{2}{n\pi}\sin(n\pi x)$

E. $\sum_{n=1}^{\infty}(1-\cos(\frac{n\pi}{2}))\frac{1}{n\pi}\sin(n\pi x)$

11. The Fourier series for the following periodic function:

$$f(x) = \begin{cases} 1, & \text{if } -1 \le x < 0, \\ 2, & \text{if } 0 \le x < 1, \end{cases}, \qquad f(x+2) = f(x) \text{ for all } x$$

is

A. $\boxed{\frac{3}{2} + \sum_{n=1}^{\infty}(1-(-1)^n)\frac{1}{n\pi}\sin n\pi x}$

B. $\frac{3}{2} + \sum_{n=1}^{\infty}(1-(-1)^n)\frac{1}{n\pi}\cos n\pi x$

C. $1 + \sum_{n=1}^{\infty}(1-(-1)^n)\frac{1}{n\pi}(\cos n\pi x + \sin n\pi x)$

D. $1 + \sum_{n=1}^{\infty}(1-(-1)^n)\frac{1}{n\pi}(\cos nx + \sin nx)$

E. $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x$

12. Let $f(x)$ be defined as

$$f(x) = \begin{cases} -1, & -2 \leq x < 0 \\ 1, & 0 \leq x < 2 \end{cases}$$
$$f(x+4) = f(x).$$

Denote by $S_N(x)$ the finite sum of Fourier series of $f(x)$, that is

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(\frac{n\pi}{2}x) + \sum_{n=1}^{N} b_n \sin(\frac{n\pi}{2}x).$$

For fixed integer k, which of the following statement is correct?

A. $\lim_{N\to\infty} S_N(2k+1) = (-1)^{k+1}$

B. $\lim_{N\to\infty} S_N(2k+1) = (-1)^{N+1}$

C. $\lim_{N\to\infty} S_N(2k+1) = 0$

D. $\boxed{\lim_{N\to\infty} S_N(2k+1) = (-1)^k}$

E. The limit of $S_N(2k+1)$ as $N \to \infty$ does not exist

13. Which $u(x,t)$ does NOT satisfy the following equations

$$u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = 0, \quad u(1,t) = 0$$

A. $u(x,t) = 0$

B. $u(x,t) = e^{-9\pi^2 t}\sin(3\pi x)$

C. $\boxed{u(x,t) = e^{-\pi^2 t}\sin(2\pi x)}$

D. $u(x,t) = e^{-\pi^2 t}\sin(\pi x) - e^{-4\pi^2 t}\sin(2\pi x)$

E. $u(x,t) = -5e^{-4\pi^2 t}\sin(-2\pi x)$

14. Let $u(x,t)$ satisfy the heat equation

$$u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = 0, \quad u(1,t) = 2$$
$$u(x,0) = 2x + \sin(\pi x).$$

Then $u(\frac{1}{2}, 1) =$

A. 1

B. 2

C. $\boxed{1 + e^{-\pi^2}}$

D. $1 - \dfrac{1}{2}e^{-\pi^2}$

E. $1 + 2e^{-\pi^2}$

15. Let a silver bar 20cm long be initially at uniform temperature 40^0C. Suppose that at time $t = 0$ the end $x = 0$ is cooled to 0^0C while the end $x = 20$ is heated to 60^0C, and both thereafter maintained at those temperatures. For silver, the thermal diffusivity is known to be 1.71 cm^2/sec. Then the temperature $u(x, t)$ at a point x centimeters from the left end, at time t seconds satisfies which of the following?

(1) $u(x, 0) = 0, \quad u(x, 20) = 60$ for $0 < x < 20$

(2) $u(0, t) = 0, \quad u(20, t) = 60$ for $t > 0$

(3) $u(x, 0) = 40$ for $0 < x < 20$

(4) $1.71 u_{xx} = u_t$ for $0 < x < 20, t > 0$

(5) $1.71 u_{xx} = u_{tt}$ for $0 < x < 20, t > 0$

A. (1), (2), (4)

B. (1), (3), (4)

C. $\boxed{\text{(2), (3), (4)}}$

D. (2), (3), (5)

E. (1), (3), (5)

16. Solve the temperature $u(x, t)$ which satisfies the heat equation and the following initial and boundary conditions,

$$\begin{cases} u_t = u_{xx}, \quad t > 0, \quad 0 < x < 2 \\ u_x(0, t) = 0, \quad u_x(2, t) = 0, \quad t > 0 \\ u(x, 0) = 10, \quad 0 < x < 2 \end{cases}$$

A. $\boxed{u(x, t) = 10}$

B. $\frac{1}{2}\sin(4x)(\sin(4t) + \frac{1}{2}\sin(8x)\sin(8t)$

C. $\sin(2x)(\sin(4t) + \sin(4x)\sin(8t)$

D. $\displaystyle\sum_{n=1}^{\infty}(1 - (-1)^n)\frac{20}{n^2\pi}\sin(nx)e^{-nt}$

E. $\displaystyle\sum_{n=1}^{\infty}(1 - (-1)^n)\frac{20}{n\pi}\sin(\frac{n\pi x}{5})e^{-\frac{n\pi t}{5}}$

17. Let $v(x)$ be the steady-state solution of the heat conduction problem:

$$\begin{cases} u_{xx} = u_t, \\ u(0,t) = 50, \\ u(30,t) = 20. \end{cases}$$

What is $v(10)$?

A. 10

B. 20

C. 30

D. $\boxed{40}$

E. 50

18. Which $u(x,t)$ does NOT satisfy the following equations

$$\begin{aligned} &4u_{xx} = u_{tt}, \quad 0 < x < \pi, \quad t > 0 \\ &u(0,t) = 0, \quad u(\pi,t) = 0 \end{aligned}$$

A. $u(x,t) = 0$

B. $\boxed{u(x,t) = \frac{1}{2}\sin(4x)(\sin(4t) + \frac{1}{2}\sin(8x)\sin(8t)}$

C. $u(x,t) = \sin(2x)(\sin(4t) + \sin(4x)\cos(8t)$

D. $u(x,t) = \frac{\pi}{2}\sin(2x)(\sin(4t) + \pi\sin(4x)\cos(8t)$

E. $u(x,t) = \frac{1}{2}\sin(2x)(\sin(4t) + \frac{1}{2}\sin(4x)\sin(8t)$

19. Find the solution $u(x,t)$ of the wave equation

$$4u_{xx} = u_{tt}, \qquad 0 < x < \pi, \quad t > 0,$$

satisfying the conditions

$$\begin{cases} u(0,t) = u(\pi,t) = 0 \qquad \text{when} \quad t > 0, \\ u(x,0) = 0 \qquad \text{when} \quad 0 \le x \le \pi, \\ u_t(x,0) = 2\sin 2x + 4\sin 4x \qquad \text{when} \quad 0 \le x \le \pi. \end{cases}$$

A. $\frac{1}{2}\sin(4x)(\sin(4t) + \frac{1}{2}\sin(8x)\sin(8t)$

B. $\sin(2x)(\sin(4t) + \sin(4x)\sin(8t)$

C. $\frac{\pi}{2}\sin(2x)(\sin(4t) + \pi\sin(4x)\sin(8t)$

D. $\frac{\pi}{2}\sin(2x)(\sin(2t) + \pi\sin(4x)\sin(4t)$

E. $\boxed{\frac{1}{2}\sin(2x)(\sin(4t) + \frac{1}{2}\sin(4x)\sin(8t)}$

20. A string $L = 5$ meter long and fixed at both ends is giving an initial velocity of $g(x) = 2m/s$ from equilibrium. The string has no initial displacement. Find an expression for the displacement function $u(x,t)$ valid for all x in $0 \le x \le 5$ and all $t \ge 0$. (Assume $\alpha = 1$, where α is the constant in the wave equation.)

A. $\boxed{\sum_{n=1}^{\infty}(1-(-1)^n)\frac{20}{n^2\pi^2}\sin(\frac{n\pi x}{5})\sin(\frac{n\pi t}{5})}$

B. $\sum_{n=1}^{\infty}(1-(-1)^n)\frac{20}{n^2\pi}\sin(nx)\sin(nt)$

C. $\sum_{n=1}^{\infty}(1-(-1)^n)\frac{20}{n\pi}\sin(\frac{n\pi x}{5})\cos(\frac{n\pi t}{5})$

D. $\sum_{n=1}^{\infty}(1-(-1)^n)\frac{4}{n^2\pi^2}\sin(nx)(\sin nt + \cos(n\alpha t))$

E. $\sum_{n=1}^{\infty}(1-(-1)^n)\frac{4}{n^2\pi^2}\sin(\frac{nx}{5})\sin(\frac{nt}{5})$

21. Which is the solution for

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le 1,\ y \ge 0 \\
&u(0,y) = 0, \qquad u(1,y) = 0 \\
&u(x,0) = 2\sin(3\pi x), \qquad \lim_{y\to\infty} u(x,y) = 0?
\end{aligned}$$

A. $2e^{-3\pi y}\sinh(3\pi x)$

B. $\boxed{2e^{-3\pi y}\sin(3\pi x)}$

C. $2\sinh(3\pi y)\sin(3\pi x)$

D. $2\sinh(3\pi y)\cos(3\pi x)$

E. none of the above.

Formula sheet

Fourier series: For a $2L$-periodic function $f(x)$, the Fourier series for f is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L},$$

where for $n=1,2,\cdots,$

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx.$$

Heat equation 1: The solution of the heat equation $\alpha^2u_{xx}=u_t$, $0<x<L$, $t>0$, satisfying the (fixed temperature) homogeneous boundary conditions $u(0,t)=u(L,t)=0$ for $t>0$ with initial temperature $u(x,0)=f(x)$ has the general form

$$u(x,t)=\sum_{n=1}^{\infty}c_ne^{-n^2\pi^2\alpha^2t/L^2}\sin\frac{n\pi x}{L},\qquad\text{where}\qquad c_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

Heat equation 2: The solution of the heat equation $\alpha^2u_{xx}=u_t$, $0<x<L$, $t>0$, satisfying the insulated boundary conditions $u_x(0,t)=u_x(L,t)=0$ for $t>0$ with initial temperature $u(x,0)=f(x)$ has the general form

$$u(x,t)=\frac{c_0}{2}+\sum_{n=1}^{\infty}c_ne^{-n^2\pi^2\alpha^2t/L^2}\cos\frac{n\pi x}{L},\qquad\text{where}\qquad c_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx.$$

Wave equation: The solution of the wave equation $\alpha^2u_{xx}=u_{tt}$, $0<x<L$, $t>0$, satisfying the homogeneous boundary conditions $u(0,t)=u(L,t)=0$ for $t>0$ and initial conditions $u(x,0)=f(x)$ and $u_t(x,0)=g(x)$ for $0\le x\le L$ has the general form

$$u(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(c_n\cos\frac{n\pi\alpha t}{L}+k_n\sin\frac{n\pi\alpha t}{L}\right)$$

where

$$c_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx\quad\text{and}\quad k_n=\frac{2}{n\pi\alpha}\int_0^L g(x)\sin\frac{n\pi x}{L}dx.$$

Laplace equation: The solution of the Laplace equation $u_{xx}+u_{yy}=0$, $0<x<a$, $0\le y\le b$, satisfying the boundary conditions $u(x,0)=u(x,b)=0$ for $0<x<a$ and $u(0,y)=0$ and $u(a,y)=f(y)$ for $0\le y\le b$ has the general form

$$u(x,y)=\sum_{n=1}^{\infty}c_n\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}\quad\text{where}\quad c_n=\frac{2}{b\sinh(\frac{n\pi a}{b})}\int_0^b f(y)\sin\frac{n\pi y}{b}dy.$$

Figure 1: Laplace Transform Table

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$
2.	e^{at}	$\dfrac{1}{s-a}$
3.	t^n	$\dfrac{n!}{s^{n+1}}$
4.	$t^p \ (p > -1)$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sin at$	$\dfrac{a}{s^2+a^2}$
6.	$\cos at$	$\dfrac{s}{s^2+a^2}$
7.	$\sinh at$	$\dfrac{a}{s^2-a^2}$
8.	$\cosh at$	$\dfrac{s}{s^2-a^2}$
9.	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
10.	$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$
11.	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
12.	$u_c(t)$	$\dfrac{e^{-cs}}{s}$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right) \ \ c > 0$
16.	$\displaystyle\int_0^t f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$