MA 440 (Honors)
Practice Problems For Final (Revised on Dec. 11, 2009)

NAME ________________________ ID number ________________________

The practice problems for final include the problems from homeworks, quizzes, midterms and the following. The majority problems on the final will be similar to the problems in these 4 sets of problems.

1. If $\xi \in \mathbb{R}$ is irrational and $r \in \mathbb{Q}$ and $r \neq 0$, show $r + \xi$ is irrational.

2. If $a > -1, a \in \mathbb{R}$, show that $(1+a)^n \geq 1 + na$ for all $n \in \mathbb{N}$ by using mathematical induction.

3. If $a > -1, a \in \mathbb{R}$, show that $(1+a)^r \geq 1 + ra$ for all $r \geq 1$.

4. State the *Supremum Property*

5. Prove the *Archimedean Property*, namely, show for every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$, such that $x < n$.

6. State the *Nested Cells Property*

7. (Schwarz inequality) Let V be an inner product space. Define

$$\|x\| = \sqrt{x \cdot x} \text{ for } x \in V$$

show $x \cdot y \leq \|x\|\|y\|$.

8. Let S be a set in $\mathbb{R}^p$. State the definition that a point x is a boundary point of S. State the definition that a point x is a cluster point of S. What are the differences?

9. Give an example such that x is a cluster point, but not a boundary point. Also give an example that x is a boundary point, but not a cluster point.

10. Show that if F is closed, then any cluster point of F is in F.

11. Show that if F is closed, then any boundary point of F is in F.

12. Prove that the set of all cluster points of $A \subset \mathbb{R}^p$ is closed.

13. Let S be a subset in $\mathbb{R}^p$ and denote ∂S be the set of all boundary points of S, Show that ∂S is closed. **Answer:** Show the complement is open

14. Show that if $S \subset \mathbb{R}$ is open, then it is the union of a countable collection of open intervals.

15. State the definition for a set K to be compact. Show directly from definition that $K = \{(x, y) : |x| + |y| < 1\}$ is not compact.

16. Show that if a set K in $\mathbb{R}^p$ is compact, then it is bounded.

Answer: Let $G_n = B_n(0)$ which is the open ball centered at 0 with radius n, then $\cup_{n=1}^{\infty} G_n = \mathbb{R}^p \supset K$. So $\{G_n : n \in \mathbb{N}\}$ is an open covering of K. Since K is compact, there is a finite open covering, i.e. there exists m, such that

$$K \subset \cup B_{n_1} \cup B_{n_2} \cup \cdots \cup B_{n_m} \subset B_L$$

where $L = \max\{n_1, n_2, \cdots, n_m\}$. Therefore K is bounded.

17. Show that if a set K in $\mathbb{R}^p$ is compact, then it is closed.

18. Show that if a set K in $\mathbb{R}^p$ is compact, then for a sequence (a_n) in K, if (a_n) converges to a, then a is in K.

19. Let D be a subset in $\mathbb{R}^p$, give the definition for D to be disconnected.

Answer: There exist two **open** sets A, B such that $A \cap D$ and $B \cap D$ are disjoint, non-empty and have union D

20. Using the fact that $\mathbb{R}^p$ is connected, show that the only subsets of $\mathbb{R}^p$ which are both open and closed are empty set ϕ and $\mathbb{R}^p$.

21. Give an example that A and B are connected subsets in $\mathbb{R}^p$, but $A \cap B$ is disconnected.

Answer: $A = \{(x, y) : x^2 + y^2 = 1\}, B = \{(x, y) : y = 0\}$, then $A \cap B = \{(1, 0), (-1, 0)\}$

22. Let K be a compact subset of $\mathbb{R}^p$ and let x be any point in $\mathbb{R}^p$ such that x is not in K. Prove that there exist open sets U and V, where U and V are disjoint, U contains K and V contains x.

Answer: K is compact, from Heine-Borel Theorem, K is closed, and then $\mathcal{C}(K)$–the compliment of K–is open. Since $x \in \mathcal{C}(K)$ which is open, there exists a $\epsilon > 0$, such that $B_\epsilon(x) \subset \mathcal{C}(K)$. Now let $V = B_{\epsilon/2}(x)$ and $U = \{y : \|x - y\| > \epsilon/2\}$, then U and V are open and disjoint, V contains x and U contains K because $K \subset \mathcal{C}(B_\epsilon(x)) = \{y : \|x - y\| \geq \epsilon\} \subset U$.

23. Let K_1 and K_2 be compact subsets of $\mathbb{R}^p$. Then there exist $x_1 \in K_1$ and $x_2 \in K_2$ such that for all $z_1 \in K_1$ and $z_2 \in K_2$, $\|z_1 - z_2\| \geq \|x_1 - x_2\|$.

Answer: Let

$$r = \inf_{z_1 \in K_1, z_2 \in K_2} \|z_1 - z_2\|.$$

By the definition of the infimum, there exist $a_n \in K_1, b_n \in K_2$, such that

$$\lim_{n \to \infty} \|a_n - b_n\| = r.$$

By the Bolzano-Weierstrass theorem, a subsequence (a_{n_k}) of (a_n) will converge to a $x_1 \in K_1$. Apply the Bolzano-Weierstrass theorem to the subsequence (b_{n_k}), there is a subsequence $(b_{n_{k_l}})$ that will converge to $x_2 \in K_2$. Therefore

$$a_{n_{k_l}} - b_{n_{k_l}} \to x_1 - x_2.$$

So, from homework 14.D

$$\|a_{n_{k_l}} - b_{n_{k_l}}\| \to \|x_1 - x_2\|$$

and $LHS \to r$ yields $\|x_1 - x_2\| = r$.

24. Show that if a monotone increasing sequence (x_n) in $\mathbb{R}$ is bounded, then it is convergent. Also $\lim_{n\to\infty} x_n = \sup x_n$.

25. Show *Bolzano-Weierstrass Theorem.* Namely, let (x_n) be a bounded sequence in $\mathbb{R}^p$ contains infinite distinct values. Then it has a convergent subsequence.

26. State the definition for (x_n) to be a Cauchy sequence. Show that (s_n) where $s_n = \sum_{k=1}^{n} \frac{1}{k}$ is not a Cauchy sequence.

27. Show that a bounded divergent sequence (x_n) must has two convergent subsequences which converge to different values.

28. Let (x_n) be a sequence in a compact set $K \subset \mathbb{R}^p$ that is not convergent. Show there are two subsequences of this sequence that are convergent to different limit points. **Answer:** same as previous

29. Let $s_n = (-2)^{(-2)^n}$. Find limsup s_n and liminf s_n and justify your answer. **Answer:** ∞, 0

30. Let(x_n) be a positive sequence and $\lim_{n\to\infty} x_n^{1/n} < 1$, show that there exists a r with $0 < r < 1$, $0 \le x_n < r^n$ for sufficiently large $n \in \mathbb{N}$.

31. Give the definition for $u \in \mathbb{R}$ to be an infimum of a non-empty subset S of $\mathbb{R}$.

Answer: u is greater than any other lower bound.

32. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be given and satisfy

$$x_n \le y_n \le z_n, \quad \lim x_n = \lim z_n = L$$

Prove by definition $\lim y_n = L$.

Answer:

$$\forall \epsilon > 0, \exists n_1, \text{ for all } n > n_1, -\epsilon < x_n - L < \epsilon$$

$$\text{there also } \exists n_2, \text{ for all } n > n_2, -\epsilon < z_n - L < \epsilon$$

since

$$x_n \le y_n \le z_n,$$

we have

$$-\epsilon < x_n - L \le y_n - L \le z_n - L \le \epsilon \text{ for all } n > \max\{n_1, n_2\}$$

namely

$$\|y_n - L\| < \epsilon \text{ for all } n > \max\{n_1, n_2\}.$$

33. Show that if $\sum a_n$ converges and $a_n \geq 0$, then $\sum \frac{\sqrt{a_n}}{n}$ converges. **Answer:** $\frac{\sqrt{a_n}}{n} \leq a_n + \frac{1}{n^2}$

34. Show that if $\sum a_n$ diverges and $a_n \geq 0$, then $\sum \frac{1+a_n}{a_n}$ diverges. **Answer:** $\frac{1+a_n}{a_n} \geq 1$, divergence test gives the result.

35. Let $f(x)$ be continuous with domain $D(f)$, $K \subset D(f)$ and K is compact. Show $f(K)$ is bounded.

36. Let $f(x)$ be continuous with domain $D(f)$, $K \subset D(f)$ and K is compact. Show $f(K)$ is closed.

37. Show that if $f(x)$ is a contraction from R^p to R^p, then $f(x)$ has a fixed point.

38. Show that if $(f_n(x))$ converges uniformly to $f(x)$ and $(f_n(x))$ are continuous on D where D is a compact set in $\mathbb{R}$, then $f(x)$ is continuous on D. (Where the "uniformly" is used?)

39. If a sequence $(f_n(x))$ converges uniformly to a function $f(x)$ on $[a, b]$, and each $f_n(x)$ is continuous and bounded. Show that $f(x)$ is continuous and bounded. **Answer:** $f(x)$ is continuous by uniform convergence and $f(x)$ is bounded by $f(x)$ is continuous on a compact set.

40. If a sequence $(f_n(x))$ converges uniformly to a function $f(x)$ on $[a, b]$, and each $f_n(x)$ is continuous and bounded. Show directly by definition that $f(x)$ is uniform continuous.

41. Show that if $f_n'(x)$ converges uniformly to $g(x)$ in $J = [a, b]$ and $f_n(x)$ converges at x_0, then $f_n(x)$ converges to $f(x)$ where $f'(x) = g(x)$.

42. Let

$$g(x) = \begin{cases} x^2 & \text{for} \quad 0 \leq x < 2, \\ x^3 & \text{for} \quad 2 \leq x < 3 \end{cases}.$$

Evaluate the Riemann-Stieltjes integral

$$\int_0^3 x dg(x)$$

and briefly justifying your computation. **Answer:** $\frac{745}{12}$

43. Let

$$g_n(x) = \begin{cases} nx & \text{for} \quad 0 \leq x \leq 1/n, \\ \frac{n}{n-1}(1-x) & \text{for} \quad 1/n < x \leq 1. \end{cases}$$

Show that (g_n) converges pointwise on $[0, 1]$ and find the limit function. Does it converge uniformly?

44. Let (x_n) be a sequence of real numbers such that $|x_n| \leq \frac{1}{2^n}$, and set $y_n = x_1 + x_2 + \cdots + x_n$. Show the sequence (y_n) converge.

Answer: We will show that y_n is Cauchy. For any $j > k$:

$$|y_j - y_k| \leq \sum_n = k + 1^j \frac{1}{2^n} = \frac{1}{2^j} - \frac{1}{2^k} \leq \frac{1}{2^j}$$

For any $\epsilon > 0$, let K be $2^K = \epsilon$ ($K = \frac{\ln 2}{\ln \epsilon}$), then for $j, k > K$ $|y_j - y_k| < \epsilon$.

45. Show that if f is continuous and bounded on $[a, b]$, then f is Riemann integrable.

46. Show that if f is a bounded function on $[0, 1]$ and if for every $a > 0$, f is Riemann integrable on $[a, 1]$, then f is integrable on $[0, 1]$.

47. State Taylor's Theorem. Give Taylor's Formula using 3 terms (including the remainder) with $f(x) = \sqrt{x}$ and $x_0 = 1$. In the remainder term, find the point at which the second derivative is evaluated.

48. Prove that if f has a continuous third derivative and satisfies $f(0) = f'(0) = f''(0) = 0$ and $f'''(x) \leq 1$ for $x \geq 0$, then $f(x) \leq x^3/3$ for $x \geq 0$.

49. Let

$$f_n = \frac{(-1)^n}{2^n} cos(2\pi n x^2), x \in [0, 1], n \in \mathbb{N}$$

show that $\sum_{n=1}^{\infty} f_n(x)$ converges.

50. Prove or disprove the series $\sum_{n=1}^{\infty} \sin(n^{-2}) \cos(n^{-1})$ converges. **Answer:** yes. Use $\sin(n^{n^{-2}}) < 1/n^2$

51. Let f, f', f'' be bounded and continuous in $\mathbb{R}$ and $f(0) = f'(0) = 0$. Show that $\sum_{n=1}^{\infty} f(\frac{x}{n})$ converges. **Answer:** using Taylor's theorem $\sum_{n=1}^{\infty} f(\frac{x}{n}) \leq Mx^2 \sum \frac{1}{n^2}$

52. Let $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$. Compute $f'(\frac{1}{3})$ and justify each steps which leads to the result. **Answer:** $3log(\frac{3}{2})$. Steps: $f(x)$ exists for $|x| \leq 1$. $S_n'(x)$ converges, so one can do term by term. Therefore $f'(x)$ exists. $f'(x) = \frac{1}{x} \sum \frac{x^k}{k} = \frac{1}{x} \sum \int_0^x t^{k-1} dt = \frac{1}{x} \int_0^x \sum t^{k-1} dt = \frac{1}{x} \int_0^x \frac{1}{1-t} dt$.

53. Show there does not exist a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, such that

(a) $f\{(x, y) : |x| \leq 1, |y| \leq 2\} = \mathbb{Q} \cap [0, 1]$

(b) $f\{(x, y) : |x| \leq 1, |y| \leq 2\} = [0, \infty)$

(c) $f^{-1}\{x : |x| < 1\} = \{|x| \leq 1, |y| \leq 2\}$

(d) $f^{-1}\{x : |x| \leq 1\} = \{|x| < 1, |y| < 2\}$

Answer: For (a), (b) f maps compact set to compact set. For (c), (d): f^{-1} (open set) =open set and f^{-1} (closed set) =closed set

54. Let A be a non-compact subset of the real line. Show that there exists a continuous function on A that is unbounded on A. **Answer:** If A is unbounded, then let $f(x) = x$. If A is not closed, then there exists x_n in A and $x_n \to x$ where x is not in A. Let $f(y) = \frac{1}{y-x}$ then f is continuous but unbounded.

55. Prove that $2\pi \sin(x) = 1 + x^2$ has at least two real roots and locate disjoint intervals (a, b), (c, d) which contain them.

56. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $\lim_{|x|\to\infty} f(x) = 0$. Show that $f(x)$ is uniformly continuous.

57. Let $f(x)$ be continuous on $[0, 1]$. Show that

$$h(x) = \sum \frac{f(x)^n}{(1 + |f(x|)^n}$$

is also continuous on $[0, 1]$. **Answer:** Let $g(x) = \frac{f(x)}{(1+|f(x|)}$. Then $g(x) < 1$ and continuous on a compact set. There exists $C < 1$, $|g(x)| \leq C < 1$. Therefore

$$\sum \frac{f(x)^n}{(1 + |f(x|)^n}$$

converges absolutely and uniformly (detail). So $h(x)$ is continuous on $[0, 1]$.

58. Let $f_n(x) = \frac{xn}{n+1}$ and let $f(x)$ be the limit function of $f_n(x)$. Find $f(x)$ and show that $f_n(x)$ does not converge to $f(x)$ uniformly. **Answer:** $f(x) = x$ and $\|f_n - f\|_{\mathbb{R}} = \infty$

59. Let (x_n) be a sequence in R^p with the property that there exists a real number $0 < r < 1$, and an integer N_0 such that

$$\|x_{n+1} - x_n\| \leq r\|x_n - x_{n-1}\| \text{ for } n \geq N_0.$$

Then prove (x_n) converges.

60. Let (x_n) be a sequence in R^p with the property that there exists an integer N_0 such that

$$\|x_{n+1} - x_n\| < \|x_n - x_{n-1}\| \text{ for } n \geq N_0$$

Can you show (x_n) converges? Justify your answer.

61. Let (x_n) be an unbounded monotone increasing sequence, show that $\lim x_n = +\infty$.

62. True or False. Justify your answer.

(a) Every sequence has a nondecreasing subsequence.

(b) Every sequence has a bounded subsequence.

(c) Every bounded sequence has a monotonic subsequence.

(d) Every subsequence of a bounded monotonic sequence converges.

(e) Every bounded sequence has a convergent subsequence.