THIS EXAM IS CLOSED TO BOOKS AND NOTES.
CALCULATORS ARE NOT ALLOWED.
Use the back of the **previous** page if more space is needed!

MA366 FINAL EXAMINATION Practice

Name______________________________ ID #__________________ Section #______

There are 23 Problems on this booklet. For All problems, mark the answers clearly.

1. Using Euler method with $h = 0.1$ on

$$y'(t) = v(t)^2 + t^2, \quad y(1) = 1$$
$$v'(t) = -y(t), \quad v(1) = -1.$$

The approximate value for $y(1.2)$ is ______. The approximate value for $v(1.2)$ is ______.

2. Which one of the following is a solution to the initial value problem

$$yy' = t, \quad y(1) = -1$$

a) $y(t) = t$

b) $y(t) = -t$

c) $y(t) = -1$

d) $y(t) = 1$

e) $y(t) = -\sqrt{t+1}$

3. Let $y = t^r$ be a solution to

$$t^2 y'' - 2ty' + 2y = 0.$$

What are the values of r? ______________

4. Find the explicit solution of

$$y' = 4t(y^2 + 2y), \quad y(0) = -1$$

5. Find the explicit solution of

$$y' = 4t(y^2 + 2y), \quad y(0) = -2$$

6. Find the explicit solution of

$$y' = \frac{2t}{2y-1}, \quad y(0) = 0$$

7. Find the explicit solution of

$$ty' - 2y = (te^t)^3, \quad y(1) = 0$$

8. If $y'' + 2y' + 5y = 0,\ y(0) = 1,\ y'(0) = 2$, then $y(\frac{\pi}{2}) =$

a) 1.

b) 0.

c) e^{π}.

d) $e^{\frac{\pi}{2}}$.

e) $-e^{-\frac{\pi}{2}}$.

9. Find the general solution of

$$y'' + 2y' + 2y = 5e^{-2t}\sin(t)$$

10. Transform the following third-order equation

$$y'''(t) - 3ty' + \sin(2t)y = 7e^{-t}$$

into a first-order system

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}' = A \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \mathbf{f}(t)$$

by setting $u_1(t) = y(t)$, $u_2(t) = y'(t)$ and $u_3(t) = y''(t)$. Find the matrix A and vector $\mathbf{f}(t)$.

11. Given $A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 4 & -2 & -3 \end{pmatrix}$ has eigenvalues -1, $-1+i$, $-1-i$ and corresponding eigenvector

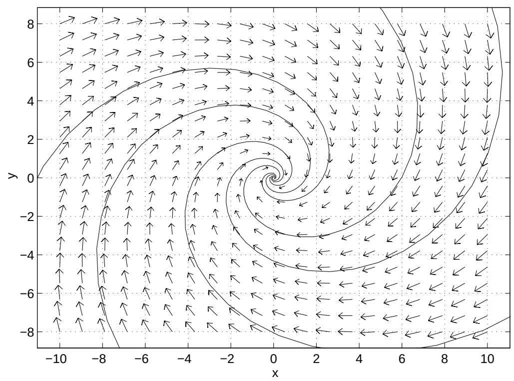
$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1+i \\ i \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1-i \\ -i \\ 2 \end{pmatrix}.$$

Find the **real** (meaning: does not involve complex numbers) general solution of $y' = Ay$.

12. The phase portrait for a linear system of the form

$$\bar{x}' = A\bar{x},$$

where A is a 2×2 matrix is as follows.



(i) Which of the following best describes the eigenvalues of A:

(a) two real distinct positive eigenvalues
(b) two real distinct negative eigenvalues
(c) two real eigenvalues of opposite signs
(d) two complex eigenvalues with positive real part
(e) two complex eigenvalues with negative real part
(f) one positive repeated eigenvalue
(g) one negative repeated eigenvalue

(ii) Describe the type and the stability of the critical point $(0, 0)$:

(a) a unstable spiral point
(b) an asymptotically stable spiral point
(c) a unstable saddle point
(d) an asymptotically stable node
(e) a unstable node
(f) a stable center

13. Find all values of k for which the equilibrium point $(0,0)$ of the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & k \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is a saddle point.

a) $k > 0$

b) $k > 1$

c) $k < 0$

d) $k < 1$

e) $k \geq 1$

14. Find the solution of the linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

a) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2}e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2}e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2}e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

d) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

e) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2}e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

15. Find the solution of the linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

a) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2} e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2} e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2} e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{2} e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

d) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

e) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-3t} \begin{pmatrix} 3+4t \\ 2+4t \end{pmatrix}$

16. The general solution of $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} + t \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ is

a) $\mathbf{x}(t) = C_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

b) $\mathbf{x}(t) = C_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3/2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3/4 \end{pmatrix}$

c) $\mathbf{x}(t) = C_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -4 \\ -3 \end{pmatrix}$

d) $\mathbf{x}(t) = C_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -4 \\ 3/2 \end{pmatrix} + \begin{pmatrix} -4 \\ 3/4 \end{pmatrix}$

e) $\mathbf{x}(t) = C_1 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4/3 \\ 3/2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

17. One of the critical points (equilibrium solutions) of

$$\frac{dx}{dt} = e^{-x+y} - \cos(x),$$
$$\frac{dy}{dt} = \sin(x - 3y)$$

is $(0, 0)$.

a) Find the corresponding linear system near $(0, 0)$.

b) Describe the type ______________ (saddle, node, spiral, center) and the stability __________________ (asymptotically stable, stable, unstable) of the critical point $(0, 0)$ for the linearized system.

c) The critical point (0,0) is ______________ (stable, unstable, indeterminate).

18. The following points are critical points (equilibrium solutions) of

$$\frac{dx}{dt} = (1 + x)\sin(y),$$
$$\frac{dy}{dt} = 1 - x - \cos(y)$$

a) (0, 0), $(2, \pi)$, $(0, 2\pi)$

b) (0, 0), (1, 1), (2, 2)

c) (0, 0), $(1, \pi)$, $(2, 2\pi)$

d) (0, 0), $(2, -\frac{\pi}{2})$, $(0, -\pi)$

e) (0, 0), $(2, \pi)$, $(4, 2\pi)$

19. One of the critical points (equilibrium solutions) of

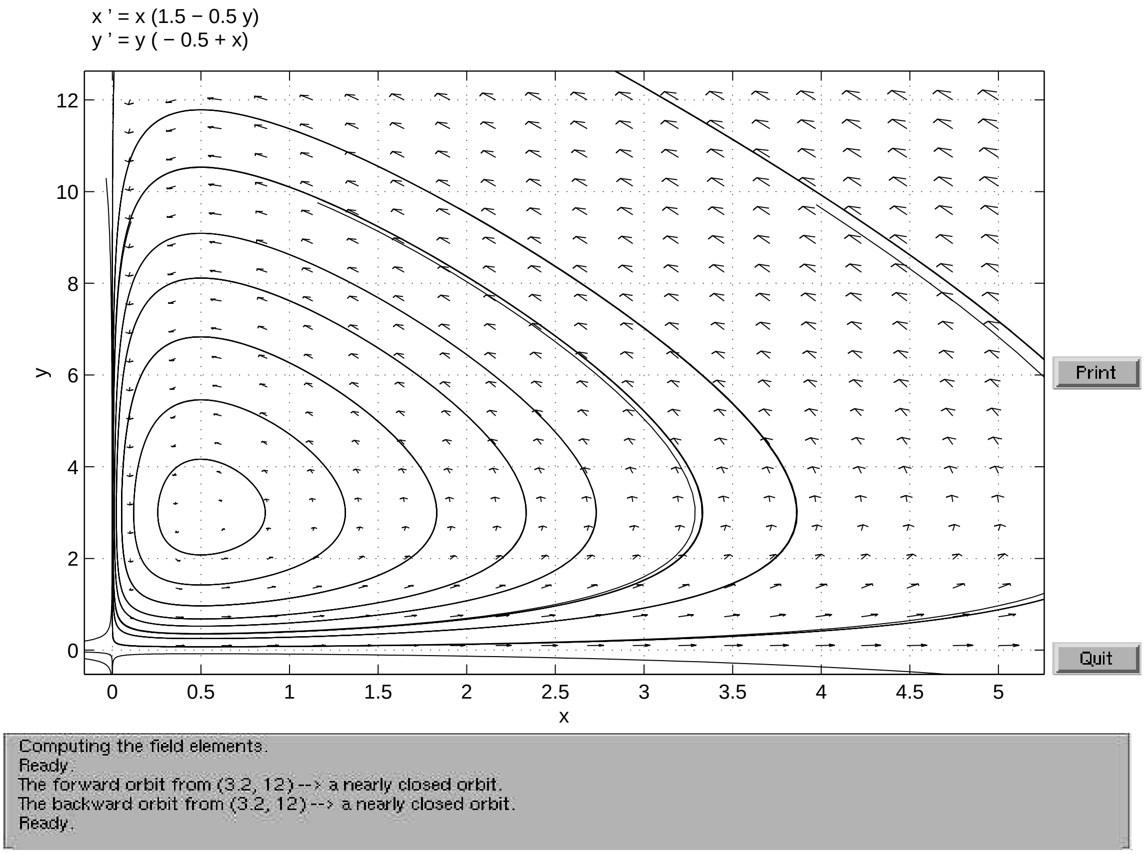
$$\frac{dx}{dt} = y - z,$$
$$\frac{dy}{dt} = x - z,$$
$$\frac{dz}{dt} = x^2 + y^2 - 2z,$$

is $(1, 1, 1)$.

a) Find the corresponding linear system near $(1, 1, 1)$.

b) The critical point $(1, 1, 1)$ is ______________ (asymptotically stable, unstable, indeterminate).

20. The phase portrait of a predator-prey model is



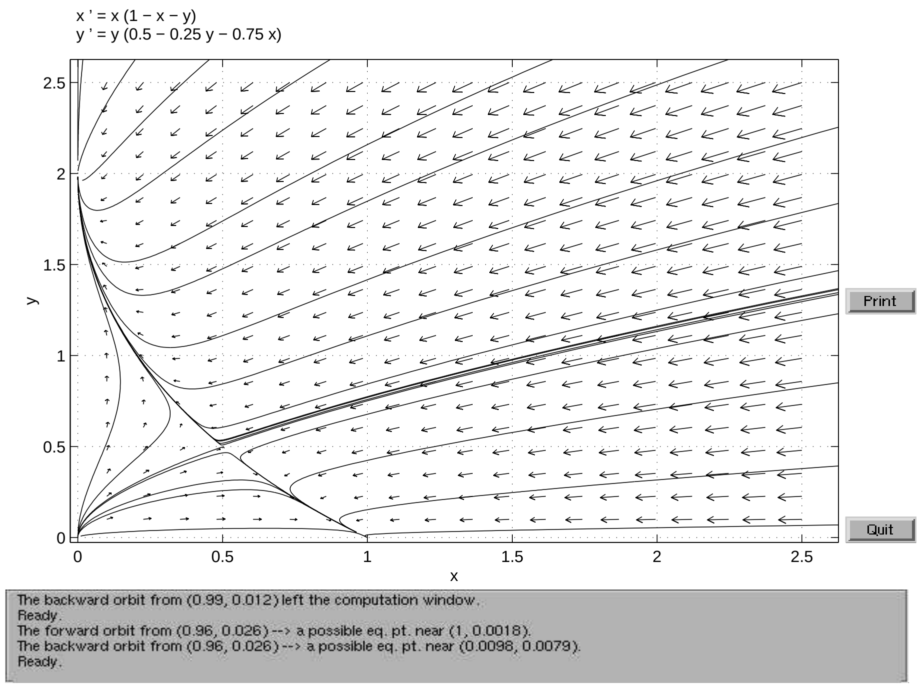
where $x(t)$ and $y(t)$ are population densities of the two species.

a) Based on the fact that when there is no predator, the population density of prey will increase and when there is no prey, the population density of predator will decrease. $x(t)$ is the population density of ______________ (predator, prey).

b) If $x(t_0) = 2.5, y(t_0) = 1.8$, then $x(t)$ is ______________ (increasing, decreasing) at $t = t_0$.

c) If $x(t_0) = 2.5, y(t_0) = 1.8$, then $y(t)$ is ______________ (increasing, decreasing) at $t = t_0$.

21. Suppose that in some closed environment there are two similar species competing for a limited food supply. Let $x(t)$ and $y(t)$ be the population of the two species. The phase portrait is as follows,



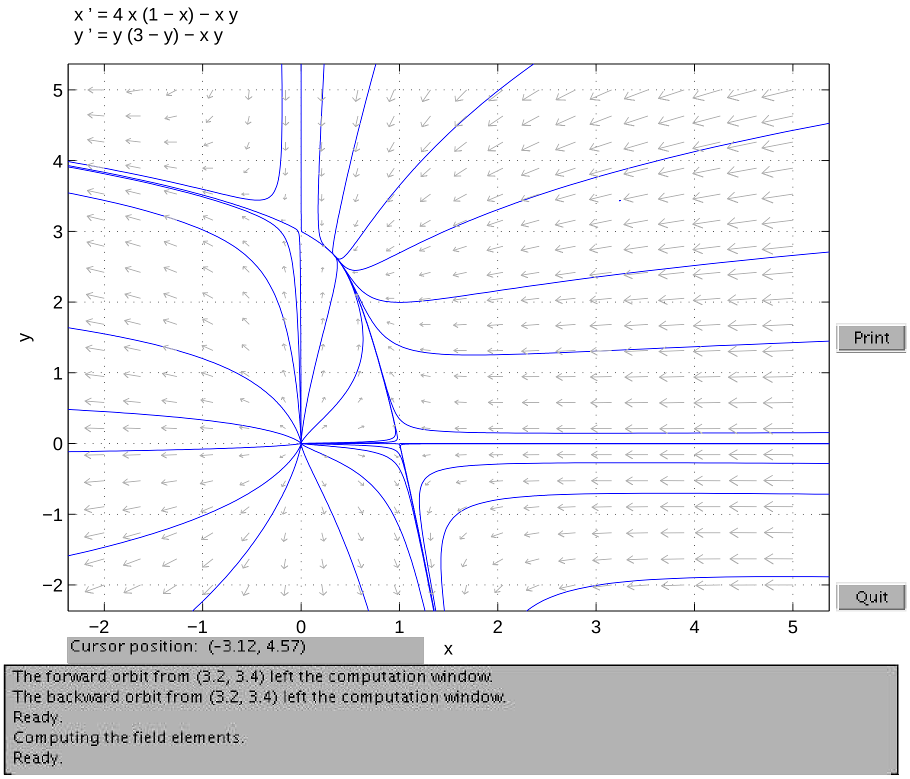
a) if $x(0) = 2, y(0) = 2, \lim_{t\to\infty}(x(t), y(t)) = (\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_)$.

b) for any initial data $x(0) > 0$ and $y(0) > 0$, what are the possible limiting values of $(x(t), y(t))$ when $t \to \infty$?

22. The system

$$\frac{dx}{dt} = 4x(1-x) - xy,$$
$$\frac{dy}{dt} = y(3-y) - xy$$

has the following phase portrait



(i) Find the equilibrium solutions; (ii) By inspecting the phase portrait, classify each as stable or unstable; (iii) classify each as saddle point, nodal source or nodal sink.

23.

EXAMINATION (Answer)

1. $1.442, \quad -1.22$
2. B
3. $r = 1, \quad 2$
4. $y = \dfrac{-2e^{4t^2}}{1+e^{4t^2}}$
5. $y = -2$
6. $y = \dfrac{1-\sqrt{1+4t^2}}{2}$
7. $y = \dfrac{t^2}{3}e^{3t} - \dfrac{1}{3}e^3t^2$
8. E
9. $y = c_1e^{-t}\cos(t) + c_2e^{-t}\sin(t) + e^{-2t}(2\cos(t)+\sin(t))$
10. $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sin(2t) & 3t & 0 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ 0 \\ 7e^{-t} \end{pmatrix}.$
11. $y = c_1e^{-t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} \cos(t)-\sin(t) \\ -\sin(t) \\ 2\cos(t) \end{pmatrix} + c_3e^{-t}\begin{pmatrix} \sin(t)+\cos(t) \\ \cos(t) \\ 2\sin(t) \end{pmatrix}$
12. (e) and (b)
13. C
14. D
15. E
16. B
17. (a) $y' = Ay$ where $A = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix}$, (b) node, asymptotically stable, (c) asymptotically stable
18. A
19. (a) $y' = Ay$ where $A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & -2 \end{pmatrix}$, (b) Asymptotically stable
20. (a) prey; (b) increasing; (c) increasing
21. (a) $\lim_{t\to\infty}(x(t), y(t)) = (0,2)$; (b) (0,2), (1,0), $(\frac{1}{2}, \frac{1}{2})$
22. (a) Equilibrium points: (0,0), (0, 3), (1, 0), (1/3, 8/3)

(b) (in the same ordering as in (a): Unstable, Unstable, Unstable, Asymptotically stable

(c) (in the same ordering as in (a): Nodal source, Saddle, Saddle, Nodal sink