MA 440 (Honors)
Midterm Examination
Oct, 7, 2009

NAME ______________________ ID number ______________________

THIS EXAM IS CLOSED TO BOOKS AND NOTES.

Points awarded

1. (12 pts) ____________
2. (4 pts) ____________
3. (4 pts) ____________
4. (4 pts) ____________
5. (6 pts) ____________
6. (10 pts) ____________
7. (10 pts) ____________
8. (10 pts) ____________
9. (10 pts) ____________

Total Points: ____________/70

1. (12 points) Decide whether the following statements are true or false.

(a) (x, y irrational $\Rightarrow x + y$ irrational).
Answer: False

(b) (x^2 irrational $\Rightarrow x$ irrational).
Answer: True

(c) (x irrational $\Rightarrow x^2$ irrational).
Answer: False

(d) (A sequence is bounded and monotone, then it is Cauchy).
Answer: True

(e) (A sequence is Cauchy, then it is monotone and bounded).
Answer: False

(f) (Let A_1, A_2, A_3 be closed subsets of $[0, 1]$, then $A_1 \cup A_2 \cup A_3$ is compact.)
Answer: True

2. (4 points)

(a) Let $S = (0, 1)$ and $C = \{A_\alpha = B_\epsilon(\alpha), \alpha \in S\}$, where $B_\epsilon(\alpha)$ is the open cell centered at α with radius $1 > \epsilon > 0$. What is the least number of A_α required to cover S? ____________

Answer: $1 + [1/\epsilon]$

(b) Let $S = (0, 1)$ and $C = \{(\frac{1}{j}, 1),\ \text{for } j \in \mathbb{N}\}$. What is the least number of sets in C required to cover S? ____________
Answer: ∞

3. (4 points) Give an example that A and B are connected subsets in $\mathbb{R}^p$, but $A \cap B$ is disconnected.

Answer: $A = \{(x, y) : x^2 + y^2 = 1\}, B = \{(x, y) : y = 0\}$, then $A \cap B = \{(1, 0), (-1, 0)\}$

4. (4 points) Consider the following statements:
(i) For every $M \in \mathbb{R}$, there is an $N_0 \in \mathbb{N}$, such that for all $n > N_0$, $a_n > M$.
(ii) For every $M \in \mathbb{R}$, there is an $N_0 \in \mathbb{N}$, such that $a_{N_0} > M$.

(a) Find a sequence $\{a_n\}$ which satisfies (i).

(b) Find a sequence $\{a_n\}$ which **satisfies (ii) but not (i)**.

Answer: (a), $a_n = n, n \in \mathbb{N}$, (b) $a_n = (-1)^n n, n \in \mathbb{N}$

5. (6 points)

(a) Let D be a subset in $\mathbb{R}^p$, give the definition for D to be disconnected.

Answer: There exist two **open** sets A, B such that $A \cap D$ and $B \cap D$ are disjoint, non-empty and have union D

(b) Give the definition for $u \in \mathbb{R}$ to be an infimum of a non-empty subset S of $\mathbb{R}$.

Answer: u is greater than any other lower bound.

6. (10 points) Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be given and satisfy

$$x_n \le y_n \le z_n, \quad \lim x_n = \lim z_n = L$$

Prove by definition $\lim y_n = L$.

Answer:

$$\forall \epsilon > 0, \exists n_1, \text{ for all } n > n_1, -\epsilon < x_n - L < \epsilon$$

$$\text{there also } \exists n_2, \text{ for all } n > n_2, -\epsilon < z_n - L < \epsilon$$

since

$$x_n \le y_n \le z_n,$$

we have

$$-\epsilon < x_n - L \le y_n - L \le z_n - L \le \epsilon \text{ for all } n > \max\{n_1, n_2\}$$

namely

$$\|y_n - L\| < \epsilon \text{ for all } n > \max\{n_1, n_2\}.$$

7. (10 points) Prove by using the definition that if $K \in \mathbb{R}^p$ is compact, then K is bounded.

Answer: Let $G_n = B_n(0)$ which is the open ball centered at 0 with radius n, then $\cup_{n=1}^{\infty} G_n = \mathbb{R}^p \supset K$. So $\{G_n : n \in \mathbb{N}\}$ is an open covering of K. Since K is compact, there is a finite open covering, i.e. there exists m, such that

$$K \subset \cup B_{n_1} \cup B_{n_2} \cup \cdots \cup B_{n_m} \subset B_L$$

where $L = \max\{n_1, n_2, \cdots, n_m\}$. Therefore K is bounded.

8. (10 points) Let K be a compact subset of $\mathbb{R}^p$ and let x be any point in $\mathbb{R}^p$ such that x is not in K. Prove that there exist open sets U and V, where U and V are disjoint, U contains K and V contains x.

Answer: K is compact, from Heine-Borel Theorem, K is closed, and then $\mathcal{C}(K)$–the compliment of K–is open. Since $x \in \mathcal{C}(K)$ which is open, there exists a $\epsilon > 0$, such that $B_\epsilon(x) \subset \mathcal{C}(K)$. Now let $V = B_{\epsilon/2}(x)$ and $U = \{y : \|x - y\| > \epsilon/2\}$, then U and V are open and disjoint, V contains x and U contains K because $K \subset \mathcal{C}(B_\epsilon(x)) = \{y : \|x - y\| \geq \epsilon\} \subset U$.

9. (10 points) Let K_1 and K_2 be compact subsets of $\mathbb{R}^p$. Then there exist $x_1 \in K_1$ and $x_2 \in K_2$ such that for all $z_1 \in K_1$ and $z_2 \in K_2$, $\|z_1 - z_2\| \geq \|x_1 - x_2\|$.

Answer: Let

$$r = \inf_{z_1 \in K_1, z_2 \in K_2} \|z_1 - z_2\|.$$

By the definition of the infimum, there exist $a_n \in K_1, b_n \in K_2$, such that

$$\lim_{n \to \infty} \|a_n - b_n\| = r.$$

By the Bolzano-Weierstrass theorem, a subsequence (a_{n_k}) of (a_n) will converge to a $x_1 \in K_1$. Apply the Bolzano-Weierstrass theorem to the subsequence (b_{n_k}), there is a subsequence $(b_{n_{k_l}})$ that will converge to $x_2 \in K_2$. Therefore

$$a_{n_{k_l}} - b_{n_{k_l}} \to x_1 - x_2.$$

So, from homework 14.D

$$\|a_{n_{k_l}} - b_{n_{k_l}}\| \to \|x_1 - x_2\|$$

and $LHS \to r$ yields $\|x_1 - x_2\| = r$.