MA 30300
Examination II (practice)

NAME ______________________ ID number ______________________

Lecture Time______________________

THIS EXAM IS CLOSED TO BOOKS AND NOTES.

No partial credit will be given for multiple choice problems. But any disputes about grades or grading will be settled by examining your written work on the booklet.

Points awarded

1. (10 pts) ____________
2. (10 pts) ____________
3. (10 pts) ____________
4. (10 pts) ____________
5. (10 pts) ____________
6. (10 pts) ____________
7. (10 pts) ____________
8. (10 pts) ____________
9. (10 pts) ____________
10. (10 pts) ____________

Total Points: ____________/100

1. The general solution for

$$y'' - \frac{3}{t}y' + \frac{4}{t^2}y = 0, \quad t > 0.$$

is $y(t) = C_1 t^2 + C_2 t^2 \ln(t)$. The equivalent first order system of equations with $x_1(t) = y(t)$ and $x_2(t) = y'(t)$ is

$$\begin{cases} x_1' = x_2 \\ x_2' = \frac{3}{t}x_2 - \frac{4}{t^2}x_1 \end{cases}.$$

What is the general solution of the system?

A. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} t^2 \\ 2t \end{pmatrix} + C_2 \begin{pmatrix} t^2 \ln(t) \\ 2t\ln(t) + t \end{pmatrix}$

B. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} t^2 \\ t^2 \ln(t) \end{pmatrix} + C_2 \begin{pmatrix} t^3 \\ t^3 \ln(t) \end{pmatrix}$

C. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} t^3 \\ t^3 \ln(t) \end{pmatrix} + C_2 \begin{pmatrix} t^2 \\ t^2 \ln(t) \end{pmatrix}$

D. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} t^2 \\ 2t \end{pmatrix} + C_2 \begin{pmatrix} t^2 \ln(t) \\ 2t\ln(t) \end{pmatrix}$

E. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} 2t \\ t^2 \end{pmatrix} + C_2 \begin{pmatrix} 2t\ln(t) + t \\ t^2 \ln(t) \end{pmatrix}$

2. The equivalent first order system of equations with $x_1(t) = y(t)$ and $x_2(t) = y'(t)$ for

$$y'' - \frac{3}{t}y' + \frac{4}{t^2}y = 0, \quad t > 0$$

is

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x} = (x_1(t), x_2(t))^T.$$

What is the matrix A?

A. $\begin{pmatrix} 0 & 1 \\ -\frac{4}{t^2} & \frac{3}{t} \end{pmatrix}$

B. $\begin{pmatrix} 1 & 0 \\ \frac{3}{t} & -\frac{4}{t^2} \end{pmatrix}$

C. $\begin{pmatrix} 1 & 1 \\ \frac{3}{t} & -\frac{4}{t^2} \end{pmatrix}$

D. $\begin{pmatrix} -\frac{4}{t^2} & \frac{3}{t} \\ 0 & 1 \end{pmatrix}$

E. $\begin{pmatrix} 0 & 1 \\ \frac{4}{t^2} & -\frac{3}{t} \end{pmatrix}$

3. Which of the following is a solution for the following system?

$$x' = \begin{pmatrix} 0 & 0 & 0 \\ 5 & 7 & 2014 \\ 0 & 0 & 0 \end{pmatrix} x$$

A. $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{7t} + \begin{pmatrix} 2024 \\ -14 \\ -7 \end{pmatrix}$

B. $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{7t} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2014 \\ -7 \\ -5 \end{pmatrix} t$

C. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{7t} + \begin{pmatrix} 2021 \\ -5 \\ -5 \end{pmatrix}$

D. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{7t} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2014 \\ -7 \\ -5 \end{pmatrix} t$

E. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{7t} + \begin{pmatrix} 2014 \\ -7 \\ -5 \end{pmatrix}$

4. Consider the system

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ k & 3 \end{pmatrix} \mathbf{x}.$$

For which of the following values of k is the origin a saddle point?

A. -10

B. -5

C. 0

D. 5

E. 10

5. The general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} \mathbf{x}$$

is

A. $c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} -3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} \right]$

B. $c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} -3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{-3t} \right]$

C. $c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{-3t} \right]$

D. $c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} \right]$

E. $c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{3t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{3t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} \right]$

6. Solve the initial value problem

$$x' = \begin{pmatrix} 2 & 4 \\ -6 & -8 \end{pmatrix} x, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

A. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 2e^{-2t} \\ -2e^{-2t} + e^{-4t} \end{pmatrix}$

B. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 3e^{-2t} - e^{-4t} \\ -6e^{-2t} + 5e^{-4t} \end{pmatrix}$

C. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 4e^{-2t} - 2e^{-4t} \\ -4e^{-2t} + 3e^{-4t} \end{pmatrix}$

D. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 5e^{-2t} - 3e^{-4t} \\ -3e^{-2t} + 2e^{-4t} \end{pmatrix}$

E. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 6e^{-2t} - 4e^{-4t} \\ -2e^{-2t} + e^{-4t} \end{pmatrix}$

7. Consider a system of linear equations $x = Ax$ and suppose that A has eigenvalues $r = 3 \pm 2i$, and one of the eigenvectors corresponding to the eigenvalue $r = 3 + 2i$ is $\begin{pmatrix} -2 \\ 1-i \end{pmatrix}$.

Find a solution of this system that satisfies the initial condition $\mathbf{x}(0) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$.

A. $\mathbf{x}(t) = e^{3t} \begin{pmatrix} 2\sin(2t) - 2\cos(2t) \\ 2\cos(2t) \end{pmatrix}$

B. $\mathbf{x}(t) = e^{3t} \begin{pmatrix} -2\cos(2t) \\ 2\cos(2t) \end{pmatrix}$

C. $\mathbf{x}(t) = e^{3t} \begin{pmatrix} \sin(2t) - 2\cos(2t) \\ 2\sin(2t) + 2\cos(2t) \end{pmatrix}$

D. $\mathbf{x}(t) = e^{3t} \begin{pmatrix} \sin(2t) - 2\cos(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix}$

E. $\mathbf{x}(t) = e^{3t} \begin{pmatrix} 2\sin(2t) - 2\cos(2t) \\ \sin(2t) + 2\cos(2t) \end{pmatrix}$

8. What is the correct form of a particular solution for the following equation?

$$\mathbf{X}' = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} \mathbf{X} + \begin{pmatrix} 3e^t \\ 2e^t \end{pmatrix}.$$

A. $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t$

B. $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} t$

C. $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} te^t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

D. $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} te^t$

E. $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} te^t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t$

9. Which of the following is a *particular* solution to the first order system:

$$x' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x + \begin{pmatrix} 2e^{2t} \\ 3e^{2t} \end{pmatrix}.$$

A. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$

B. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 3e^{2t} \\ -6e^{-2t} \end{pmatrix}$

C. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 4e^{-2t} - 2e^{2t} \\ e^{-2t} \end{pmatrix}$

D. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{2t} \\ -3e^{-2t} \end{pmatrix}$

E. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{5}{3}e^{2t} \\ 2e^{2t} \end{pmatrix}$

10. Find the general solution to the following first order system:

$$x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 3t \end{pmatrix},$$

given that the eigenvalues of the matrix $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ are $r_1 = 3, r_2 = -1$, and the corresponding eigenvectors are $v^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

A. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t$

B. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} t$

C. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

D. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

E. $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

11. (**Hint**: By observing the correct form of the solution, you should be able to find the answer for this problem)
The matrix $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ has eigenvalues 1 and -1, and their corresponding eigenvectors are respectively $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Which of the following is the solution $\mathbf{X}$ to the initial value problem

$$\mathbf{X}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{X} + \begin{bmatrix} e^t \\ 5e^t \end{bmatrix}, \qquad \mathbf{X}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A. $\begin{bmatrix} -e^t + e^{-t} + te^t \\ 2e^t - 2e^{-t} + te^t \end{bmatrix}$

B. $\begin{bmatrix} 3e^t + e^{-t} - te^t \\ e^t - e^{-t} + te^t \end{bmatrix}$

C. $\begin{bmatrix} -e^t + e^{-t} - 2te^t \\ -2e^t + 2e^{-t} - 2te^t \end{bmatrix}$

D. $\begin{bmatrix} e^t - e^{-t} - te^t \\ 3e^t - 3e^{-t} - te^t \end{bmatrix}$

E. $\begin{bmatrix} e^t - e^{-t} - 2te^t \\ -3e^t + 3e^{-t} - 2te^t \end{bmatrix}$

12.

$$\begin{cases} y' = 1 + t - y \\ y(0) = 3 \end{cases}$$

What is the approximate value of $y(2)$ computed using the forward Euler method with step size $h = 1$?

A. 0

B. 1

C. 2

D. 3

E. 4

13. Use the Euler method with $h = 0.1$ to find an approximate value for $x(0.2)$ in the solution of the initial value problem

$$\begin{cases} x' = 2y - t + 3x \\ y' = x - 3t \end{cases}$$

with initial value $x(0) = 1$ and $y(0) = 1$.

A. 4.19

B. 1.68

C. 3.32

D. 4.59

E. 2.16

14. The approximate value of the solution at $t = 2$ of

$$y' = 1 - t + 6y, \quad y(1) = 2$$

is evaluated using the Improved Euler method with $h = 1$. We recall the formula of Improved Euler method is for $n = 0, 1, 2, \cdots$,

$$t_{n+1} = t_n + h;$$

$$k_{1n} = f(t_n, y_n)$$
$$k_{2n} = f(t_{n+1}, y_n + hk_{1n})$$
$$y_{n+1} = y_n + \frac{h}{2}(k_{1n} + k_{2n})$$

The approximate value of $y(2)$ is

A. 2.52

B. -5.8

C. 46

D. 5.34

E. 49.5

15. For the two-point boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = y'(3\pi) = 0,$$

the eigenvalues are $\lambda_n = (\frac{n}{3})^2$ and the corresponding eigenfunctions are $y_n = \cos\left(\frac{nx}{3}\right)$ for $n = 0, 1, 2, \cdots$. Which of these are true statements:

(1) $y = 0$ is a solution for $y'' + \frac{4}{9}y = 0, \quad y'(0) = y'(3\pi) = 0$

(2) $y = 0$ is a solution for $y'' + \frac{9}{4}y = 0, \quad y'(0) = y'(3\pi) = 0$

(3) $y = 5\cos(\frac{2}{3}x)$ is a solution for $y'' + \frac{4}{9}y = 0, \quad y'(0) = y'(3\pi) = 0$

(4) $y = 5\cos(\frac{2}{3}x)$ is a solution for $y'' + \frac{9}{4}y = 0, \quad y'(0) = y'(3\pi) = 0$

(5) $y = 5\cos(\frac{5}{3}x)$ is a solution for $y'' + \frac{4}{9}y = 0, \quad y'(0) = y'(3\pi) = 0$

A. Only (1), (2) and (5) are true

B. Only (1) and (2) are true

C. Only (1), (2) and (3) are true

D. Only (4) and (5) are true

E. All of them are true

16. Solve the boundary value problem if possible:

$$y'' + 4y = \cos x, \quad y(0) = 0, \quad y(\pi) = 0.$$

A. $y(x) = \cos x$

B. $y(x) = \sin x + c_1 \cos x$

C. $y(x) = \cos 2x + (\cos x)/3$

D. $y(x) = \cos 2x + c_1 \sin x + (\cos x)/3$

E. No solution

17. What are the eigenvalues on positive real axis for the following boundary value problem?

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y'(L) = 0 \end{cases}$$

A. $\dfrac{n^2\pi^2}{L^2},\ n = 1, 2, \cdots$

B. $\dfrac{\left(n+\frac{1}{2}\right)^2\pi^2}{L^2},\ n = 0, 1, 2, \cdots$

C. $\dfrac{n^2\pi^2}{4L^2},\ n = 1, 2, \cdots$

D. $\dfrac{\left(n+\frac{1}{3}\right)^2\pi^2}{L^2},\ n = 0, 1, 2, \cdots$

E. $\dfrac{(2n+1)^2\pi^2}{L^2},\ n = 0, 1, 2, \cdots$

answer: AACECC AEEBDC EECEB