# NONLINEAR WAVE PHENOMENON
# DRAFT, 8, 2003

November 7, 2003

Preface: These notes are concerned with some aspects of theoretical fluid mechanics, especially wave propagation. Various problems arising in fluid mechanics are treated in detail, calling on methods from modern functional analysis, the theory of partial differential equations and numerical analysis and simulation.

# Contents

# Chapter 1

# Introduction and a Brief Review of the History

Model equations for waves that take account of both nonlinearity and dispersion have their genesis in the discovery of the solitary wave by John Scott Russell. The story of Scott Russell's encounter with the solitary wave in 1834 has been retold many times.

While working as a consultant for the xxx on the Edinburgh-Glasgow canal, Scott Russell witnessed a heavily laden barge drawn by a pair of horses come suddenly to rest, owing to an obstruction in the canal. This sudden cessation of forward motion created various disturbances on the water's surface, including a long-crested wave some 18 inches in elevation that went rolling off down the canal in the direction the barge had been traveling. Scott Russell gave chase by horse and observed the wave, which was more or less uniform in the spanwise direction, propagated with constant speed and without change of shape. Fascinated, Scott Russell went on to conduct a set of laboratory experiments on this phenomenon which he reported in 1841 and 1844 to the British Association (see Scott Russell 1845 [10]). Among other appellations he called such waves solitary waves.

The more theoretically inclined scientists interested in fluid mechanics soon took Scott Russell to task. The Astronomer Royal, Sir George Airy addressed the issue of whether or not it was possible to have a steadily propagating wave of permanent form on the surface of water. He concluded such waves were not possible on the basis of analysis to be described presently.

Stokes, who was later also accorded the title Sir George, analyzed waves on the surface of water, concluding on the basis of forthcoming analysis that

such wave motion was not possible.

Despite the mathematical theory, the experimental evidence in favor of solitary waves was convincing. The issue lay unresolved until the seminal work of Boussinesq in the 1870's [2, 3, 4]. With the hindsight derived from Boussinesq's work, one sees clearly that both Airy and Stokes were on the right track, and both had part of the issue in hand, as will become apparent in the next section.

Lord Rayleigh also addressed the issue of existence of solitary waves, and concluded in a long article on waves published in 1876 that there were such motions. His paper is curious because, in addition to an approximate analysis dealing successfully with solitary waves, he also has material analogous to Airy's shallow-water theory which was taken as evidence that such traveling waves do not exist.

In the 1890's, a couple of other significant papers were published. McKeon (1894) [?]. The famous paper of Korteweg and de Vries (1895) [7] appeared the next year. These Dutch scientists were apparently ignorant of the work of Boussinesq, for they refer to Stokes' much earlier paper (Stokes 1845 ) [11]. In a clear account which is very readable more than 100 years after it was written, Korteweg and de Vries lay out the essential modelling and mathematical issues that go into the 19th century analysis of Scott Russell's solitary waves.

At the turn of the century, it seems fair to say that Scott Russell, who died in 18 , was vindicated in his view of traveling waves of elevation existing on the surface of water. It is worth note that Stokes reversed himself in print regarding whether or not solitary waves exist.

In the first half of the 20th century, solitary waves and related evolution equations were not a major topic of scientific conversation. The notion of a solitary wave was used in a descriptive manner, but it does not appear as a central issue in theoretical discussion. For example, Lamb's (1932)[9] rendering of solitary waves accords Boussinesq a footnote, does not mention the Korteweg-de Vries equation, but centers around Lord Rayleigh's development, which in retrospect was probably the least interesting approach since he did not derive an evolution equation which could countenance a range of disturbances, but rather passed directly to a traveling-wave description.

The oceanographer Keulegan pioneered the use of the idea of a solitary wave, particularly solitary internal waves, in geophysical applications. Keulegan with Patterson (1940) wrote an article [6] that reviewed some of Boussinesq's ideas. As the original was somewhat inaccessible, this proved to be a

very helpful endeavor.

The linear heat equation features infinite speed of propagation. In principle, a candle lit in Austin, Texas could be detected immediately in Florianopolis, Brasil with sufficiently accurate instruments. In fact, heat does not propagate at infinite speed. Enrico Fermi was looking for a model for heat conduction that featured finite speed of propagation. With John Pasta and Stanislaw Ulam, he put forward a discrete spring and mass model such as one encounters in elementary physics courses. The difference was the springs were not Hooksian, but instead the restoring force had a quadratic dependence on the extension. Gravity is ignored, and so Newton's laws lead to a coupled system of nonlinear ordinary differential equations. Exact solutions were not available, so they resorted to numerical simulation using Los Alamos Laboratory's ENIAC computer. What they found did not correspond well to heat conduction; it seems this simple mass and spring systems features near recurrence of initial states, and not the kind of thermalization one expects. A Los Alamos report was duly constructed and the issue then lay dormant. Fermi died in 195... holding the opinion that these numerical simulations were somehow important, but not knowing exactly why.

A few years later, Gardner and Morikawa (1960) [?], studied the stability of a cold collisionless plasma as it arises in a putative description of nuclear fusion. Starting from the full Magneto-Hydrodynamic equations, and making simplifying assumptions about the motion of the plasma, they derived the same equations as had Boussinesq and Korteweg - de Vries, although the physical context was different. Their work appeared initially as an NYU report, but was published in the permanent literature only many years later.

At the Plasma Physics Laboratory in Princeton University, Martin Kruskal knew of the work of Gardner and Morikawa. He also knew about the work of Fermi, Pasta and Ulam and at a certain stage, in collaboration with Norman Zabusky, he revisited their model. Kruskal and Zabusky took a continuum limit of the original discrete system. The system of ordinary differential equations goes over to a partial differential equation in this limit, and the equation in question was the Boussinesq-Korteweg-de Vries equation again. A well-conceived sequence of numerical experiments for the spatially-periodic initial-value problem was carried out and reported (Kruskal and Zabusky 196 [8]). These experiments showed some of the same fascinating properties that Fermi, Pasta and Ulam had seen earlier. The Korteweg-de Vries equation had now arisen as a description of three, distinct physical systems.

Further study of the Korteweg-de Vries equation led to the inverse-scattering

theory for the initial-value problem. This imaginative leap was first described by Gardner, Greene, Miura and Kruskal (1967) and later amplified in a series of papers entitled Korteweg-de Vries Equation and Generalization (see [?] ). Peter Lax (1968) [?] made a fundamental step forward by providing a mathematical framework in which to consider the inverse-scattering theory as it applies to initial-value problems for partial differential equations.

Shortly afterward, the subject began to assume industrial proportions and it quickly becomes difficult to trace the developments. Indeed, many areas of mathematics, physics and mechanics have been influenced by the elaboration and extension of the ideas just outlined.

# Chapter 2

# Model Equations for Waves in Dispersive Media

The Euler equation of fluid dynamics consists of Newton's laws of motion for continuous and homogeneous matter in the fluid state. We consider a body of water of finite depth under the influence of gravity, bounded below by an impermeable surface. Ignoring the effects of viscosity and assuming the flow is incompressible and irrotational, the fluid motion is taken to be governed by the Euler equations together with suitable boundary conditions on the rigid surface and on the water-air interface. After briefly explaining the Euler equations, further approximations are introduced and analyzed, leading to a set of model equations formally valid for small-amplitude long wavelength motion.

## 2.1 Derivation of Model Equations for Waves in Dispersive Media

Let $oxyz$ be a right-handed (Cartesian) coordinate system with $oy$ pointing in the direction opposite to that of gravity, $ox$ to the right, and $oz$ toward us from the page. Let $\mathbf{u}(\mathbf{x}, t) = (u, v, w)$ denote the velocity vector of the mass at point $\mathbf{x} = (x, y, z)$ and at time $t$. We first introduce the definition of convective derivatives.

**Convective derivatives (or material derivatives)**    Convective deriva-
tive, denoted by $\frac{DF}{Dt}$, calculates the rate of a function $F(x, y, z, t)$ varies for
a moving particle, namely the derivative following the motion.

Let a particle P be at $(x, y, z)$ at time $t$.  At $t + \delta t$, where $\delta t$ is an
infinitesimal increment of time, the location of P is $(x + u\delta t, y + v\delta t, z + w\delta t)$.
It is clear then that

$$
\begin{aligned}
\frac{DF}{Dt} &= \lim_{\delta t \to 0} \frac{F(x + u\delta t, y + v\delta t, z + w\delta t) - F(x, y, z, t)}{\delta t} \\
&= \frac{\partial F}{\partial x}u + \frac{\partial F}{\partial y}v + \frac{\partial F}{\partial z}w + \frac{\partial F}{\partial t}.
\end{aligned}
$$

Therefore,

$$
\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\mathbf{u} \cdot \nabla)F. \tag{2.1}
$$

*graph coordinate and point* $(\mathbf{x}, t)$ *and* $(\mathbf{x} + \mathbf{u}\delta t, t + \delta t)$

**Convective derivative of a volume element**    Consider a rectangular
prism volume element which at time $t$ has endpoints P, L, M, N, A, B, C,
and D, where

$$
P = (x, y, z), L = (x + \delta x, y, z), M = (x, y + \delta y, z), N = (x, y, z + \delta z),
$$

so the volume $Q = \delta x \delta y \delta z$. At $t + \delta t$, the same element will form an oblique
parallelepiped. The corresponding endpoints are

$$
\begin{aligned}
P' &= (x + u_p\delta t, y + v_p\delta t, z + w_p\delta t), \\
L' &= (x + \delta x + u_L\delta t, y + v_L\delta t, z + w_L\delta t), \\
M' &= (x + u_M\delta t, y + \delta y + v_M\delta t, z + w_M\delta t), \\
N' &= (x + u_N\delta t, y + v_N\delta t, z + \delta z + w_N\delta t)
\end{aligned}
$$

and A', B', C', D', where $\mathbf{u}_P = (u_P, v_P, w_P)$ is the velocity at point $P$ and similar notations are used for velocities at other points. The edge P'L' is $(\delta x + (u_L - u_p)\delta t = \delta x(1 + \frac{\partial u}{\partial x}\delta t), \frac{\partial v}{\partial x}\delta x\delta t, \frac{\partial w}{\partial x}\delta x\delta t)$. So, at $t + \delta t$, the length of P'L' is

$$\delta x \sqrt{(1 + u_x\delta t)^2 + (v_x\delta t)^2 + (w_x\delta t)^2}.$$

Therefore,

$$\frac{D\delta x}{Dt} = \lim_{\delta t \to 0} \frac{|P'L'| - |PL|}{\delta t} = u_x\delta x.$$

Similarly,

$$\frac{D\delta y}{Dt} = v_y\delta y, \quad \frac{D\delta z}{Dt} = w_z\delta z.$$

Using product rule,

$$\frac{DQ}{Dt} = \frac{D\delta x}{Dt}\delta y\delta z + \frac{D\delta y}{Dt}\delta x\delta z + \frac{D\delta z}{Dt}\delta x\delta y = (u_x + v_y + w_z)Q.$$

Therefore,

$$\frac{1}{Q}\frac{DQ}{Dt} = \nabla \cdot \mathbf{u} = \mathbf{div} \cdot \mathbf{u}. \tag{2.2}$$

*graph element Q and Q'*

**Exercise.** Show rigorously

$$\frac{1}{Q}\frac{DQ}{Dt} = \nabla \cdot \mathbf{u} = \mathbf{div} \cdot \mathbf{u}.$$

by computing the change of volume from $t$ to $t + \delta t$.

**Mass conservation law** Since mass conservation means that

$$\frac{D(\rho Q)}{Dt} = 0,$$

one obtains
$$\frac{1}{Q}\frac{DQ}{Dt} + \frac{1}{\rho}\frac{D\rho}{Dt} = 0.$$

Using (2.1) and (2.2)

$$\rho\nabla\cdot\mathbf{u} + \frac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho = 0$$

therefore
$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0. \tag{2.3}$$

If the consideration is restricted to incompressible fluid, $\rho = constant$, so

$$\nabla\cdot\mathbf{u} = 0. \tag{2.4}$$

**Momentum conservation law**   Newton's second law of motion states that: the rate of change of momentum is equal to the net applied forces. So,

$$\frac{D}{Dt}(m\mathbf{u}(\mathbf{x}, t)) = \mathbf{f}$$

where $m$ is the mass of the volume element and $\mathbf{f}$ is the net force on it. Assume that the fluid is inviscid and the mass is only acted upon by the pressure and the gravity force . So along x-coordinate,

$$m\frac{Du}{dt} = (-P(x + \delta x, y, z) + P(x, y, z))\delta y\delta z = -\frac{\partial P}{\partial x}Q,$$

where $P = P(x, y, z, t)$ is the pressure. Using $m = \rho Q$, one obtains

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0.$$

Similarly, along y-coordinate,

$$m\frac{Dv}{dt} = -\frac{\partial P}{\partial y}Q - \rho Qg,$$

where $g$ is the gravity constant. Therefore

$$\frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial P}{\partial y} + g = 0.$$

Combining three components, one obtains

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla P + g\mathbf{j} = 0$$

or

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla P + g\mathbf{j} = 0, \tag{2.5}$$

where $\mathbf{j} = (0, 1, 0)$ is the unit vector in direction opposite to gravitation.

Combining the mass conservation equation (2.4) with (2.5), we have four equations for four unknowns $\mathbf{u}$ and $P$. With appropriate boundary conditions, one should be able to solve the problem.

**Remark 2.1.1.** *At first sight this doesn't look very wavy. The waves come from the effects of the free surface which are discussed next.*

**Water wave equations for irrotational flow**  Assume that the fluid is irrotational, it follows that

$$curl\,\mathbf{u} = \nabla \times \mathbf{u} = 0. \tag{2.6}$$

In consequence, there is a velocity potential $\phi = \phi(x, y, z, t)$ such that

$$\mathbf{u} = \nabla \phi. \tag{2.7}$$

Combining with (2.4), it then follows that

$$\triangle \phi = 0. \qquad\qquad\qquad -\text{ Laplace equation} \tag{2.8}$$

Thus we are reduced to solving (2.8) with the appropriate boundary conditions and then the velocity field $\mathbf{u}$ may be read off from (2.7).

**The relationship between pressure P and the velocity potential $\phi$**
Recall that the conservation of momentum is expressed mathematically by the relation,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P - g\mathbf{j}, \tag{2.9}$$

where $P = P(x, y, z, t)$ is the pressure, $\mathbf{j} = (0, 1, 0)$ is the unit vector in the direction opposite gravitation, and $g$ is the gravity constant.

Using (2.6) and

$$\nabla(\mathbf{u} \cdot \mathbf{u}) = 2(\mathbf{u} \cdot \nabla)\mathbf{u} + 2\mathbf{u} \times (\nabla \times \mathbf{u}),$$

conservation of momentum (2.9) may be rewritten as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) = -\frac{1}{\rho}\nabla P - g\mathbf{j}. \tag{2.10}$$

Combining (2.10) with (2.7), we come to the conclusion

$$\nabla\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{\rho}P + gy\right] = 0, \tag{2.11}$$

since $\nabla y = \mathbf{j}$. The gradient of the quantity in square bracket vanishes in the flow domain, and assuming the latter is simply connected, it follows that

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{\rho}P + gy = B(t), \tag{2.12}$$

where $B(t)$ is a constant independent of the spatial coordinates $(x, y, z)$. The latter expression may be written in another form, namely

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{1}{\rho}(P - P_0) + gy = B(t), \tag{2.13}$$

where $P_0$ is the pressure in the air near to the surface of the liquid. This quantity will be taken to be constant in the present consideration. Let $\tilde{\phi}(x, y, z, t) = \phi(x, y, z, t) - \int_0^t B(s)\, ds$, and rewrite (2.13) in terms of $\tilde{\phi}$, viz.

$$\frac{\partial \tilde{\phi}}{\partial t} + \frac{1}{2}\nabla\tilde{\phi} \cdot \nabla\tilde{\phi} + \frac{1}{\rho}(P - P_0) + gy = 0. \tag{2.14}$$

Dropping the tilde from $\tilde{\phi}$ and rearranging the order gives

$$\frac{P - P_0}{\rho} = -\frac{\partial \phi}{\partial t} - \frac{1}{2}\nabla\phi \cdot \nabla\phi - gy. \tag{2.15}$$

Therefore, the pressure $P$ can be read off from (2.15) once $\phi$ is obtained.

**Boundary condition on the free surface**    Suppose the free surface of
the liquid is described by an equation of the form

$$f(x, y, z, t) = 0. \tag{2.16}$$

Since the fluid doesn't cross this surface, the velocity of the fluid at the
surface must be the velocity of the surface. Therefore

$$\frac{Df}{Dt} = 0,$$

which leads to

$$f_t + u f_x + v f_y + w f_z = 0. \qquad - \text{ kinematic boundary condition.} \tag{2.17}$$

If the free surface can be described by a single-valued function of $(x, z)$ for
some time interval, say,

$$f(x, y, z, t) = \eta(x, z, t) - y,$$

then the kinematic boundary condition (2.17) above becomes

$$\eta_t + u\eta_x + w\eta_z - v = 0,$$

or what is the same,

$$\eta_t + \phi_x \eta_x + \phi_z \eta_z = \phi_y. \tag{2.18}$$

There is also a dynamical condition on the free surface. Since the surface
has no mass, and if surface tension is neglected, the pressure in the water and
the air pressure must be equal on the free surface. Of course a disturbance in
the surface imparts some motion of the air. We argue, because of the small
density of air relative to the density of the water, that the air pressure is not
changed significantly, and so may be approximated by its undisturbed value.
Hence the second boundary condition on the free surface is

$$P = P_0 \quad \text{at} \quad y = \eta(x, z, t), \tag{2.19}$$

where $P = P(x, \eta, z, t)$ is the pressure at the surface. Using (2.15) in con-
junction with (2.19), it is seen that

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0 \qquad \text{for} \qquad y = \eta, \qquad - \text{ Bernoulli condition} \tag{2.20}$$

Because the lower, containing boundary is impermeable, the velocity normal to the bottom must be zero, which is to say, there is no flow through the bottom. If the bottom profile is $y = -h_0(x, z)$, then $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n} = (h_{0x}, 1, h_{0z})$ is the normal direction to the bottom; hence

$$\phi_x h_{0x} + \phi_z h_{0z} + \phi_y = 0 \qquad \text{at} \quad y = -h_0(x, z). \tag{2.21}$$

**Summary**   Summarize all the equations obtained so far, it might appear that our system is a little overdetermined since we have $\triangle\phi = 0$ inside the flow domain, one boundary condition on the bottom, but two on the free surface. This is contrary to what we know about elliptic equations. The resolution of this conondrum lies in the free surface not being prescribed in advance, but instead constituting part of solution of the problem. So, in summary, assuming the free surface and the bottom profile can be described as single-valued function of $(x, z, t)$, the motion of the perfect liquid may be described by the system:

$$
\begin{aligned}
&\triangle\phi = 0 && \text{in the flow domain} - h_0 < y < \eta, \\
&\left.\begin{aligned}
&\eta_t + \phi_x \eta_x + \phi_z \eta_z = \phi_y \\
&\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0
\end{aligned}\right\} && \text{at the free surface} \quad y = \eta, \\
&\phi_x h_{0x} + \phi_z h_{0z} + \phi_y = 0 && \text{on the bottom} \quad y = -h_0(x, z).
\end{aligned}
\tag{2.22}
$$

It is sometimes interesting and appropriate to specialize to the case of two-dimensional flow; i.e. motions which are independent of $z$ say. Suppose additionally that $h_0$ is constant, so the bottom is flat and horizontal. Then the system (2.22) above reduces to

$$
\begin{aligned}
&\phi_{xx} + \phi_{yy} = 0 && \text{in the domain} \quad -h_0 < y < \eta, \\
&\left.\begin{aligned}
&\eta_t + \phi_x \eta_x = \phi_y \\
&\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta = 0
\end{aligned}\right\} && \text{at the free surface} \quad y = \eta, \\
&\phi_y = 0 && \text{on the bottom} \quad y = -h_0,
\end{aligned}
\tag{2.23}
$$

together with appropriate initial conditions and other boundary conditions if a lateral surface intrudes.

## 2.2 Linear Dispersion Relation, Phase and Group Velocities

If the propagation of infinitesimal waves is considered, then it is warranted to linearize the equations of motion around the rest solution. Let $\tilde{y} = y + h_0$, so in the new coordinate, the bottom of the channel is at $\tilde{y} = 0$ and the still water surface is at $\tilde{y} = h_0$. In the new coordinate, the linearized Euler equation (2.23) reads (the tilde was dropped)

$$
\begin{aligned}
\triangle\phi = 0 \qquad & \text{in} \quad 0 < y < h_0, \\[4pt]
\left.\begin{array}{l}
\eta_t = \phi_y \\
\phi_t + g\eta = 0
\end{array}\right\} \qquad & \text{on} \quad y = h_0, \\[4pt]
\phi_y = 0 \qquad & \text{at} \quad y = 0.
\end{aligned}
\qquad (2.24)
$$

We start by looking for a particular traveling-wave solution of the form $\phi(x, y, t) = \psi(y)e^{i(kx-\omega t)}$. Substituting this form into (2.24) and simplifying gives

$$
\begin{cases}
\psi'' - k^2\psi = 0, \\
\psi'(0) = 0, \\
\dfrac{\omega^2}{g}\psi(h_0) - \psi'(h_0) = 0.
\end{cases}
\qquad (2.25)
$$

It follows that

$$
\psi(y) = c \, \sinh(ky) + d \, \cosh(ky). \qquad (2.26)
$$

As $\psi'(0) = 0$, $c = 0$ and $\psi(y) = d \, \cosh(ky)$. Applying the second boundary condition leads to the *dispersion relation*

$$
\omega^2 = g\frac{\psi'(h_0)}{\psi(h_0)} = gk\frac{\sinh(kh_0)}{\cosh(kh_0)} = gk \, \tanh(kh_0).
$$

Thus the frequency $\omega$ is

$$
\omega(k) = \pm\sqrt{gk \, \tanh(kh_0)}. \qquad \text{- Dispersion relation} \qquad (2.27)
$$

**Remark 2.2.1.** *At first, one might think that the linearized Euler equation*

*should be*

$$\triangle\phi = 0 \qquad\qquad in \qquad 0 < y < h_0 + \eta,$$

$$\left.\begin{array}{l} \eta_t = \phi_y \\ \phi_t + g\eta = 0 \end{array}\right\} \qquad on \qquad y = h_0 + \eta, \qquad\qquad (2.28)$$

$$\phi_y = 0 \qquad\qquad at \qquad y = 0.$$

*By looking for solution of the form* $\phi(x, y, t) = \psi(y)e^{i(kx-\omega t)}$, *(2.28) reduces to*

$$\begin{cases} \psi'' - k^2\psi = 0, \\ \psi'(0) = 0, \\ \dfrac{\omega^2}{g}\psi(h_0 + \eta) - \psi'(h_0 + \eta) = 0. \end{cases} \qquad\qquad (2.29)$$

*It follows that*

$$\psi(y) = c \, \sinh(ky) + d \, \cosh(ky). \qquad\qquad (2.30)$$

*As* $\psi'(0) = 0$, $c = 0$ *and* $\psi(y) = d \, \cosh(ky)$. *Applying the second boundary condition leads to*

$$\omega^2 = g\frac{\psi'(h_0 + \eta)}{\psi(h_0 + \eta)} = gk \, \tanh(k(h_0 + \eta)).$$

*By keeping the leading term around the rest state, one recovers the dispersion relation (2.27).*

There are two important velocities associated to a propagating wave. One is *phase velocity* defined by

$$c_p(k) =: \frac{\omega(k)}{k} \qquad\qquad \text{- Phase velocity}$$

and the other is *group velocity.* defined by

$$c_g(k) =: \frac{d\omega(k)}{dk} \qquad\qquad \text{- Group velocity}$$

Let us first consider a pure traveling sinusoidal wave,

$$A(t, x) = A_0 \cos(kx - \omega t)$$

where $A_0$ is the wave amplitude. The angular frequency $\omega$ of a wave is the number of radians per unit time at a fixed position, whereas the wave number

$k$ is the number of radians per unit distance at a fixed time. We use the cosine function rather than the sine merely for convenience, the difference being only a matter of phase. The minus sign denotes the fact the wave is propagating in the positive $x$ direction. Reversing the sign gives $A_0 \cos(kx + \omega t)$, which is the equation of a wave propagating in the negative $x$ direction. Since $\omega$ is the number of radians of the wave that pass a given location per unit time, and $\frac{1}{k}$ is the spatial length of the wave per radian, it follows that $\frac{\omega}{k}$ is the speed at which the shape of the wave is moving, i.e., the speed at which any fixed phase of the cycle is displaced. Consequently this is called the phase velocity of the wave, denoted by $c_p$.

In practice and common usage, though, we tend to define the "phase" of a signal with respect to the intervals between consecutive local maxima (or minima, or zero crossings). To illustrate, consider a signal consisting of two superimposed cosine waves with slightly different frequencies and wavelengths, i.e., a signal with the amplitude function

$$\cos((k+dk)x - (\omega+d\omega)t) + \cos((k-dk)x - (\omega-d\omega)t)$$
$$= 2\cos(kx - \omega t)\cos((dk)x - (d\omega)t). \tag{2.31}$$

It is clear that the phase velocity of this propagating wave is approximately $\frac{\omega}{k}$ when $dk$ and $d\omega$ are small.

Formula (2.31) shows that the combination of two slightly unequal tones produces a "beat". It is a simple sinusoidal wave with the angular velocity $w$, the wave number $k$, and the modulated amplitude $2\cos((dk)x - (d\omega)t)$. In other words, the amplitude of the wave is itself a wave, and the phase velocity of this modulation wave is $\frac{d\omega}{dt}$. Since each amplitude wave contains a group of internal waves, this speed is usually called the group velocity, denoted by $c_g$.

A typical plot of such a signal is shown below for the case

$$\omega = 6\text{rad/sec}, \quad k = 6\text{rad/meter},$$
$$d\omega = 0.1\text{rad/sec}, dk = 0.3\text{rad/meter}.$$

The "phase velocity" of the internal oscillations is $\omega/k = 1$ meter/sec, whereas the amplitude envelope wave (indicated by the dotted lines) has a phase velocity of $d\omega/dk = 0.33$ meter/sec.

As a result, if we were riding along with the envelope, we would observe the internal oscillations moving forward from one group to the next. The propagation of information or energy in a wave always occurs as a change
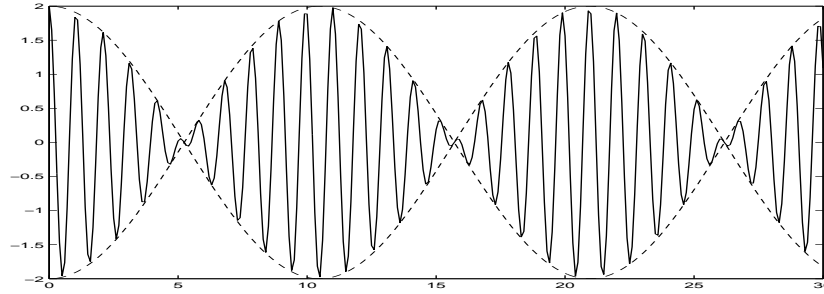
Figure 2.1:  A  function  of  form  (2.31)  The  dashed  line  is  the  envelope
$2\cos((dk)x)$.

in the wave.  The most obvious example is changing the wave from being
absent to being present, which propagates at the speed of the leading edge of
a wave train, which is the group velocity.  Incidentally, since we can contrive
to make the "groups" propagate in either direction, it's not surprising that we
can also make them stationary.  Two identical waves propagating in opposite
directions at the same speed are given by function

$$A_0 \cos(kx \pm \omega t) + A_0 \cos(kx \mp \omega t) = 2A_0 \cos(kx) \cos(\omega t).$$

Superimposing these two waves propagating in opposite directions yields
a standing pure wave.

For the linearized Euler equation, the *phase velocity* is

$$c_p(k) = \pm\sqrt{gh_0}\sqrt{\frac{\tanh(kh_0)}{kh_0}},$$

and the *group velocity* is

$$c_g(k) = \frac{1}{2\omega}(g\tanh(kh_0) + gkh_0\mathrm{sech}^2(kh_0)).$$

The quantity $c_p(k)$ is the speed of individual crests.  The quantity $\sqrt{gh_0}$
is the so-called kinematic wave velocity, the velocity of extremely long waves.
According to the linearized theory, long waves travel faster than short wave-
length disturbances.

**Exercise:**    Write a program to show the evolution of wave profile (2.31)
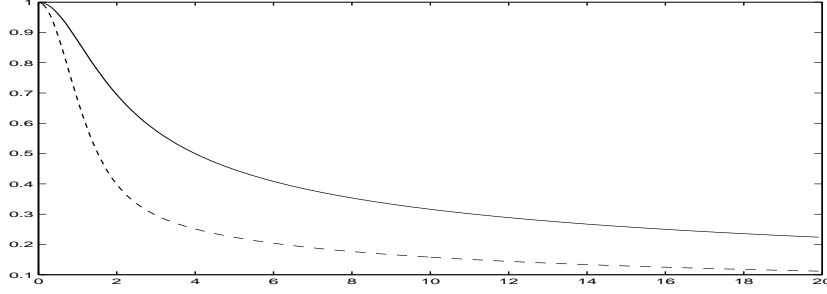with

Figure 2.2: The phase velocity (solid line) and the group velocity of linearized Euler equations. $g$ and $h_0$ are scaled to 1.

- Stationary group where the group velocity is zero. $k = 12, w = 2, dk = 1, dw = 0$

- Moving group where the phase velocity is zero. $k = 12, w = 0, dk = 1, dw = 2$

- Moving group and phase where both velocities are nonzero. $k = 12, w = 5, dk = 1, dw = 1.5$

## 2.3  Nonlinear Boussinesq Equations

To study waves that are not infinitesimally small, the nonlinear effects have to be included in the consideration. We return to the nonlinear Euler equation (2.23) with the object of further simplifying it. Let $a = sup_{x \in \mathbb{R}, t \geq 0} |\eta(x, t)|$ be the maximum amplitude of the contemplated wave motion, $l$ a typical wavelength in the wave motion, and $c_0 = \sqrt{gh_0}$ the kinematic wave velocity. Assume that $a << h_0$ and that $h_0 << l$. It is natural to non-dimensionalize the variables to bring these assumptions to the fore: let

$$x' = lx, \quad y' = h_0(y - 1), \quad t' = \frac{lt}{c_0}, \quad \eta' = a\eta, \quad \phi' = g\frac{la}{c_0}\phi.$$

Here, the primed variables connote the original coordinates, while the un-primed quantities are the new dimensionless variables. In the new variables,

the system (2.23) becomes

$$
\begin{cases}
\beta\phi_{xx} + \phi_{yy} = 0 & \text{in} \quad 0 < y < 1 + \alpha\eta, \\[2mm]
\phi_y = 0 & \text{at} \quad y = 0, \\[2mm]
\left.
\begin{aligned}
&\eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_y = 0 \\[2mm]
&\eta + \phi_t + \frac{1}{2}\alpha\phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_y^2 = 0
\end{aligned}
\right\} & \text{at} \quad y = 1 + \alpha\eta,
\end{cases}
\tag{2.32}
$$

where $\alpha = \dfrac{a}{h_0}$ and $\beta = \dfrac{h_0^2}{l^2}$.

A formal expansion of $\phi$ in a power series in $y$ is posited:

$$
\phi(x, y, t) = \sum_{m=0}^{\infty} f_m(x, t)y^m.
\tag{2.33}
$$

From the Laplace equation in (2.32), there follows

$$
\begin{aligned}
0 &= \beta\phi_{xx} + \phi_{yy} \\
&= \beta\sum_{m=0}^{\infty} f_m'' y^m + \sum_{m=2}^{\infty} m(m-2)f_m y^{m-2} \\
&= \sum_{m=0}^{\infty} \left( \beta f_m'' + (m+2)(m+1)f_{m+2} \right) y^m,
\end{aligned}
\tag{2.34}
$$

whence

$$
\beta f_m'' = -(m+2)(m+1)f_{m+2} \qquad \text{for} \quad m \geq 0.
\tag{2.35}
$$

Since $\phi_y(x, 0) = 0$ is specified in the first boundary condition in (2.32), $f_1(x, t) = 0$, and so by recursion $f_3 = f_5 = f_7 = \ldots = f_{2n+1} = 0$ for all $n \geq 0$. If we write $f(x, t)$ for $f_0(x, t)$, then $f_2 = -\frac{\beta}{2!}f''$, $f_4 = -\frac{\beta}{4\cdot 3}f_2'' = \frac{\beta^2}{4!}f''''$ and so forth. Thus, the Laplace equation together with the boundary condition at the bottom leads to

$$
\phi(x, y, t) = \sum_{m=0}^{\infty} f_{2m}(x, t)y^{2m} = \sum_{m=0}^{\infty} (-1)^m \beta^m \frac{y^{2m}}{(2m)!} f^{(2m)}(x, t).
\tag{2.36}
$$

Substituting (2.36) into the non-dimensional version of the Euler equations in (2.32), the kinematic boundary condition on the free surface yields

$$
\eta_t + \alpha\eta_x[f_x - \beta\frac{(1+\alpha\eta)^2}{2}f_{xxx}] - \frac{1}{\beta}[-\beta(1+\alpha\eta)f_{xx} + \frac{\beta^2(1+\alpha\eta)^3}{6}f_{xxxx}] + O(\beta^2) = 0.
\tag{2.37}
$$

Ignoring terms quadratic in $\alpha$ and $\beta$, this simplifies first to

$$
\begin{aligned}
&\eta_t + ((1 + \alpha\eta)f_x)_x \\
&\quad - \left\{ \frac{1}{6}(1 + \alpha\eta)^3 f_{xxxx} + \frac{1}{2}\alpha(1 + \alpha\eta)^2 \eta_x f_{xxx} \right\} \beta + O(\beta^2, \alpha\beta) = 0
\end{aligned}
\tag{2.38}
$$

and then even further to

$$
\eta_t + ((1 + \alpha\eta)f_x)_x - \frac{\beta}{6} f_{xxxx} + O(\beta^2, \alpha\beta) = 0.
\tag{2.39}
$$

The Bernoulli condition on the free surface gives, after simplifying,

$$
\eta + f_t + \frac{1}{2}\alpha f_x^2 - \frac{1}{2}\beta f_{xxt} + O(\beta^2, \alpha\beta) = 0.
\tag{2.40}
$$

Since $f_x$ is the horizontal velocity at the bottom, it is a variable with a direct physical interpretation. Writing $w$ for $f_x$ and combining (2.39) and (2.40) gives one version of the Boussinesq equations, namely

$$
\begin{cases}
\eta_t + [(1 + \alpha\eta)w]_x - \dfrac{1}{6}\beta w_{xxx} = 0, \\[2mm]
w_t + \eta_x + \alpha w w_x - \dfrac{1}{2}\beta w_{xxt} = 0.
\end{cases}
\tag{2.41}
$$

**Remark 2.3.1.** *If $\beta << \alpha$, the nonlinear effect dominates. Dropping terms that are $O(\beta)$ in the above pair of equations yields a version of shallow water theory*

$$
\begin{cases}
\eta_t + [(1 + \alpha\eta)w]_x = 0, \\
w_t + \eta_x + \alpha w w_x = 0.
\end{cases}
\tag{2.42}
$$

*which is used to characterize near shore zone hydrodynamics.*

*If $\alpha << \beta$, the dispersive effect dominates. We would be tempted to drop the nonlinear terms and thereby arrive at the linear system*

$$
\begin{cases}
\eta_t + w_x - \dfrac{1}{6}\beta w_{xxx} = 0, \\[2mm]
w_t + \eta_x - \dfrac{1}{2}\beta w_{xxt} = 0.
\end{cases}
\tag{2.43}
$$

*The behavior of solutions of such linear systems is determined by their dispersion relation. This is obtained in a straightforward way by first eliminating $\eta$ to reach the single equation*

$$
w_{tt} - w_{xx} + \frac{1}{6}\beta w_{xxxx} - \frac{1}{2}\beta w_{xxtt} = 0.
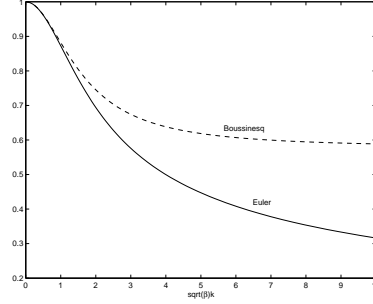\tag{2.44}
$$

Figure 2.3:  Comparison between the phase velocities from the linearized Euler equations and a linearized Boussinesq system (2.43).

*We refer to this as the linear Boussinesq equation.  Substituting the form $w(x,t) = w_0 e^{i(kx-\omega t)}$ into (2.44) leads to*

$$-\omega^2 + k^2 + \frac{\beta}{6}k^4 - \frac{\beta}{2}\omega^2 k^2 = 0,$$

*so*

$$\omega^2 = k^2 \left[ \frac{1 + \frac{\beta}{6}k^2}{1 + \frac{\beta}{2}k^2} \right]$$

*and*

$$c(k) = \frac{\omega(k)}{k} = \pm \left[ \frac{1 + \frac{\beta}{6}k^2}{1 + \frac{\beta}{2}k^2} \right]^{\frac{1}{2}} = \pm(1 - \frac{1}{6}\beta k^2 + \frac{5}{72}\beta^2 k^4 + \cdots).$$

*This agrees with the dispersion relation*

$$\sqrt{\frac{\tanh(\beta^{\frac{1}{2}}k)}{\beta^{\frac{1}{2}}k}} = 1 - \frac{1}{6}\beta k^2 + \frac{19}{120}\beta^2 k^4 + \cdots$$

*for the full, linearized Euler equations to the second order in k.  But there is a difficulty associated with large wavenumbers (small wavelengths), which will be discussed presently.*

*If $\alpha \approx \beta$, namely when $\frac{\alpha}{\beta} = O(1)$, the equations (2.41) is nonlinear and dispersive.  The solitary wave solutions might exist because the balance between nonlinear and dispersive effects.*

## 2.4 Effects of different terms

We now consider some simple equations and see what are the effects of different terms.

**Example 1.** Dispersive equation

$$u_t + u_x + u_{xxx} = 0, \quad u(x,0) = f(x). \tag{2.45}$$

By taking Fourier transform, one finds

$$\hat{u}_t + (ik)\hat{u} + (ik)^3\hat{u} = 0, \quad \hat{u}(k,0) = \hat{f}(k)$$

where the Fourier transform of a function $f$ of the spatial variable $x$ is

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx.$$

Solving the ordinary differential equation on $\hat{u}$,

$$\hat{u}(k,t) = e^{-ik(1-k^2)t}\hat{f}(k).$$

Taking the inverse Fourier transform, the solution of (2.45) is obtained

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-ik(1-k^2)t} \hat{f}(k)dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(kx-\omega t)} \hat{f}(k)dk$$

where

$$\omega(k) = k(1 - k^2)$$

is the dispersion relation.

If we let $f(x) = e^{imx}$, then

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-i(k-m)x}dx = 2\pi\delta(k-m)$$

and

$$u(x,t) = \int_{-\infty}^{\infty} e^{-i(kx-k(1-k^2)t)}\delta(k-m)dk = e^{i(mx-\omega(m)t)}$$

is the solution.

Since the wave travels with phase speed $\frac{\omega(k)}{k}$ which is a function of wave number $k$, equation (2.45) is often called dispersive.

**Example 2.** Dissipative equation

$$u_t + u_x - u_{xx} = 0.$$

By searching for the solution of type

$$u(x,t) = e^{i(kx - \omega t)}$$

one finds that

$$\omega = k - ik^2. \tag{2.46}$$

Therefore

$$u(x,t) = e^{-k^2 t + ik(x-t)}$$

is a solution with initial condition $u(x,0) = e^{ikx}$. It is clear that $u(x,t)$ is decaying with time. Therefore, equation (2.46) is often called dissipative. Notice that the sign in front of $u_{xx}$ term is important here.

**Example 3.** Nonlinear equation (Burger's equation)

$$u_t + u_x + uu_x = 0, \quad u(x,0) = f(x). \tag{2.47}$$

Since

$$\frac{du}{dt} = u_t + \frac{dx}{dt} u_x,$$

$u$ is constant along the characteristic line $x = (1 + u)t + c$. Therefore, the solution of (2.47) is

$$u = f(x - (1 + u)t).$$

Above process offers an implicit solution for $u$ and it is not always trivial to solve $u$. But one can sometimes construct the solution geometrically. Let's consider a special case where

$$f(x) = \cos(x).$$

The solution at $t = 0, 1, 2$ are shown in figure (2.4). From the graph, one can easily see that $u$ is a single valued function only for a finite time. After that, the solution will be either multi-valued or develop a discontinuity. In general, the nonlinearity makes the wave steeper and eventually develop discontinuity.

It is worth to note that another effect of nonlinearity is that the superposition of two solutions is not a solution.

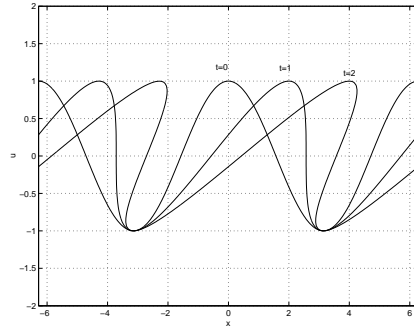**Exercises 1:**    Find the solutions of

Figure 2.4: Solution of a nonlinear equation (2.47) with $f(x) = \cos(x)$ at $t = 0, 1, 2$.

- $u_t + u_x = 0$,

- $u_t - u_x = 0$,

- $u_t + u_x + u = 0$,

- $u_t + u_x - u = 0$,

- $u_t + u_x - u_{xx} = 0$,

- $u_t + u_x + u_{xx} = 0$,

- $u_t + u_x + u_{xxx} = 0$,

- $u_t + u_x + u_{xxt} = 0$

with initial value
$$u(x, 0) = f(x).$$

**Exercises 2:**   Solve the equations in problem 1 with $f(x) = \cos(x) + \cos(3x)$ and plot each solution for $t = 0 : 0.4 : 5$. Use one plot for one solution.

**Exercises 3:**   Find the solution of

$$u_t + u_x + uu_x = 0$$

with initial condition

$$u(x,0) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and plot out the solution for $t = 0, 0.1, \cdots, 1$.

**Exercises 4:**   Compare the dispersion relation for the equation

$$\begin{aligned} u_t + u_x + u_{xxx} = 0 \\ u_t + u_x - u_{xxt} = 0 \end{aligned} \tag{2.48}$$

particularly in the limiting cases of long and short waves.

**Exercises 5:**   Find the dispersion relation for the Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0.$$

## 2.5    One-way Propagation

Here, the Boussinesq system of equations is specialized to the description of waves propagating just to the right. At the very lowest order where even the terms of order $\alpha$ and $\beta$ are dropped, there appears a factored version of the one-dimensional wave equation, *viz.*

$$\begin{cases} \eta_t + w_x = 0, \\ w_t + \eta_x = 0, \\ \eta(x, 0) = f(x), \\ w(x, 0) = g(x), \end{cases} \tag{2.49}$$

posed with initial conditions on both $\eta$ and $w$. The solution of (2.49) is

$$\eta(x,t) = \frac{1}{2}[f(x + t) + f(x - t)] + \frac{1}{2}[g(x + t) - g(x - t)]$$

and

$$w(x,t) = \frac{1}{2}[g(x + t) + g(x - t)] + \frac{1}{2}[f(x + t) - f(x - t)].$$

As the left-propagating component must vanish, it is required that $f = g$, whence $\eta(x, t) = f(x - t) = w(x, t)$. Thus, at the lowest order, we have

$$w = \eta + O(\alpha, \beta) \quad \text{and} \quad \eta_t + \eta_x = O(\alpha, \beta). \tag{2.50}$$

The next step is to extend the relations just obtained via the linear wave equation to obtain a model correct to order $\alpha$ and $\beta$ while still maintaining one-way propagation. Whatever the extensions are that lead to the next order, it seems clear they will involve terms of order $\alpha$ and $\beta$. It is therefore natural to try the Ansatz $w = \eta + \alpha A + \beta B$, where $A = A(\eta, \eta_x, \eta_t, \cdots)$, and $B = B(\eta, \eta_x, \eta_t, \cdots)$. Putting this relation into the Boussinesq system (2.41) results in the pair of equations

$$\eta_t + \eta_x + \alpha A_x + \beta B_x + \alpha \left\{ \eta[\eta + \alpha A + \beta B] \right\}_x - \frac{1}{6}\beta[\eta + \alpha A + \beta B]_{xxx} = 0,$$

$$\eta_t + \alpha A_t + \beta B_t + \eta_x + \alpha(\eta + \alpha A + \beta B)(\eta_x + \alpha A_x + \beta B_x)$$

$$- \frac{1}{2}\beta(\eta_{xxt} + \alpha A_{xxt} + \beta B_{xxt}) = 0.$$

$$\tag{2.51}$$

Collecting terms featuring the same power of $\alpha$ and $\beta$ leads to the relations

$$\begin{cases} \eta_t + \eta_x + \alpha(A_x + 2\eta\eta_x) + \beta(B_x - \frac{1}{6}\eta_{xxx}) = O(\alpha^2, \alpha\beta, \beta^2), \\ \eta_t + \eta_x + \alpha(A_t + \eta\eta_x) + \beta(B_t - \frac{1}{2}\eta_{xxt}) = O(\alpha^2, \alpha\beta, \beta^2), \end{cases}$$

or, dropping terms quadratic in $\alpha$ and $\beta$ and using (2.50)

$$\begin{cases} \eta_t + \eta_x + \alpha(A_x + 2\eta\eta_x) + \beta(B_x - \frac{1}{6}\eta_{xxx}) = 0, \\ \eta_t + \eta_x + \alpha(-A_x + \eta\eta_x) + \beta(-B_x - \frac{1}{2}\eta_{xxt}) = 0. \end{cases} \tag{2.52}$$

This pair of equations can be made consistent by choosing $A_x = -\frac{1}{2}\eta\eta_x$, or $A = -\frac{1}{4}\eta^2$, and $B_x = \frac{1}{12}\eta_{xxx} - \frac{1}{4}\eta_{xxt}$, or $B = \frac{1}{12}\eta_{xx} - \frac{1}{4}\eta_{xt}$. It is worthwhile noting that from the lowest-order theory, $\eta_t = -\eta_x + O(\alpha, \beta)$ as $\alpha, \beta \to 0$. In consequence, we may use $\eta_t$ and $-\eta_x$ interchangeably in terms whose formal order is $\alpha$ or $\beta$ without affecting the overall level of the approximation. Thus, at the formal level,

$$B = \frac{1}{12}\eta_{xx} - \frac{1}{4}\eta_{xt} = \frac{1}{3}\eta_{xx} + O(\alpha, \beta) = -\frac{1}{3}\eta_{xt} + O(\alpha, \beta) \tag{2.53}$$

as $\alpha, \beta \to 0$. Because $B$ appears in (2.52) multiplied by $\beta$, the dispersive terms in (2.52) could have either of the forms $\frac{1}{6}\eta_{xxx}$ or $-\frac{1}{6}\eta_{xxt}$, or, indeed, any convex combination of theses two forms. Taking only the pure forms $\eta_{xxx}$ or $-\eta_{xxt}$, we come to

$$
\begin{cases}
w = \eta - \dfrac{1}{4}\alpha\eta^2 + \dfrac{1}{3}\beta\eta_{xx} + \text{terms quadratic in } \alpha, \beta, \\[2mm]
\eta_t + \eta_x + \dfrac{3}{2}\alpha\eta\eta_x + \dfrac{1}{6}\beta\eta_{xxx} = \text{terms quadratic in } \alpha, \beta,
\end{cases}
\tag{2.54}
$$

or

$$
\begin{cases}
w = \eta - \dfrac{1}{4}\alpha\eta^2 - \dfrac{1}{3}\beta\eta_{xt} + \text{terms quadratic in } \alpha, \beta, \\[2mm]
\eta_t + \eta_x + \dfrac{3}{2}\alpha\eta\eta_x - \dfrac{1}{6}\eta_{xxt} = \text{terms quadratic in } \alpha, \beta.
\end{cases}
\tag{2.55}
$$

We thus have two separate model equations for unidirectional propagation of long waves of small amplitude. In fact, more models could be constructed using the observation that $\partial_t = -\partial_x + \text{order}(\alpha, \beta)$, namely

$$
\eta_t + \eta_x + \left\{ \begin{array}{l} +\dfrac{3}{2}\alpha\eta\eta_x \\[3mm] -\dfrac{3}{2}\alpha\eta\eta_t \end{array} \right\} + \left\{ \begin{array}{l} +\dfrac{1}{6}\beta\eta_{xxx} = 0 \\[3mm] -\dfrac{1}{6}\beta\eta_{xxt} \\[3mm] +\dfrac{1}{6}\beta\eta_{xtt} = 0 \\[3mm] -\dfrac{1}{6}\beta\eta_{ttt} = 0 \end{array} \right\} = 0.
\tag{2.56}
$$

There are eight different model equations here, without doing anything more complicated (like changing the dependent variable or allowing convex combinations of the individual nonlinear and dispersive terms).

Omitting the nonlinear terms yields four possibilities,

$$
\eta_t + \eta_x + \frac{1}{6}\beta \left\{ \begin{array}{l} +\eta_{xxx} \\[2mm] -\eta_{xxt} \\[2mm] +\eta_{xtt} \\[2mm] -\eta_{ttt} \end{array} \right\} = 0.
\tag{2.57}
$$

Trying $\eta = e^{i(kx-\omega t)}$ leads to the linearized dispersion relations

$$\omega(k) = \begin{cases} k(1 - \dfrac{\beta}{6}k^2), \\[2mm] \dfrac{k}{1 + \frac{\beta}{6}k^2}, \\[2mm] \dfrac{3}{\beta k}\left[\pm\sqrt{1 + \dfrac{2}{3}\beta k^2} - 1\right], \\[2mm] \text{solution of a cubic equation.} \end{cases} \qquad (2.58)$$

The associated phase speeds are

$$c(k) = \begin{cases} 1 - \dfrac{\beta}{6}k^2, \\[2mm] \dfrac{1}{1 + \frac{\beta}{6}k^2}, \\[2mm] \dfrac{3}{\beta k^2}\left[\pm\sqrt{1 + \dfrac{2}{3}\beta k^2} - 1\right], \\[2mm] \text{solution of a cubic equation.} \end{cases} \qquad (2.59)$$

The first two, the third with a + sign and the fourth if the right branch is taken, all agree to order $k^2$ with the linearized dispersion relation for the full two-dimensional Euler equations.

Consider the pure initial-value problem posed on $\mathbb{R}$ for the above models, namely

$$\begin{cases} \eta_t + \eta_x + \dfrac{\beta}{6}Lu = 0, & x \in \mathbb{R}, t \geq 0, \\[2mm] \eta(x,0) = g(x), & x \in \mathbb{R}, \end{cases} \qquad (2.60)$$

where $L$ represents one or another of the dispersion operator $\partial_x^3$, $-\partial_x^2\partial_t$, $\partial_x\partial_t^2$, or $-\partial_t^3$, at least for small values of $\beta$ and order-one initial data. This should represent a well posed problem if one is to take the equation seriously as a model of physical phenomena. For the moment, attention is given over to the cases where $L$ is $\partial_x^3$ or $-\partial_x^2\partial_t$. The other two cases are less interesting because there is apparently insufficient data to initiate the motion uniquely. They will be discussed in Appendix A.

Taking the Fourier transform in the spatial variable $x$ for the linearized Korteweg deVries equation where $L = \partial_x^3$ gives

$$\hat{\eta}_t + i(k - \frac{\beta}{6}k^3)\hat{\eta} = 0,$$

whence

$$\hat{\eta}(k, t) = \hat{\eta}(k, 0)e^{-i(k - \frac{\beta}{6}k^3)t}.$$

Computing similarly for the linear regularized long-wave equation (RLW equation or BBM equation where $L = -\partial_x^2\partial_t$) leads to

$$(1 + \frac{\beta}{6}k^2)\hat{\eta}_t + ik\hat{\eta} = 0,$$

and so

$$\hat{\eta}(k, t) = \hat{\eta}(k, 0)e^{-i\frac{k}{1 + \frac{\beta}{6}k^2}t}.$$

For these two models, the frequency $\omega(k)$ dispersion is modeled by $k - \frac{\beta}{6}k^3$ and $\frac{k}{1 + \frac{\beta}{6}k^2}$, respectively. In terms of the phase speed $c = c(k) = \frac{\omega(k)}{k}$, these are the two alternatives

$$c(k) = \begin{cases} \dfrac{1}{1 + \frac{\beta}{6}k^2} & \text{RLW -} \quad L = -\partial_{xxt}, \\[3mm] 1 - \dfrac{\beta}{6}k^2 & \text{KdV -} \quad L = \partial_{xxx}. \end{cases}$$

For values of $k$ in the range $|k| \le 1$, which is appropriate in the present scaling, these two dispersion relations differ by less than $\frac{\beta^2}{36}$. As for the nonlinear term $\eta\eta_x$ versus $\eta\eta_t$, the conservation laws

$$\eta_t + \eta_x + \begin{cases} \dfrac{3}{2}\alpha\eta\eta_x = 0, \\[3mm] -\dfrac{3}{2}\alpha\eta\eta_t = 0, \end{cases} \tag{2.61}$$

correspond to the characteristic equations

$$\frac{dx}{dt}\Big|_{\eta=constant} = \begin{cases} 1 + \dfrac{3}{2}\alpha\eta, \\[3mm] \dfrac{1}{1 - \frac{3}{2}\alpha\eta}, \end{cases} \tag{2.62}$$

respectively. For values of $\eta$ with $|\eta| \leq \frac{2}{3}$, say, which is consistent with the small-amplitude presumption in force, these differ by less than $\alpha^2$.

Thus for small values of $\alpha$ and $\beta$, these models appear likely to present nearly identical outcomes. Nevertheless, there might be a marginal preference for the choices $-\eta_{xxt}$ and $\eta\eta_x$. As far as the preference for $\eta_{xxt}$ goes, observe that short-wave components for the linear equation (2.57) with $\eta_{xxx}$ can propagate in the $-x$ direction with arbitrarily large phase velocities (and the group velocity is likewise unbounded), whereas the $\eta_{xxt}$ term has bounded (and positive) phase velocities (and bounded group velocity). Regarding the nonlinear term, whilst one cannot really distinguish between the two possibilities in (2.61) for $\eta$ small, as $\eta$ gets large, the $\eta\eta_t$ term has singular characteristics, whilst $\eta\eta_x$ just propagates larger amplitude waves faster.

On the basis of these arguments, the model equations

$$\begin{cases} \eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{\beta}{6}\eta_{xxt} = 0, \\ \eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{\beta}{6}\eta_{xxx} = 0, \end{cases} \quad (2.63)$$

are singled out for study. The second one is the famous KdV equation, first derived by Boussinesq in 1871 and later by Korteweg and deVries in 1895.

With these formalities in front of us, the historical perspective presented in Section 1 may be given more precision. The model put forward by Airy in 1845 corresponds to taking $\alpha$ small and $\beta = 0$ in the present notation. Thus, Airy put forward what we would now call shallow water theory as a model for what Scott Russel observed. This is a model where small, but finite amplitude effects are contemplated, but finite wavelength effects are ignored. It is a model that retains validity only for waves of extreme length. Indeed, it is an easy exercise to see that the evolution equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x = 0 \quad (2.64)$$

does not possess a traveling-wave solution $\eta(x,t) = \phi(x-ct)$, $c > 0$ a positive constant, that has the form of a solitary wave of elevation. Stokes, on the other hand, viewed the regime in which Scott Russel made his experiments as corresponding to infinitesimal waves. He ignored finite-amplitude effects by taking $\alpha = 0$. However, he kept the effects of finite wavelength on wave speed by taking $\beta$ small, but non-zero. He thus put forward the model

$$\eta_t + \eta_x + \frac{\beta}{6}\eta_{xxx} = 0,$$

in the present notation. Fourier analysis shows that this model also has no
solution of the form $\eta(x,t) = \phi(x - ct)$ where $\phi$ is an even function decaying
rapidly to zero at $\pm\infty$. There are periodic wavetrains traveling at constant
velocity, but a heap of water would decompose into components traveling at
different speeds, and so continuously spreading. Described in terms of the
Stokes or Ursell number (Stokes 1845 [11], Ursell 1953 [?])

$$S = \frac{\alpha}{\beta^2},$$

Airy took this quantity to be infinite, Stokes took it to be zero, where as
the presumption that corresponds to Scott Rusell's observations is $S \sim 1$.
In the latter regime, the nonlinear versus dispersive effects come in at the
same order, hence the equations in (2.63). In general, $S$ is a rough measure
of the relative importance of nonlinear versus dispersive effects, with $S$ small
corresponding to a linear system and $S$ large a much more nonlinear regime.

Once the equations in (2.63) have been obtained, the need for the small
parameters disappears. For mathematical analysis, it is convenient to dis-
pense with $\alpha$, $\beta$ and the coefficients $\frac{3}{2}$ and $\frac{1}{6}$. This may be accomplished by
redefining the variables $\eta$, $x$ and $t$, viz. $\tilde{\eta}(\tilde{x}, \tilde{t}) = \frac{3}{2}\alpha\eta(\sqrt{\frac{\beta}{6}}\tilde{x}, \sqrt{\frac{\beta}{6}}\tilde{t})$, namely
by introducing $\tilde{\eta} = \frac{3}{2}\alpha\eta$, $x = \sqrt{\frac{\beta}{6}}\tilde{x}$ and $t = \sqrt{\frac{\beta}{6}}\tilde{t}$. Dropping the tildes, the
dimensionless equations

$$\eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0 \qquad\qquad (2.65)$$

and

$$\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0 \qquad\qquad (2.66)$$

emerge. The small parameters are not really absent, however; they appear
in the imposition of auxiliary conditions. For example, if it is supposed the
waveform is known initially, then we are concerned with the pure initial-value
problem with $\eta(x, 0)$ given. In the variables appertaining to (2.63), $\eta(x, 0)$ is
of order one along with its derivatives, whereas in the (2.65)-(2.66) variable,
$\eta(x, 0)$ has the form $\frac{3}{2}\alpha g(\sqrt{\frac{\beta}{6}}x)$ to be physically relevant.

# Chapter 3

# Mathematical Theory for the Initial-value Problems

In this chapter, the standard notation will be used. The $L_p(\mathbb{R})$ norm will be written $|\cdot|_p$ for $1 \leq p \leq \infty$. If $f \in H^s = H^s(\mathbb{R})$, where $s \geq 0$, the Sobolev-class of $L_2$-functions whose first $s$ derivatives also lie in $L_2$, then its norm is written $\|f\|_s$. If $s$ is not an integer, the notion is extended via the Fourier transform in the usual way,

$$\|f\|_s = \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\widehat{f}(k)|^2 (1+k^2)^s dk \right)^{\frac{1}{2}} \tag{3.1}$$

is a norm on $H^s(\mathbb{R})$ which is equivalent to the usual norm

$$\left( \sum_{j=0}^{s} \left\| f^{(j)}(x) \right\|_0^2 \right)^{\frac{1}{2}}$$

when $s$ is a positive integer (cf. [5]). When $s = 0$, Parseval's formula implies

$$\|f\|_0 = |f|_2 = \left( \int_{-\infty}^{+\infty} |f(x)|^2 dx \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} |\widehat{f}|_2.$$

If $X$ is any other Banach space, its norm will be denoted, unabbreviated, as $\|\cdot\|_X$. The product space $X \times X$ will be abbreviated by $X^2$ and it carries the norm

$$\|\mathbf{f}\|_{X^2} = \left( \|f_1\|_X^2 + \|f_2\|_X^2 \right)^{\frac{1}{2}}$$

35

for $\mathbf{f} = (f_1, f_2)$. We denote by $B(X, Y)$ the set of all bounded linear operators from $X$ to $Y$. The associated norm is denoted by $\|\cdot\|_{X,Y}$. The domain of an operator $T$ is written $D(T)$. If $X$ is a Banach space, the continuous mappings $w : [a, b] \to X$, equipped with the maximum norm

$$\max_{a \le t \le b} \|w(t)\|_X$$

is again a Banach space denoted by $C(a, b; X)$.

## 3.1   Theory for the BBM-RLW equation

The discussion of rigorous theory begins with the initial-value problem

$$\begin{cases} \eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0 & \text{for} \quad t \ge 0, x \in \mathbb{R} \\ \eta(x, 0) = g(x), & \text{for } x \in \mathbb{R}. \end{cases} \tag{3.2}$$

The following formal calculation gives an indication of some of the mathematics that follows. Suppose $\eta$ is a smooth solution of (3.2), that, with all its derivatives, decays to 0 at $\pm\infty$. Multiply the equation (3.2) by $\eta$ and integrate over $\mathbb{R}$ to obtain

$$0 = \int_{\mathbb{R}} (\eta\eta_t + \eta\eta_x + \eta^2\eta_x - \eta\eta_{xxt}) \, dx = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} (\eta^2 + \eta_x^2) \, dx$$

after integration by parts and imposition of zero boundary conditions at $\pm\infty$. This is equivalent to

$$\int_{-\infty}^{\infty} \left[\eta(x, t)^2 + \eta_x(x, t)^2\right] dx = \|\eta(\cdot, t)\|_{H^1}^2$$
$$= \|g\|_{H^1}^2 = \int_{-\infty}^{\infty} \left[g(x)^2 + g_x(x)^2\right] dx. \tag{3.3}$$

Thus the $H^1$-norm of solutions is a conserved quantity; the law (3.3) corresponds to conservation of momentum in some physical systems. Similarly, conservation of mass is expressed in the form

$$\int_{-\infty}^{\infty} u(x, t) \, dx = \int_{-\infty}^{\infty} g(x) \, dx. \tag{3.4}$$

Rewrite (3.2) in the form

$$(1 - \partial_x^2)u_t = -(u + \frac{1}{2}u^2)_x,$$

view it as an ordinary differential equation $v - v'' = f$, where $f = -(u + \frac{1}{2}u^2)_x$, and take Fourier transform on the equation

$$(1 + k^2)\hat{u}_t = \hat{f}$$

so

$$u_t = f * \mathcal{F}^{-1}(\frac{1}{1 + k^2})$$

where $*$ denote the convolution and $\mathcal{F}^{-1}$ is the inverse Fourier transform. By breaking the integral $\int_{-\infty}^{\infty}$ into $\int_{-\infty}^{0}$ and $\int_{0}^{\infty}$, and integration by parts,

$$\mathcal{F}(\frac{1}{2}e^{-|x|}) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{2}e^{-|x|}e^{-ikx}dx = \int_{0}^{\infty}\cos(ku)e^{-u}du = \frac{1}{1 + k^2},$$

one therefore obtain the formula

$$u_t(x, t) = M * f = -\int_{-\infty}^{\infty} M(x - y)\left[u_y(y, t) + u(y, t)u_y(y, t)\right] dy,$$

where $M(x) = \frac{1}{2}e^{-|x|}$. Other methods such as variation of constants can also be used to obtain the above formula. Provided that $u$ is bounded (or at least not exponentially growing as $x \to \pm\infty$), integration by parts gives the alternative

$$u_t(x, t) = \int_{-\infty}^{\infty} K(x - y)\left[u(y, t) + \frac{1}{2}u^2(y, t)\right] dy, \tag{3.5}$$

where

$$K(z) = \frac{1}{2}\text{sgn}(z)e^{-|z|}.$$

**Remark 3.1.1.** *To obtain (3.5), break the integral in the previous equation at $y = x$ and integrate these two by parts separately, viz.*

$$\int_{-\infty}^{\infty} e^{-|x-y|}f'(y)\, dy = \int_{-\infty}^{x} e^{-(x-y)}f'(y)\, dy + \int_{x}^{\infty} e^{-(y-x)}f'(y)\, dy$$

$$= e^{y-x}f(y)\Big|_{-\infty}^{x} - \int_{-\infty}^{x} e^{y-x}f(y)\, dy + e^{x-y}f(y)\Big|_{x}^{\infty} + \int_{x}^{\infty} e^{x-y}f(y)\, dy$$

$$= f(x) - \int_{-\infty}^{x} e^{y-x}f(y)\, dy - f(x) + \int_{x}^{\infty} e^{x-y}f(y)\, dy$$

$$= -\int_{-\infty}^{\infty} sgn(x - y)e^{-|x-y|}f(y)\, dy.$$

A formal integration with respect to time $t$ in (3.5) yields

$$u(x,t) - u(x,0) = \int_0^t \int_{-\infty}^{\infty} K(x-y)\left[u(y,\tau) + \frac{1}{2}u^2(y,\tau)\right] dy\, d\tau,$$

or, since $u(x,0)$ is known,

$$u(x,t) = g(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y)\left[u(y,\tau) + \frac{1}{2}u^2(y,\tau)\right] dy\, d\tau. \qquad (3.6)$$

Write (3.6) in the form

$$u = Au, \qquad (3.7)$$

where $A$ is the integral operator defined by the right-hand side of (3.6); that is, if $v = v(x,t)$ is a bounded continuous function, say, then

$$Av(x,t) = g(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y)\left[v(y,\tau) + \frac{1}{2}v^2(y,\tau)\right] dy\, d\tau. \qquad (3.8)$$

The question of existence of a solution for the initial-value problem (3.2) has thereby been converted into the issue of existence of a fixed point. At least for some small values of $t$, existence of a fixed point follows from the Contraction Mapping Principle.

**Contraction Mapping Principle.** Let $M$ be a complete metric space with metric $d$ and $A : M \to M$ such that

$$d(Ax, Ay) \leq \theta d(x,y)$$

for any $x, y \in M$, where $\theta < 1$. Then there exists a unique $x_0 \in M$ such that $Ax_0 = x_0$. Moreover, if $x_1 \in M$ is arbitrary, and we define $x_{j+1} = Ax_j$ for $j \geq 1$, then $\{x_j\}_{j=1}^{\infty}$ converges to $x_0$.

This result applies in a straightforward manner to (3.7)-(3.8). Let $T > 0$ and let $C_T$ be the Banach space

$$C_T = C_T(\mathbb{R} \times [0,T])$$
$$= \left\{ v : \mathbb{R} \times [0,T] \to \mathbb{R}, v \text{ is continuous and } \sup_{x,t} |v| < +\infty \right\},$$

normed by $||v||_{C_T} = \sup_{x \in \mathbb{R}, 0 \leq t \leq T} |v(x,t)|$. For the space $M$, choose $B_R = \{v : ||v||_{C_T} \leq R\}$ in $C_T$, where the constants $R > 0$ and $T > 0$ remain to be chosen. For any $R > 0$, the set $B_R$ is a closed subspace of the Banach space $C_T$, so it is certainly a complete metric space.

**Theorem 3.1.2.** *Let $g$ be a bounded and continuous function, say, $\sup |g(x)| \leq b$. Then there is $T = T(b) > 0$ such that the integral equation (3.6) has a solution in $C_T$.*

*Proof.* The idea is to show that $A$ is a contraction mapping of $B_R$ into itself for suitable choices of $R$ and $T$. The result then follows from the Contraction Mapping Principle.

Step 1. $v \in C_T$ implies $Av \in C_T$ since $g$ and $v$ are bounded and continuous and $K \in L_1(\mathbb{R})$. Indeed, we have

$$\|Av\|_{C_T} \leq \sup_{x \in \mathbb{R}} |g(x)| + T(\|v\|_{C_T} + \frac{1}{2}\|v\|^2_{C_T}) < \infty \qquad (3.9)$$

since $\int |K(z)|\, dz = 1$.

Step 2. If $v, w \in C_T$, then

$$\|Av - Aw\|_{C_T} = \sup_{x,t} \left| \int_0^t \int_{\mathbb{R}} K(x - y)\left[ (v - w) + \frac{1}{2}(v^2 - w^2)\right] dy\, d\tau \right|$$

$$\leq T\|v - w\|_{C_T} + \frac{1}{2}\left( \|v\|_{C_T} + \|w\|_{C_T} \right)\|v - w\|_{C_T}$$

$$\leq T\left( 1 + \frac{1}{2}(\|v\|_{C_T} + \|w\|_{C_T}) \right)\|v - w\|_{C_T}.$$

$$(3.10)$$

Step 3. Now suppose $v, w \in B_R$. Then (3.10) implies

$$\|Av - Aw\|_{C_T} \leq T(1 + R)\|v - w\|_{C_T}.$$

To apply the Contraction Mapping Principle, first demand that $T$ and $R$ are such that

$$\Theta = T(1 + R) = \frac{1}{2},$$

say. Then choose $R = 2b$ where $b = \sup_{x \in \mathbb{R}} |g(x)|$ and notice this choice means that

$$\|Au\|_{C_T} \leq \|Au - A0\|_{C_T} + \|A0\|_{C_T} \leq \theta\|u - 0\|_{C_T} + b = \frac{1}{2}\|u\|_{C_T} + \frac{1}{2}R \leq R$$

if $u \in B_R$. Thus $A$ is a contraction of $B_R$ and the result follows. $\square$

**Remark 3.1.3.** *Notice that*

$$T = \frac{1}{2(1+R)},$$

*so as the initial data gets larger, the interval of existence $T$ obtained by the above argument gets smaller.*

To insure the fixed point $u$ of (3.6) is a solution of the initial-value problem (3.2), the regularity of $u$ is brought into focus.

Let $C_b^k(\mathbb{R})$ be the Banach space

$$C_b^k = C_b^k(\mathbb{R})$$
$$= \left\{ v : \mathbb{R} \to \mathbb{R}, v^j \text{ is continuous and } \sup_{x,t} |v^j| < +\infty, \text{ for } 0 \le j \le k \right\},$$

**Proposition 3.1.4.** *If $g \in C_b^2(\mathbb{R})$ and $u$ is a solution in $C_T$ of the integral equation (3.6), then $u, u_x$ and $u_{xx}$ are infinitely smooth functions of $t$, and $u$ solves (3.2) pointwise. More precisely, $\partial_t^m u \in C_T$, $\partial_t^m u_x \in C_T$, $\partial_t^m u_{xx} \in C_T$ for all $m \ge 0$, $\lim_{t\to 0} u(x,t) = g(x)$ in $C_b^2(\mathbb{R})$, and the continuous function $u_t + u_x + u u_x - u_{xxt}$ is identically equal to zero for $(x,t) \in \mathbb{R} \times [0,T]$.*

*Proof.* We use bootstrap-type arguments.

Step 1. $\partial_t^m u \in C_T$, for all $m \ge 0$. Since

$$u(x,t) = g(x) + \int_0^t \int_{\mathbb{R}} K(x-y)(u + \frac{1}{2}u^2)\, dy\, d\tau$$

where $K(x) = \frac{1}{2} sgn(x) e^{-|x|}$, then plainly $u$ is differentiable with respect to $t$ and

$$u_t = \int_{\mathbb{R}} K(x-y)(u + \frac{1}{2}u^2)\, dy \in C_T$$

is a bounded and continuous function. Elementary considerations then imply that $u_{tt}$ exists and

$$u_{tt} = \int_{\mathbb{R}} K(x-y)(u_t + u u_t)\, dy \in C_T$$

since $u_t \in C_T$. An inductive argument leads to the conclusion $\partial_t^m u \in C_T$ for all $m \ge 0$.

Step 2. $\partial_t^m u_x \in C_T$, $\partial_t^m u_{xx} \in C_T$ for all $m \geq 0$. Now write the integral equation (3.6) as

$$u(x,t) = g(x) + \int_0^t \int_{-\infty}^x K(x-y)(u+\frac{1}{2}u^2)\,dy\,d\tau + \int_0^t \int_x^\infty K(x-y)(u+\frac{1}{2}u^2)\,dy\,d\tau$$

and use Leibnitz rule

$$\frac{d}{dx}\int_{u_1(x)}^{u_2(x)} F(x,\alpha)d\alpha = \int_{u_1(x)}^{u_2(x)} \frac{\partial}{\partial x}F(x,\alpha)d\alpha$$
$$+ F(x, u_2(x))\frac{du_2(x)}{dx} - F(x, u_1(x))\frac{du_1(x)}{dx}$$

for the differentiation of the integrals to obtain

$$u_x = g' + \int_0^t \left[ u(x,\tau) + \frac{1}{2}u^2(x,\tau) \right] d\tau$$
$$- \int_0^t \int_{-\infty}^\infty \frac{1}{2}e^{-|x-y|}(u + \frac{1}{2}u^2)\,dy\,d\tau. \tag{3.11}$$

This is plainly in $C_T$ since $u \in C_T$ and $g \in C_b^1(\mathbb{R})$. Note that $u_x$ is expressed in terms of $u$, so another inductive argument demonstrates that $\partial_t^m u_x \in C_T$ for $m \geq 0$. A similar argument shows $u_{xx}$ to exist and to be given by

$$u_{xx} = g''(x) + \int_0^t [u_x(x,\tau) + u(x,\tau)u_x(x,\tau)]\,d\tau$$
$$+ \int_0^t \int_{-\infty}^\infty K(x-y)(u + \frac{1}{2}u^2)\,dy\,d\tau. \tag{3.12}$$

The right-hand side clearly lies in $C_T$, and again, as $\partial_t^m u_x \in C_T$ for $m \geq 0$, so also $\partial_t^m u_{xx} \in C_T$.

Step 3. $\lim_{t\to 0} u(x,t) = g(x)$ in $C_b^2(\mathbb{R})$, and $u_t + u_x + uu_x - u_{xxt} = 0$ for $(x,t) \in \mathbb{R} \times [0,T]$. Using (3.6) in (3.12) gives

$$u_{xx} = g'' + \int_0^t (u_x + uu_x)\,d\tau + u - g(x). \tag{3.13}$$

Differentiating the last expression with respect to $t$, there appears

$$u_{xxt} = u_x + uu_x + u_t,$$

as hoped.

The fact that $u(x,t)$ converges to $g(x)$ as $t \downarrow 0$ is obvious and the proposition is established. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Proposition 3.1.5.** *If $g \in C_b^k(\mathbb{R})$ for any $k \geq 2$ and $u$ is a solution in $C_T$ of the integral equation (3.6), then $u$ solves (3.2) pointwise and $\partial_t^m \partial_x^p u \in C_T$, for all $m \geq 0$ and $0 \leq p \leq k$.*

*Proof.* By using previous proposition and taking derivatives on (3.13), the result is straightforward by induction.

**Remark 3.1.6.** *In fact, $u$ is analytic function of $t$; i.e. $u(x,t)$ can be expanded as $\sum_{m=0}^{\infty} u_m(x)t^m$ for suitable functions $\{u_m\}_{m \geq 0}$, and the series has a positive radius of convergence [?].*

**Remark 3.1.7.** *Note that a solution cannot acquire more spatial regularity than that of the initial data, namely, for any $k \geq 0$, if $g \in C_b^k(\mathbb{R})$ but $g \notin C_b^{k+1}(\mathbb{R})$, then $u(\cdot,t) \notin C_b^{k+1}(\mathbb{R})$ for any $t > 0$.*
*Suppose that for some $t > 0$, $u(\cdot,t) \in C_b^{k+1}(\mathbb{R})$, then at this value of $t$,*

$$g(x) = u(x,t) - \int_0^t \int_{\mathbb{R}} K(x-y)(u + \frac{1}{2}u^2)(y,s)\, dy\, ds.$$

*At time $t$, $u(\cdot,t) \in C_b^{k+1}(\mathbb{R})$, and since $u(\cdot,t) \in C_b^k(\mathbb{R})$ for all $t$, so is $u + \frac{1}{2}u^2$. Hence after convolution with $K$, there obtains a function in $C_b^{k+1}$ in the spatial variable. The integration with respect to $t$ does not change the spatial regularity, and consequently it is adduced that $g \in C_b^{k+1}(\mathbb{R})$, contrary to assumption.*

**Remark 3.1.8.** *What did we need for the contraction mapping argument? The answer is that we could have used any Banach space $X = X(\mathbb{R})$ such that if $f \in X$, then $\|f\|_\infty \leq C\|f\|_X$, where $c$ is some universal constant. Let $Y = C(0,T;X)$, then*

$$\|u\|_Y \leq \|g\|_X + T\|K\|_{L_1}\|u + \frac{1}{2}u^2\|_\infty$$
$$\leq \|g\|_X + T(\|u\|_Y + \|u\|_Y^2),$$

$$\|Av - Aw\|_Y \leq T(\|v - w\|_Y + \frac{1}{2}(\|v\|_Y + \|w\|_Y)\|v - w\|_Y)$$
$$\leq T(1 + \frac{1}{2}(\|v\|_Y + \|w\|_Y))\|v - w\|_Y.$$

*So letting $M = \{u \in Y : \|u\|_Y \leq R\}$, and requiring*

$$\|g\|_X + T(R + R^2) \leq R$$
$$T(1 + R) = \theta < 1$$

*the contraction mapping argument applies.*

The issue in front of us now is how to extend the local existence theory to arbitrary time intervals. The following result will be helpful in pursuit of this goal.

**Lemma 3.1.9.** *Let $u$ be the solution in $C_T$ of the integral equation (3.6) constructed from the contraction mapping theorem, corresponding to given initial data $g \in C_b^k(\mathbb{R})$ for some $k \geq 0$. Suppose additionally that for some $p \leq k$,*

$$g, g', ...., g^{(p)} \to 0 \qquad as \ x \to \pm\infty. \tag{3.14}$$

*Then for $0 \leq t \leq T$,*

$$\partial_x^l \partial_t^m u \to 0 \qquad as \ x \to \pm\infty, \ for \ 0 \leq l \leq p \ and \ any \ m \geq 0. \tag{3.15}$$

*Proof.* For $n \geq 1$. let $u_n = Au_{n-1}$ where $A$ is defined in (3.8) and let $u_0(x, t) = g(x)$, for $0 \leq t \leq T$. By assumption, $u_0$ is null at $\pm\infty$.

Claim 1. If $v \in C_T^0$, so is $\int_\mathbb{R} e^{-|x-y|} v(y, t) \, dy$ and $\int_\mathbb{R} K(x-y) v(y, t) \, dy$. Let $\epsilon > 0$ be given. For $x \geq \xi$,

$$\left| \int_\mathbb{R} e^{-|x-y|} v(y, t) \, dy \right|$$
$$\leq e^{-x} \int_{-\infty}^\xi e^y |v(y, t)| \, dy + \sup_{y \geq \xi} |v(y, t)| \left( \int_\xi^x e^{-|x-y|} dy + \int_x^\infty e^{-|x-y|} dy \right)$$
$$\leq e^{-x+\xi} \sup_{y,t} |v(y, t)| + 2 \sup_{y \geq \xi} |v(y, t)|.$$

Since $v \to 0$ at $\pm\infty$, there exists $\xi$ such that $|v(y, t)| \leq \frac{\epsilon}{4}$ for $y \geq \xi$. At the same time, $x$ can be chosen large enough such that the first term is made smaller than $\frac{\epsilon}{2}$, hence the sum is smaller than $\epsilon$. A similar argument applies as $x \to -\infty$.

Hence by Claim 1, $u_1 = Au_0$ is null at $\pm\infty$ and inductively, $u_n = Au_{n-1}$ is null at $\pm\infty$.

By $A$ is a contraction mapping, $u_n \to u$ in $C_T$.

Claim 2. For $v_n \in C_T^0$ where

$$C_T^0 = \{v : v \in C_T \text{ and } v \to 0 \text{ as } x \to \pm\infty\}$$

and $v_n \to v$ in $C_T$, then $v \to 0$ at $\pm\infty$. To see this, fix $t \in [0, T]$ and write

$$|v(x, t)| \le |v(x, t) - v_n(x, t)| + |v_n(x, t)|.$$

Let $\epsilon > 0$ be given and choose a corresponding $n_0$ so large that

$$\sup_{x,t} |v(x, t) - v_{n_0}(x, t)| \le \frac{\epsilon}{2}.$$

Since $v_{n_0}$ is known to be null at $\pm\infty$, there exists $M$ such that $|v_{n_0}(x, t)| \le \frac{\epsilon}{2}$ for $|x| \ge M$. Therefore if $|x| \ge M$, $|v(x, t)| \le \epsilon$, which is to say $v \to 0$ at $\pm\infty$.

By Claim 2, $u$ is asymptotically null.

Now

$$u_t = \int K(x - y)(u + \frac{1}{2}u^2) \, dy$$

is asymptotically null by step 2, and by induction, so too are higher temporal derivatives. As discovered already, if $p \ge 1$,

$$u_x = g' - \int_0^t \int_{\mathbb{R}} e^{-|x-y|}(u + \frac{1}{2}u^2) \, dy \, d\tau + \int_0^t \left[ u + \frac{1}{2}u^2 \right] \, d\tau.$$

Now $u + \frac{1}{2}u^2$ is bounded and asymptotically null, and hence so is $\int_0^t \left[ u + \frac{1}{2}u^2 \right] \, d\tau$ by the Dominated Convergence Theorem. Thus $u_x$ is asymptotically null by Step 2. Then

$$u_{xx} = g'' + \int_0^t (u_x + uu_x) \, d\tau + u - g(x)$$

is asymptotically null and so on. A double induction finishes the proof. $\square$

**Lemma 3.1.10.** *Suppose $g \in C_b^k(\mathbb{R}), k \ge 2$ and $g \in H^1(\mathbb{R})$. Then there exists $T > 0$ such that the solution $u$ of the initial-value problem*

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} = 0, \\ u(x, 0) = g(x) \end{cases}$$

*lies in $H^1(\mathbb{R})$ for all $0 \le t \le T$, and*

$$\|u(\cdot, t)\|_{H^1} = \|g(\cdot)\|_{H^1}$$

*Proof.* As we already know, $u$ is a classical solution of the partial differential equation. The existence of the solution in $H^1$ is obtained by Remark 3.1.8 and using Lemma 3.1.9. Multiply (3.2) by $u$, then integrate with respect to $x$ over $\mathbb{R}$ and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(u^2(x,t) + u_x^2(x,t))\,dx = 0.$$

It follows that for all $t$ for which the solution exists on $[0, t]$,

$$\int_{\mathbb{R}}(u^2(x,t) + u_x^2(x,t))\,dx = \int_{\mathbb{R}}(u^2(x,0) + u_x^2(x,0))\,dx,$$

which is to say

$$\|u(\cdot, t)\|_{H^1(\mathbb{R})} = \|g\|_{H^1(\mathbb{R})}. \qquad \square$$

This last point suffices to establish a global existence and uniqueness theorem, after we introduce the Gronwall Lemma.

**Lemma 3.1.11.** *(**Gronwall Lemma**)* *Let $g, h, y$ be three locally integrable function on $(t_0, \infty)$ such that $y'$ is locally integrable on $(t_0, \infty)$ and which satisfy*

$$\frac{dy}{dt} \leq gy + h \quad for \quad t \geq t_0, \tag{3.16}$$

*then*

$$y(t) \leq e^{\int_{t_0}^t g(\tau)d\tau}\left(y(t_0) + \int_{t_0}^t h(s)e^{-\int_{t_0}^s g(\tau)d\tau}\,ds\right), \quad t \geq t_0.$$

*Proof.* Multiply (3.16) by $\exp(-\int_{t_0}^t g(\tau)d\tau)$ and observe that the resulting inequality reads

$$\frac{d}{dt}(y(t)e^{-\int_{t_0}^t g(\tau)d\tau}) \leq h(t)e^{-\int_{t_0}^t g(\tau)d\tau}$$

The result follows by integration between $t_0$ and $t$. $\qquad \square$

**Theorem 3.1.12.** *Suppose $g \in H^1(\mathbb{R}) \cap C_b^2(\mathbb{R})$, then there exists a unique global solution $u$ of the initial-value problem (3.2) such that the solution $u$ satisfies $\partial_t^m \partial_x^k u \in C_T$ for all $T > 0$, $0 \leq k \leq 2$, and $m \geq 0$, and asymptotically null as well.*

*Proof.* First, note that if $g \in C_b^2 \cap H^1$ then $g, g' \to 0$ at $\pm\infty$ (see (3.17)). Hence there exists a local solution of the desired type at least on a small interval $[0, T]$, where $T$ depends only on $\sup_x |g(x)| = b$.

To extend this local solution to global, repeat this argument by using $u(x, T)$ as new data to extend the range of the solution. This would get us out to $T = \infty$ if $\sup_x |u(x, t)|$ is bounded on bounded time intervals. In fact, instead of controlling $\sup_x |u(x, t)|$ directly, we control $\|u(\cdot, t)\|_{H^1}$, then use the property $\sup |u| \leq \|u\|_{H^1}$. The latter inequality follows because for $f(x) \in C_b^2 \cap H^1$,

$$
\begin{aligned}
f^2(x) &= 2 \int_{-\infty}^x f(y) f'(y) \, dy \leq \int_{-\infty}^x (f^2 + f'^2) \, dy \\
&\leq \int_{-\infty}^\infty (f^2 + f'^2) dy = \|f\|_{H^1}^2.
\end{aligned}
\tag{3.17}
$$

So there is one bound for $u(x, t)$, independent of $t$. The contraction mapping argument may be iterated, then the global existence of the solution is proved.

Uniqueness: Let $u, v$ be two solutions of BBM (3.2), $w = u - v$, then $w \in H^1$, $w_t \in H^2$, and satisfies

$$
\begin{cases}
w_t + w_x + \dfrac{1}{2}[(u + v)w_x] - w_{xxt} = 0, \\
w(x, 0) = 0.
\end{cases}
\tag{3.18}
$$

Multiply (3.18) by $w$, then integrate over $\mathbb{R}$ :

$$
\int_{-\infty}^\infty w \left( w_t + w_x + \frac{1}{2}[(u + v)w]_x - w_{xxt} \right) dx = 0
$$

$$
\begin{aligned}
\frac{d}{dt} \int_{-\infty}^\infty (w^2 + w_x^2) dx &= \int_{-\infty}^\infty (u + v) w w_x dx \\
&\leq \frac{1}{2}|u + v|_\infty \int_{\mathbb{R}} (w^2 + w_x^2) dx \\
&\leq \|g\|_{H^1} \int_{\mathbb{R}} (w^2 + w_x^2) dx.
\end{aligned}
$$

By Gronwall lemma with $h = 0$, $g = \|g\|_{H^1}$ and $t_0 = 0$, we have

$$
\|w\|_{H^1} \leq e^{\|g\|_{H^1} t} \|w(x, 0)\|_{H^1} = 0,
$$

so, $w = u - v = 0$.                                                          □

**Remark 3.1.13.** *we can also prove the uniqueness by using implicit function theorem.*

**Corollary 3.1.14.** *Let $g \in H^k(\mathbb{R})$, for some $k \geq 1$. Then there exists a solution $u(x,t)$ of the integral equation (3.6), i.e.*

$$u(x,t) = g(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y)\left(u(y,\tau) + \frac{1}{2}u^2(y,\tau)\right) dy\, d\tau,$$

*where $K(z) = \frac{1}{2}sgn(z)e^{-|z|}$. The solution $u(x,t)$ is in $C(0,T;H^k(\mathbb{R}))$, for some $T > 0$. Moreover, $u_t \in C(0,T;H^{k+1}(\mathbb{R}))$.*

*Proof.* $H^k(\mathbb{R})$ is a Banach space embedded in $C_b(\mathbb{R})$, for $k > \frac{1}{2}$, so the continuity of the solution is obvious according to the integral equation. Also differentiate the integral equation with respect to $t$, we have

$$u_t = \int_{\mathbb{R}} K(x-y)(u + \frac{1}{2}u^2)\, dy, \tag{3.19}$$

where $u + \frac{1}{2}u^2 \in C(0,T;H^k)$, so that $K * (u + \frac{1}{2}u^2) \in C(0,T;H^{k+1})$. To see this, let $f \in C(0,T;H^k)$, then

$$\widehat{K * f} = \frac{i\xi}{1 + \xi^2}\hat{f}.$$

The variable $\xi$ (instead of $k$) is used for Fourier space so it will not confuse with the $k$ in the notation for space $H^k$. Then

$$\int_{-\infty}^{\infty}(1 + \xi^2)^{k+1}|\frac{i\xi}{1 + \xi^2}\hat{f}(\xi)|^2\, d\xi \leq \int_{-\infty}^{\infty}(1 + \xi^2)^k|\hat{f}(\xi)|^2\, d\xi < \infty.$$

It is also clear from (3.19) that $u_t$ is a continuous function of $t$. Thus if $g \in H^1(\mathbb{R})$, then $u \in C(0,T;H^1)$ and $u_t \in C(0,T;H^2)$. The result for $k > 1$ follows by taking derivatives (see the proof of Proposition 3.1.5). $\qquad\square$

**Theorem 3.1.15.** *(continuous dependence on the initial data). The mapping $g \mapsto u$ is continuous from $X = H^1(\mathbb{R}) \cap C_b^2(\mathbb{R}) \to Y = C(0,T;X)$.*

*Proof.* It suffices to check this locally near the origin of time and then iterates.
   Let $g, h \in X$, and

$$u = A_g(u) = g + \int_0^t \int_{\mathbb{R}} K(x-y)(u + \frac{1}{2}u^2)dyds$$

and $v = A_h(v)$. Since $A$ is a contraction in $W = C(0, t; X)$ for $t$ small,

$$\|u - v\|_W = \|A_g(u) - A_h(v)\|_W \leq \|A_g(u) - A_g(v)\|_W + \|A_g(v) - A_h(v)\|_W$$
$$\leq \theta\|u - v\|_W + \|g - h\|_X.$$

Therefore

$$\|u - v\|_W \leq \frac{1}{1 - \theta}\|g - h\|_X,$$

and the mapping $g \to u$ is Lipschitz continuous with constant $\frac{1}{1-\theta}$. Iterates on time then yields the desired result. $\qquad\square$

## 3.2    Bore Propagation

In this section, we consider BBM equation

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} = 0, & \forall x \in \mathbb{R}, \quad t \geq 0, \\ u(x, 0) = g(x), \end{cases} \tag{3.20}$$

where $g \in C_b^2(\mathbb{R})$, $g(x) \to a$ as $x \to +\infty$, $g(x) \to b$ as $x \to -\infty$, and $g' \in L_2(\mathbb{R})$, $(g' \to 0$ as $x \to \pm\infty)$.

As before, we still discuss the local existence, regularity, and the global existence of solution for (3.20). Notice that Theorem 3.1.12 does not apply to this case, but the local existence and regularity results in Theorem 3.1.2, Proposition 3.1.4 and 3.1.5 can be used.

We now prove the global existence of the solution. By changing variable $v = u - g$ in (3.20), (3.20) turns to be:

$$\begin{cases} v_t - v_{xxt} + (g + v + \frac{1}{2}g^2 + gv + \frac{1}{2}v^2)_x = 0, \\ v(x, 0) = 0. \end{cases} \tag{3.21}$$

Then (3.21) is equivalent to the following integral equation:

$$v = -\int_0^t M * (g + v + \frac{1}{2}g^2 + gv + \frac{1}{2}v^2)_y \, d\tau, \tag{3.22}$$

where $M(z) = \frac{1}{2}e^{-|z|}$, or

$$v = -\int_0^t M * (v_y + vv_y + gv_y + g'v) \, d\tau - \int_0^t M * (g' + gg') \, d\tau. \tag{3.23}$$

Since $g' \to 0$ at $\pm\infty$, $\int_0^t M * (g' + gg')d\tau \to 0$ at $\pm\infty$. Now we can essentially argue as in lemma 3.1.9 to conclude that $v, v_x, v_{xt} \to 0$ at $\pm\infty$.

Multiply (3.21) by $v$ and integrate over $\mathbb{R}$ and $[0, t]$ :

$$
\begin{aligned}
\frac{1}{2} &\int_{-\infty}^{\infty} (v^2 + v_x^2) \, dx \\
&= \int_{-\infty}^{\infty} \left( v^2(x, 0) + v_x^2(x, 0) \right) dx - \int_0^t \int_{-\infty}^{\infty} \left( (1+g)g'v - gvv_x \right) dx \\
&\le (1 + \|g\|_\infty) \int_0^t \|g'\|_{L^2} \|v\|_{L^2} d\tau + \|g\|_\infty \int_0^t \|v\|_{L^2} \|v_x\|_{L^2} d\tau \\
&\le \frac{C}{2} \int_0^t \int_{\mathbb{R}} (v^2 + v_x^2) dx + \frac{C_1}{2} d\tau,
\end{aligned}
\tag{3.24}
$$

where $C$ and $C_1$ are constants only dependent on $\|g\|_{H^1}$. Then a Gronwall lemma yields

$$
\int_{-\infty}^{\infty} \left[ v^2(x, t) + v_x^2(x, t) \right] \, dx \le \frac{C_1}{C} (e^{ct} - 1),
$$

which is enough to extend to $+\infty$, because for any $T > 0$, $\|v\|_{H^1}$ is bounded, independent of $t$, so solution exists on $[0, T]$.

**Exercises 1:**   Consider the two-point boundary value problem

$$
\begin{aligned}
u_t + u_x + uu_x - u_{xxt} &= 0, \quad x \in [0, L], t \ge 0, \\
u(x, 0) &= g(x), \quad x \in [0, L], \\
u(0, t) &= h_1(t), u(L, t) = h_2(t), \quad 0 \le t \le T,
\end{aligned}
$$

which satisfy the compatibility condition

$$
g^{(j)}(0) = h_1^{(j)}(0), \quad g^{(j)}(L) = h_2^{(j)}(0)
$$

for $j = 0, 1, 2$. Then if $g \in C^2(0, L)$ and $h_1, h_2 \in C^1(0, T)$, there exists a unique classical solution $u \in C^1(0, T_0, C^2([0, L]))$, where $0 < T_0 \le T$.

## 3.3  Theory for the Korteweg-de Vries equation

Without the small parameters, the KdV equation can be written as

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \forall x \in \mathbb{R}, \quad t \geq 0, \\ u(x,0) = g(x). \end{cases} \tag{3.25}$$

Let us first shift to traveling coordinates. Set $\tilde{u}(x,t) = u(x+t,t)$, then (3.25) becomes

$$\begin{cases} \tilde{u}_t + \tilde{u}\tilde{u}_x + \tilde{u}_{xxx} = 0, & \forall x \in \mathbb{R}, \quad t \geq 0, \\ \tilde{u}(x,0) = g(x). \end{cases} \tag{3.26}$$

Now drop the tildes and study this one. We are going to follow the line of proof in [1] by regularizing this differential equation as following:

$$\begin{cases} u_t + uu_x + u_{xxx} - \epsilon u_{xxt} = 0 & \forall x \in \mathbb{R}, \quad t \geq 0 \\ u(x,0) = g(x), \end{cases} \tag{3.27}$$

where $\epsilon > 0$ is fixed for the time being.

**Remark 3.3.1.** *Equation (3.27) looks a little peculiar, since the more standard regularization might be $u_t + u_x + uu_x + u_{xxt} - \epsilon u_{xxxx} = 0$ to make the equation basically parabolic. However, we can answer certain interesting questions later using the regularization in (3.27).*

**Proposition 3.3.2.** *Suppose $g \in H^s$ where $s \geq 2$. Then there exists a unique solution $u$ to the regularized KdV equation (3.27) which lies in $H_T^s$ for any finite $T > 0$. Furthermore, for $0 \leq l \leq s$, $\partial_t^l u \in H_T^{m-l}$ for any $T > 0$.*

*Proof.* The proof is straightforward by considering the transform of variables $v(x,t) = \epsilon u\left(\epsilon^{\frac{1}{2}}(x-t), \epsilon^{\frac{3}{2}}t\right)$, so (3.27) becomes:

$$\begin{cases} v_t + v_x + vv_x - v_{xxt} = 0, \\ v(x,0) = \epsilon g(\epsilon^{\frac{1}{2}}x), \end{cases} \tag{3.28}$$

and using Corollary 3.1.14.                                                      $\square$

We now study (3.27) which is easier to handle than the KdV equation for the theoretical analysis, then if we let $\epsilon \to 0$, and hope to recover a solution of KdV. To do so, we need $\epsilon$-independent bounds on the solutions of (3.28), then pass to KdV (3.25) as $\epsilon \downarrow 0$.

We might as well work with a function $g \in H^\infty(\mathbb{R})$, which are functions in $C^\infty$ in $x$ with all its derivatives in $L^2$, since we have continuous dependence results. The the solution $u$ is then in $C^\infty$ in $x$ and in $t$ and $u(\cdot, t) \in L^2(\mathbb{R})$ with all its derivatives. Therefore, everything in sights goes to 0 at $\pm\infty$.

**Proposition 3.3.3.** *The solution $u$ of (3.27) corresponding to $g$ in $H^\infty$ satisfies the inequality*

$$\|u\|_1 \leq a_1(\|g\|_1) \tag{3.29}$$

*for all $t > 0$, independent of $\epsilon \in (0, 1]$, where $a_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, monotone increasing with $a_1(0) = 0$.*

*Proof.* Multiply (3.27) by $u$, then integrated with $x$ over line $\mathbb{R}$ :

$$\int_{-\infty}^{\infty} (uu_t + u^2 u_x + uu_{xxx} - \epsilon uu_{xxt}) \, dx = 0,$$

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (u^2 + \epsilon u_x^2) \, dx = 0.$$

so,

$$\int_{-\infty}^{\infty} (u^2 + \epsilon u_x^2) \, dx = \int_{-\infty}^{\infty} (g^2 + \epsilon g'^2) \, dx \leq \|g\|_{H^1}^2,$$

where we have taken $\epsilon \leq 1$ without loss of generosity. Hence, independent of $\epsilon > 0$,

$$\|u\|_{L^2} \leq \|g\|_{H^1}. \tag{3.30}$$

Rewrite (3.27) as

$$u_t + (\frac{1}{2}u^2 + u_{xx} - \epsilon u_{xt})_x = 0,$$

and multiplying it by $\frac{1}{2}u^2 + u_{xx} - \epsilon u_{xt}$ then integrating over $\mathbb{R}$ gives:

$$\frac{d}{dt} \int_{-\infty}^{\infty} \left( \frac{1}{6}u^3 - \frac{1}{2}u_x^2 \right) \, dx = 0,$$

i.e.

$$\int_{-\infty}^{\infty} (u_x^2 - \frac{1}{3}u^3) \, dx = constant = \int_{-\infty}^{\infty} (g'^2 - \frac{1}{3}g^3) \, dx,$$

so using (3.30),

$$
\int_{\mathbb{R}} u_x^2 dx \le \frac{1}{3}|u|_\infty \int_{\mathbb{R}} u^2\, dx + \int_{\mathbb{R}} g_x^2\, dx + \frac{1}{3}|g|_\infty \int_{\mathbb{R}} g^2\, dx
$$

$$
\le \frac{1}{3}\|g\|_{H^1}\|u\|_{L^2}^{\frac{1}{2}}\|u_x\|_{L^2}^{\frac{1}{2}} + \|g\|_{H^1}^2 + \frac{1}{3}\|g\|_{H^1}^3 \tag{3.31}
$$

$$
\le \frac{1}{3}\|g\|_{H^1}^{\frac{3}{2}}\|u_x\|_{L^2}^{\frac{1}{2}} + \|g\|_{H^1}^2 + \frac{1}{3}\|g\|_{H^1}^3.
$$

Let $A = A(t) = \|u_x(\cdot, t)\|_{L_2}$, then (3.31) is rewritten by using Young's inequality

$$
ab \le \frac{a^p}{p} + \frac{b^q}{q} \text{ where } a, b > 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1
$$

as

$$
A^2 \le CA^{\frac{1}{2}} + D \le \frac{1}{4}A^2 + C_1 + D \tag{3.32}
$$

where $C = \frac{1}{3}\|g\|_{H^1}^{\frac{3}{2}}$, $D = \|g\|_{H^1}^2 + \frac{1}{3}\|g\|_{H^1}^3$ and $C_1 = \frac{3}{4}C^{\frac{4}{3}} = \frac{1}{4}\|g\|_{H^1}^2$ are only dependent on initial data $g$. So,

$$
\|u_x\|^2 = A^2 \le \frac{4}{3}(C_1 + D) = D_1(\|g\|_{H^1}).
$$

Thus in summary,

$$
\|u\|_{H^1} \le a_1(\|g\|_{H^1}). \tag{3.33}
$$

$\square$

Next is the crucial step, obtaining an $H^2$ bound, which is a little involved.

**Proposition 3.3.4.** *Let $u$ be a smooth solution of regularized KdV (3.27) corresponding to $g \in H^\infty(\mathbb{R})$. Then there exists $\epsilon_0 = \epsilon_0(T, \|g\|_{H^3})$ such that if $0 < \epsilon \le \epsilon_0$, then*

$$
\|u\|_{C(0.T;H^2)} \le a_2(\|g\|_{H^3}),
$$

*where $a_2 : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, monotone increasing with $a_2(0) = 0$.*

*Proof.* Multiply (3.27) by $u^3 + 3u_x^2 + 6uu_{xx} + \frac{18}{5}u_{xxxx}$ and integrate over $\mathbb{R}$ by parts, after several simplification, we derive the identity:

$$
\frac{d}{dt}\int_{\mathbb{R}}(\frac{9}{5}u_{xx}^2 - 3uu_x^2 + \frac{1}{4}u^4)\, dx = \epsilon\int_{\mathbb{R}}(u^3 + 3u_x^2 + 6uu_{xx} + \frac{18}{5}u_{xxxx})u_{xxt}\, dx.
$$

Integrating by parts gives over the right hand side:

$$\frac{d}{dt} \int_{-\infty}^{\infty} [(\frac{9}{5} - 3\epsilon u) u_{xx}^2 - 3u u_x^2 + \frac{1}{4} u^4 + \epsilon \frac{9}{5} u_{xxx}^2] \, dx$$
$$= -\epsilon \int_{-\infty}^{\infty} (3u_t u_{xx}^2 + 3u^2 u_x u_{xt} + 6u_x u_{xx} u_{xt}) \, dx. \tag{3.34}$$

Define

$$V(t) = \int_{\mathbb{R}} [(\frac{9}{5} - 3\epsilon u) u_{xx}^2 - 3u u_x^2 + \frac{1}{4} u^4 + \epsilon \frac{9}{5} u_{xxx}^2] \, dx,$$

then after integration of (3.34) over $[0, T]$, we have

$$V(t) = V(0) - \epsilon \int_0^t \int_{-\infty}^{\infty} [3u_t u_{xx}^2 + 3u^2 u_x u_{xt} + 6u_x u_{xx} u_{xt}] \, dx \, dt.$$

Now by (3.29) we know that, independently of $t \geq 0$, and $\epsilon \geq 0$,

$$\sup_{x \in \mathbb{R}, t \geq 0} |u(x,t)| \leq \sup_{t \geq 0} \|u\|_{H^1} \leq a(\|g\|_{H^1}).$$

Hence there exists $\epsilon_1 > 0$ such that for $0 < \epsilon \leq \epsilon_1$, we have

$$\frac{13}{5} \geq \frac{9}{5} - 3\epsilon u \geq 1,$$

i.e.

$$|3\epsilon u| \leq \frac{4}{5},$$

and

$$\int_{-\infty}^{\infty} u_{xx}^2 \, dx \leq \int_{-\infty}^{\infty} [(\frac{9}{5} - 3\epsilon u) u_{xx}^2 + \frac{1}{4} u^4 + \frac{9}{5} \epsilon u_{xxx}^2] \, dx$$
$$= V(t) + 3\epsilon \int_{-\infty}^{\infty} u u_{xx}^2 \, dx$$
$$\leq V(0) + 3 \int_{-\infty}^{\infty} |u| |u_x|^2 \, dx \tag{3.35}$$
$$+ \epsilon \int_0^t \int_{-\infty}^{\infty} |3u_t u_{xx}^2 + 3u^2 u_x u_{xt} + 6u_x u_{xx} u_{xt}| \, dx d\tau;$$

where

$$V(0) \leq \frac{13}{5} \|g\|_{H^2}^2 + 3\|g\|_{H^1}^3 + \frac{1}{4} \|g\|_{H^1}^4 + \frac{9}{5} \epsilon \|g\|_{H^3}^2 = C(\|g\|_{H^3}),$$

and
$$3 \int_{-\infty}^{\infty} |u||u_x|^2 \, dx \leq 3\|u\|_{H^1} \int_{-\infty}^{\infty} u_x^2 \leq 3\|u\|_{H^1}^3 \leq 3\bar{a}(\|g\|_{H^1}).$$

Thus,

$$V(t) \leq \bar{C}(\|g\|_{H^3})$$
$$+ \epsilon \int_0^t \left(3\|u_t\|_\infty \|u_{xx}\|^2 + 3\|u\|_\infty^2 \|u_x\|\|u_{xt}\| + 6\|u_x\|_\infty \|u_{xx}\|\|u_{xt}\|\right) \, d\tau,$$

in which $\bar{C}(\|g\|_{H^3}) = C(\|g\|_{H^3}) + 3\bar{a}(\|g\|_{H^1})$. From (3.36), (3.35) and the definition of $V(t)$, we derive

$$A^2(t) \leq \bar{C}(\|g\|_{H^3})$$
$$+ \epsilon \int_0^t \left(3\|u_t\|_\infty A^2(\tau) + 3\|u\|_\infty^2 \|u_x\|\|u_{xt}\| + 6\|u_x\|_\infty \|u_{xt}\|A(\tau)\right) \, d\tau,$$

where $A^2(t) = \int_{-\infty}^{\infty} u_{xx}^2 \, dx$.

Now, we need some bounds of these temporal derivatives. Denote $u_t = v$, then $v$ satisfies:
$$v_t + (uv)_x + v_{xxx} - \epsilon v_{xxt} = 0. \tag{3.36}$$

Multiply (3.36) by $v$ and integrate over $\mathbb{R}$,

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(v^2 + \epsilon v_x^2) \, dx = \int_{-\infty}^{\infty} uvv_x \, dx = -\frac{1}{2}\int_{-\infty}^{\infty} u_x v^2 \leq \frac{1}{2}\|u_x\|_\infty \int_{-\infty}^{\infty} v^2 \, dx.$$

Set $B^2(t) = \int_{-\infty}^{\infty}(v^2 + \epsilon v_x^2) \, dx$, then we have

$$B^2(t) \leq B^2(0) + \int_0^t \|u_x\|_\infty B^2(\tau) \, d\tau.$$

Sublemma,
$$\|u_t\|_\infty \leq \epsilon^{-\frac{1}{4}} B(t),$$
$$\|u_x\|_\infty \leq \sqrt{\|u_x\|\|u_{xx}\|} \leq CA^{\frac{1}{2}}(t),$$
$$\|u_{xt}\| \leq \epsilon^{-\frac{1}{2}} B(t).$$

**Remark 3.3.5.** *The sublemma can be proved by following:*

$$\|u_t\|_\infty^2 = \|v\|_\infty^2 \leq (\epsilon^{-\frac{1}{4}}\|v\|)(\epsilon^{\frac{1}{4}}\|v_x\|) \leq \epsilon^{-\frac{1}{2}}\|v\|^2 + \epsilon^{\frac{1}{2}}\|v_x\|^2 = \epsilon^{-\frac{1}{2}}B^2(t),$$
$$\epsilon\|u_{xt}\|^2 = \epsilon\|v_x\|^2 \leq B^2(t).$$

*Using this sublemma, we derive the pair of integral inequalities:*

$$\begin{cases} A^2(t) \leq C + \epsilon C \int_0^t [\epsilon^{-\frac{1}{4}} B(\tau) A^2(\tau) + \epsilon^{-\frac{1}{2}} B(\tau) + \epsilon^{-\frac{1}{2}} B(\tau) A^{\frac{3}{2}}(\tau)] \, d\tau, \\[2mm] B^2(t) \leq B^2(0) + C \int_0^t A^{\frac{1}{2}}(\tau) B^2(\tau) \, d\tau, \end{cases}$$

(3.37)

*where $C's$ stand for constants independent of t. Next, we need a bound on $B(0)$, so multiply (3.27) by $u_t$ and integrate over $\mathbb{R}$,*

$$\begin{aligned} B^2(t) &= \int_{-\infty}^{\infty} (u_t^2 + \epsilon u_{xt}^2) \, dx \\ &= \int_{-\infty}^{\infty} (-u_t u u_x - u_t u_{xxx}) \, dx \\ &\leq \|u_t\| \|u\|_\infty \|u_x\| + \|u_t\| \|u_{xxx}\| \\ &\leq B(t) (\|g\|_{H^1}^2 + \|u_{xxx}\|). \end{aligned}$$

*Cancel off $B(t)$ and evaluate at $t = 0$,*

$$B(0) \leq (\|g\|_{H^1}^2 + \|g\|_{H^3}).$$

*Now let's go back to (3.37) again, define $D^2(t) = A^2(t) + 1$, then it is implied that*

$$\begin{cases} D^2(t) \leq C + \epsilon^{\frac{1}{2}} C \int_0^t B(\tau) D^2(\tau) \, d\tau, \\[2mm] B^2(t) \leq C + C \int_0^t D^{\frac{1}{2}}(\tau) B^2(\tau) \, d\tau. \end{cases}$$

(3.38)

*The reason to do so is that if $D(t)$ can be bounded, then so can be $A(t)$. The claim is that the above (3.38) implies*

$$\begin{cases} D^2(t) \leq \left(\dfrac{\alpha}{1 - \epsilon^{\frac{1}{2}}}\right)^4 + \epsilon^{\frac{1}{2}} \dfrac{4\gamma}{\beta} \int_0^t B(\tau) D^2(\tau) \, d\tau, \\[3mm] B^2(t) \leq \left(\dfrac{\beta}{1 - \epsilon^{\frac{1}{2}}}\right)^2 + \dfrac{2\gamma}{\alpha} \int_0^t D^{\frac{1}{2}}(\tau) B^2(\tau) \, d\tau, \end{cases}$$

(3.39)

*where $\alpha$, $\beta$, $\gamma$ do not depend on $\epsilon \leq \epsilon_1$ provided that $\epsilon_1 < 1$, as we now assume. First choose $\epsilon_1 \leq \frac{1}{2}$ and in accordance with previous restriction. Next choose $\alpha$, $\beta$ large enough, and finally choose $\gamma$ large enough. Note $\alpha$, $\beta$, $\gamma$ do not depend on $\epsilon_1$ or on $\epsilon$.*

Define $\bar{D}$ and $\bar{B}$ by the equality in the last equality (3.39):

$$\begin{cases} \bar{D}^2 = \left(\dfrac{\alpha}{1-\epsilon^{\frac{1}{2}}}\right)^4 + \epsilon^{\frac{1}{2}}\dfrac{4\gamma}{\beta}\displaystyle\int_0^t \bar{D}^2\bar{B}\,d\tau, \\[3mm] \bar{B}^2 = \left(\dfrac{\beta}{1-\epsilon^{\frac{1}{2}}}\right)^2 + \dfrac{2\gamma}{\alpha}\displaystyle\int_0^t \bar{D}^{\frac{1}{2}}\bar{B}^2\,d\tau. \end{cases} \tag{3.40}$$

This system of ordinary differential equations can be solved exactly, and the solution bounds above $D$ and $B$, respectively.

$$\bar{D} = \left(\frac{\alpha}{1-\epsilon^{\frac{1}{2}}e^{\gamma t}}\right)^2, \qquad \bar{B} = \frac{\beta e^{\gamma t}}{1-\epsilon^{\frac{1}{2}}e^{\gamma t}}. \tag{3.41}$$

Choose $\epsilon_2$ such that

$$1 - \epsilon_2^{\frac{1}{2}}e^{\gamma t} \geq \frac{1}{2},$$

and let $\epsilon_0 = \min(\epsilon_1, \epsilon_2)$, then we see that

$$\bar{D} \leq (2\alpha)^2, \qquad \bar{B} \leq 2\beta e^{\gamma T} \qquad \text{for } 0 \leq t \leq T.$$

In particular,

$$\int_{\mathbb{R}} u_{xx}^2\,dx \leq const(\|g\|_{H3}, T) \qquad \text{for } 0 \leq t \leq T,$$

where this *const* doesn't depend on $\epsilon$, for $\epsilon$ sufficiently small.

Now the going is much easier. Let $m > 2$ and suppose inductively that we have and $\|g\|_{H^m}$. We then show bounded $\|u\|_{C(0,T;H^{m-1})}$ is bounded, independently of $\epsilon \leq \epsilon_0$, with a bound dependent only on $T$, $\epsilon_0$, that $u$ is bounded in $C(0,T;H^m)$ with a bound dependent only on $T$, $\epsilon_0$, $\|g\|_{H^m}$ and $\epsilon^{\frac{1}{2}}\|g\|_{H^{m+1}}$.

Multiply (3.27) by $u_{(m)}$ and integrate over $\mathbb{R}$ by parts,

$$
\begin{aligned}
\frac{d}{dt} &\int_{\mathbb{R}} (u_{(m)}^2 + \epsilon u_{(m+1)}^2)\, dx \\
&= -\int_{\mathbb{R}} (u^2)_{(m+1)} u_{(m)} \\
&= \epsilon_0 \int_{\mathbb{R}} u u_{(m+1)} u_{(m)}\, dx \\
&\quad + c_1 \int_{-\infty}^{\infty} \left( u_x u_{(m)}^2 + u_{(m)} \sum_{r=2}^{m-2} c_r u_{(r)} u_{(m+1-r)} + u_{(m-1)}^2 u_{(m)} \right) dx \\
&\leq |u_x|_\infty \int_{\mathbb{R}} u_{(m)}^2 + c\|u_{(m)}\|_{L^2} \\
&\leq c_1 \|u_{(m)}\|^2 + c_2 \|u_{(m)}\|.
\end{aligned}
\tag{3.42}
$$

and this does it, by Gronwall-type estimate.   $\square$

Now we look to the limit $\epsilon \downarrow 0$. There are two or three ways to handle the passages to the limit. The way we choose leads to sharp regularity results, but an easier method could be adopted, using weak compactness arguments in $L^\infty(0, T; H^?)$ for example.

**Proposition 3.3.6.** *Let $T > 0$ and $g \in H^\infty$ be given. Let $\epsilon_0$ be as in Proposition 3.3.4. Then for $\epsilon \leq \epsilon_0$, and $m \geq 3$,*

$$
\|u\|_m \leq a_3(T, \epsilon_0, \|g\|_m, \epsilon^{\frac{1}{2}} \|g\|_{m+1}).
$$

*Proof.*   $\square$

Let's fix initial data $g \in H^s$, where $s \geq 3$ and consider the regularized equation,

$$
\begin{cases}
u_t + u u_x + u_{xxx} - \epsilon u_{xxt} = 0, \\
u(x, 0) = g_\epsilon(x),
\end{cases}
\tag{$P_\epsilon$}
$$

where $\hat{g}_\epsilon(\xi) = \phi(\epsilon^{\frac{1}{6}} \xi) \hat{g}(\xi)$. The function $\phi$ is a $C^\infty$ function, with $0 \leq \phi \leq 1$ everywhere, $\phi(0) = 1$ and $\phi \to 0$ exponentially rapid at $\pm\infty$, and

$$
\psi(\xi) = 1 - \phi(\xi)
$$

has a zero of infinity order at $\xi = 0$. That is, $\phi$ is very flat at 0. An example
of such fuction is $\phi(\xi) = e^{-\xi^2 e^{-1/\xi^2}}$.

We need a lemma detailing how various Sobolev norms of $g_\epsilon$ behave. Here
it is.

**Proposition 3.3.7.** *Let $g \in H^s$, $s \geq 3$ and let $g_\epsilon$ be as above. Then $g_\epsilon \in H^\infty$,
and*

$$\|g_\epsilon\|_{H^{s+j}} = O(\epsilon^{-\frac{j}{6}}), \qquad j = 1, 2, \cdots$$

$$\|g - g_\epsilon\|_{H^{s-j}} = o(\epsilon^{\frac{j}{6}}), \qquad j = 1, 2, \cdots$$

$$\|g - g_\epsilon\|_{H^s} = o(1),$$

*as $\epsilon \downarrow 0$. The first bound holds uniformly on bounded sets and the last two
hold uniformly on compact subsets of $H^s$. (If o is replaced by O, then the
second bounds hold uniformly on bounded sets).*

Since the Lebesgue's Dominated Convergence Theorem will be used in
the proof of the proposition, we first recall it here.

**Lemma 3.3.8.** *Lebesgue's Dominated Convergence Theorem.     Suppose
that $f_n$ is a sequence of measurable functions, that $f_n \to f$ as $n \to \infty$
and that $|f_n| \leq g$ for all $n$ where $g$ is integrable. Then $f$ is integrable and
$\int f d\mu = \lim_{n\to\infty} f_n d\mu$.*

*Proof of Proposition.* This is an easy calculation in the Fourier transformed
variables. Look at

$$\epsilon^{\frac{j}{3}}\|g_\epsilon\|_{s+j}^2 \leq C\epsilon^{\frac{j}{3}} \int_{-\infty}^{\infty} [1 + \xi^{2(s+j)}]|\hat{g}_\epsilon(\xi)|^2 \, d\xi$$

$$\leq C\epsilon^{\frac{j}{3}} \int_{-\infty}^{\infty} \left[ \frac{1 + \xi^{2(s+j)}}{1 + \xi^{2s}} \phi^2(\epsilon^{\frac{1}{6}}\xi) \right] (1 + \xi^{2s})|\hat{g}(\xi)|^2 \, d\xi \qquad (3.43)$$

$$\leq C \sup_{\xi \in \mathbb{R}} \left\{ \epsilon^{\frac{j}{3}} \frac{1 + \xi^{2(s+j)}}{1 + \xi^{2s}} \phi^2(\epsilon^{\frac{1}{6}}\xi) \right\} \|g\|_{H^s}^2.$$

Let $K = \epsilon^{\frac{1}{6}}\xi$, suppose $0 < \epsilon \leq 1$ as well, then

$$\sup_{\xi \in \mathbb{R}} \epsilon^{\frac{j}{3}} \frac{1 + \xi^{2(s+j)}}{1 + \xi^{2s}} \phi^2(\epsilon^{\frac{1}{6}}\xi) = \sup_{K \in \mathbb{R}} \epsilon^{\frac{j}{3}} \frac{1 + (\epsilon^{-\frac{1}{6}}K)^{2(s+j)}}{1 + (\epsilon^{-\frac{1}{6}}K)^{2s}} \phi^2(K)$$

$$\leq \sup_{K \in \mathbb{R}} \frac{\epsilon^{\frac{s+j}{3}} + K^{2(s+j)}}{\epsilon^{\frac{s}{3}} + K^{2s}} \phi^2(K) \leq C,$$

since $\phi(K) \to 0$ exponentially as $K \to \infty$ and $\frac{\epsilon^{\frac{s+j}{3}}+K^{2(s+j)}}{\epsilon^{\frac{s}{3}}+K^{2s}}\phi^2(K)$ is bounded, independent of $\epsilon$, for $K$ bounded, so,

$$\epsilon^{\frac{j}{3}}\|g_\epsilon\|^2_{s+j} \leq C\|g\|^2_{H^s}.$$

The first estimate is proved.

To prove the second estimate,

$$\begin{aligned}
\|g - g_\epsilon\|^2_{H^{s-j}} &\leq C\int_\mathbb{R}[1+\xi^{2(s-j)}]|\hat{g}(\xi) - \hat{g}_\epsilon(\xi)|^2\,d\xi \\
&\leq C\int_\mathbb{R}[1+\xi^{2(s-j)}]|\hat{g}(\xi)|^2\psi^2(\epsilon^{\frac{1}{6}}\xi)\,d\xi \\
&\leq C\int_\mathbb{R}\frac{1+\xi^{2(s-j)}}{1+\xi^{2s}}\psi^2(\epsilon^{\frac{1}{6}}\xi)(1+\xi^{2s})|\hat{g}(\xi)|^2\,d\xi \\
&\leq C\sup_{\xi\in\mathbb{R}}\frac{1+\xi^{2(s-j)}}{1+\xi^{2s}}\psi(\epsilon^{\frac{1}{6}}\xi)\int_\mathbb{R}\psi(\epsilon^{\frac{1}{6}}\xi)(1+\xi^{2s})|\hat{g}(\xi)|^2\,d\xi.
\end{aligned} \tag{3.44}$$

Similar to above, let $K = \epsilon^{\frac{1}{6}}\xi$,

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}}\frac{1+\xi^{2(s-j)}}{1+\xi^{2s}}\psi(\epsilon^{\frac{1}{6}}\xi) &= \sup_{K\in\mathbb{R}}\frac{1+(\epsilon^{-\frac{1}{6}}K)^{2(s-j)}}{1+(\epsilon^{-\frac{1}{6}}K)^{2s}}\psi(K) \\
&= \sup_{K\in\mathbb{R}}\frac{\epsilon^{\frac{s}{3}}+\epsilon^{\frac{j}{3}}K^{2(s-j)}}{\epsilon^{\frac{s}{3}}+K^{2s}}\psi(K) = \sup_{K\in\mathbb{R}}\epsilon^{\frac{j}{3}}\frac{\epsilon^{\frac{s-j}{3}}+K^{2(s-j)}}{\epsilon^{\frac{s}{3}}+K^{2s}}\psi(K) \\
&\leq \epsilon^{\frac{j}{3}}\sup_{K\in\mathbb{R}}\frac{\epsilon^{\frac{s-j}{3}}+K^{2(s-j)}}{\epsilon^{\frac{s}{3}}+K^{2s}}\psi(K) \leq C\epsilon^{\frac{j}{3}},
\end{aligned}$$

since $0 \leq \psi \leq 1$ and $\psi(0) = 0$ $\sup_{K\in\mathbb{R}}\frac{\epsilon^{\frac{s-j}{3}}+K^{2(s-j)}}{\epsilon^{\frac{s}{3}}+K^{2s}}\psi(K)$ is bounded, independt of $\epsilon$, and

$$\|g - g_\epsilon\|^2_{H^{s-j}} \leq C\epsilon^{\frac{j}{3}},$$

that is $\|g - g_\epsilon\|_{H^{s-j}} = O(\epsilon^{\frac{j}{6}})$ uniformly on bounded sets of $H^s$. Now by dominated convergence, since the integral is $o(1)$ as $\epsilon \to 0$ uniformly on compact subsets of $H^s$ and integral is also bounded above by $\|g\|^2_s$, therefore

$$\int_\mathbb{R}\psi(\epsilon^{\frac{1}{6}}\xi)(1+\xi^{2s})|\hat{g}(\xi)|^2\,d\xi \to 0.$$

For the third inequality,

$$\|g - g_\epsilon\|_s \to 0 \text{ as } \epsilon \to 0$$

can be shown as before using dominated convergence theorem. Then note that to demonstrate the uniformity on compact subsets, it is sufficient to show that if $g_n \to g$ in $H^s$, then $\|g_{n\epsilon} - g_n\|_s \to 0$ as $\epsilon \to 0$ uniformly for $n = 1, 2, \cdots$ since sequential compactness is equivalent to compactness in a metric space.

To prove this, let $\gamma > 0$ be given. It is required to find an $\epsilon_0 > 0$ so that if $0 < \epsilon \leq \epsilon_0$, then $\|g_{n\epsilon} - g_n\|_s < \gamma$ for all $n = 1, 2, \cdots$. Notice that for all $n$,

$$\|g_{n\epsilon} - g_\epsilon\|_s^2 \leq \|g_n - g\|_s^2. \tag{3.45}$$

So, choose $N$ so large that if $n \geq N$, then $\|g_n - g\|_s \leq \frac{1}{3}\gamma$. Then choose $\epsilon_0$ so small that $\|g_{k\epsilon} - g_k\|_s < \frac{1}{3}\gamma$ for $1 \leq k \leq N$ and $\|g_\epsilon - g\|_s < \frac{1}{3}\gamma$ for $\epsilon$ in $(0, \epsilon_0]$. Then certainly

$$\|g_{n\epsilon} - g_n\|_s < \gamma$$

for $1 \leq n \leq N$. If $n > N$, then by using (3.45),

$$\|g_{n\epsilon} - g_n\|_s \leq \|g_{n\epsilon} - g_\epsilon\|_s + \|g_\epsilon - g\|_s + \|g - g_n\|_s$$
$$\leq \|g_n - g\|_s + \|g_\epsilon - g\|_s + \|g - g_n\|_s \leq \gamma$$

The proposition is therefore proved. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Corollary 3.3.9.** *Let $u_\epsilon$ be the solution of problem $P_\epsilon$, where $\epsilon$ is in $(0, 1]$ and $g \in H^s$ with $s \geq 3$. Then for each $T > 0$, and for $m \geq 0$, $\epsilon^{\frac{m}{6}} u_\epsilon$ is bounded in $C(0, T; H^{s+m})$ independently of $\epsilon$ sufficiently small.*

*Proof.* We know from our "bounds" in Proposition 3.3.4 that, for $T > 0$ given,

$$\|u_\epsilon\|_{C(0,T;H^s)} \leq C(T, \epsilon_0, \|g_\epsilon\|_{H^s}, \epsilon^{\frac{1}{2}}\|g_\epsilon\|_{H^{s+1}}),$$

so $m = 0$ is straightforward, since both $\|g_\epsilon\|_{H^s} \leq \|g\|_s$ and $\epsilon^{\frac{1}{2}}\|g_\epsilon\|_{H^{s+1}} \leq C\epsilon^{\frac{1}{3}}\|g\|_s$ are bounded. For $m \geq 1$, one needs to check that

$$\|\epsilon^{\frac{m}{6}} u_\epsilon\|_{H^{s+m}} \leq C(T, \epsilon_0, \epsilon^{\frac{m}{6}}\|g_\epsilon\|_{H^{s+m}}, \epsilon^{\frac{m}{6}+\frac{1}{2}}\|g_\epsilon\|_{H^{s+m+1}}), \tag{3.46}$$

and then the conclusion follows by using Proposition 3.3.7.

To derive the inequality (3.46), we use $s = 3$ as a model case.

Multiply the regularized KdV equation by $u_{\epsilon(8)}$, use

$$(u_\epsilon{}^2)_{(5)} = 20u_{\epsilon xx}u_{\epsilon xxx} + 10u_{\epsilon x}u_{\epsilon(4)} + 2u_\epsilon u_{\epsilon(5)}$$

and integration by parts, and use Proposition ??

$$\frac{d}{dt}\int_{\mathbb{R}}(u_{\epsilon(4)}^2 + \epsilon u_{\epsilon(5)}^2)\,dx = \int_{\mathbb{R}}(u_\epsilon^2)_{(5)}u_{\epsilon(4)}\,dx$$

$$\leq C\left(\int_{\mathbb{R}}u_{\epsilon x}u_{\epsilon(4)}^2\,dx + \int_{\mathbb{R}}u_{\epsilon xx}u_{\epsilon(3)}u_{\epsilon(4)}\,dx\right)$$

$$\leq C(\|u_\epsilon\|_\infty\|u_{\epsilon(4)}\|_{L_2}^2 + \|u_{\epsilon xx}\|_\infty\|u_{\epsilon xxx}\|_{L^2}\|u_{\epsilon(4)}\|_{L^2}$$

$$\leq C(\|g\|_{H^3})\int_{\mathbb{R}}u_{\epsilon(4)}^2\,dx + c(\|g\|_{H^3})\sqrt{\int_{\mathbb{R}}u_{\epsilon(4)}^2\,dx}$$

$$\leq c(\|g\|_{H^3})\int_{\mathbb{R}}u_{\epsilon(4)}^2\,dx + c(\|g\|_{H^3})$$

Multiply above by $\epsilon^{\frac{1}{3}}$

$$\frac{d}{dt}\int_{\mathbb{R}}(\epsilon^{\frac{1}{3}}u_{\epsilon(4)}^2 + \epsilon^{1+\frac{1}{3}}u_{\epsilon(5)}^2)\,dx \leq c_1\int_{\mathbb{R}}\epsilon^{\frac{1}{3}}u_{\epsilon(4)}^2\,dx + c_2\epsilon^{\frac{1}{3}}.$$

Applying Gronwall inequality leads to

$$\int_{\mathbb{R}}(\epsilon^{\frac{1}{3}}u_{\epsilon(4)}^2 + \epsilon^{1+\frac{1}{3}}u_{\epsilon(5)}^2)\,dx \leq e^{c_1t}\int_{\mathbb{R}}\left(\epsilon^{\frac{1}{3}}g_{(4)}^2 + \epsilon^{1+\frac{1}{3}}g_{(5)}^2\right)dx + \frac{c_2\epsilon^{\frac{1}{3}}}{c_1}(e^{c_1t} - 1)\,dx$$

Therefoer (3.46) is valid for $m = 1$.

(iii)

$$\frac{d}{dt}\int_{\mathbb{R}}\left(u_{\epsilon(5)}^2 + \epsilon u_{\epsilon(6)}^2\right)dx = \int_{\mathbb{R}}(u_\epsilon^2)_{(6)}u_{\epsilon(5)}\,dx$$

$$= \int_{\mathbb{R}}u_{\epsilon x}u_{\epsilon(5)}^2\,dx + \int_{\mathbb{R}}u_{\epsilon xx}u_{\epsilon(4)}u_{\epsilon(5)}\,dx + \int_{\mathbb{R}}u_{\epsilon xxx}^2u_{\epsilon(4)}u_{\epsilon(5)}\,dx$$

$$\leq c\int_{\mathbb{R}}u_{\epsilon(5)}^2\,dx + c\|u_{\epsilon(4)}\|\|u_{\epsilon(5)}\| + c\|u_{\epsilon(4)}\|\|u_{\epsilon(5)}\|$$

$$\frac{d}{dt}\int_{\mathbb{R}}\left(\epsilon^{\frac{2}{3}}u_{\epsilon(5)}^2 + \epsilon^{1+\frac{2}{3}}u_{\epsilon(6)}^2\right)dx \leq \int_{\mathbb{R}}\epsilon^{\frac{2}{3}}u_{\epsilon(5)}^2\,dx + c\epsilon^{\frac{1}{3}}\|u_{\epsilon(4)}\|\epsilon^{\frac{1}{3}}\|u_{\epsilon(5)}\|$$

So, $m = 2$ is proved.

In general, if we write

$$\frac{d}{dt}\left(u^2_{\epsilon\,(m)} + \epsilon u_{\epsilon\,(m+1)}\right)\,dx = \int_{\mathbb{R}} u^2_{\epsilon\,(m+1)} u_{\epsilon(m)}\,dx$$

$$\leq C(\|g\|_{H^2})\int_{-\infty}^{\infty} u^2_{\epsilon\,(m)}\,dx + D(\|g\|_{H^2})\|u_{\epsilon(m-1)}\|\|u_{\epsilon(m)}\| + E(\|g\|_{H^3})$$

$$\leq C(\|g\|)_{H^3}\int_{-\infty}^{\infty} u^2_{(m)}\,dx + D(\|g\|_{H^3})\|u_\epsilon\|^2_{(m)}$$

Doing this for $m = m + s$, we get:

$$\frac{d}{dt}\int_{\mathbb{R}}\left(u^2_{(m+s)} + \epsilon u^2_{(m+s+1)}\right)\,dx \leq c\int_{\mathbb{R}} u^2_{(m+s)}\,dx + D\|u_{(m+s+1)}\|^2 \qquad (3.47)$$

Multiply (3.47) above by $\epsilon^{\frac{m}{3}}$,

$$\frac{d}{dt}\int_{\mathbb{R}}\left(\epsilon^{\frac{m}{3}}u^2_{(m+s)} + \epsilon^{1+\frac{3}{m}}u^2_{(m+s+1)}\right)\,dx \leq C\int_{\mathbb{R}}\epsilon^{\frac{m}{3}}u^2_{(m+s)}\,dx + D\epsilon^{\frac{m}{3}}\|u\|^2_{(m+s+1)}$$

$$(3.48)$$

Notice that the D-term is bounded even tends to 0 as $\epsilon \downarrow 0$, by induction, say.

Gronwall can now finish it off.                    $\square$

**Corollary 3.3.10.** $\partial_t u_\epsilon$ *is bounded in* $C(0, T; H^{s-3})$ *and* $\epsilon^{\frac{m}{6}}\partial_x^{s+m-3}\partial_t u_\epsilon$ *is bounded in* $C(0, T; H^0)$, *independently of* $\epsilon$ *sufficiently small, and* $m = 1, 2, 3, 4, 5$.

*Proof.* Since

$$u_{\epsilon t} = (1 - \epsilon\partial_x^2)^{-1}(-u_\epsilon u_{\epsilon x} - u_{\epsilon xxx}).$$

Hence,

$$\|u_{\epsilon t}\|_{s-3} \leq \|u_\epsilon\|_{s-3}\|u_\epsilon\|_{s-2} + \|u_\epsilon\|_s$$
$$\leq \|u_\epsilon\|^2_s + \|u_\epsilon\|_s \leq C.$$

Similarly,

$$\epsilon^{\frac{m}{6}}\|\partial_x^{s+m-3}\partial_t u_\epsilon\| \leq \epsilon^{\frac{m}{6}}\|\partial_t u_\epsilon\|_{s+m-3}$$
$$\leq \epsilon^{\frac{m}{6}}\left(\|u_\epsilon\|_{s+m-3}\|u_\epsilon\|_{s+m-2} + \|u_\epsilon\|_{s+m}\right)$$
$$\leq \epsilon^{\frac{m}{6}}\|u_\epsilon\|_{s+m} + \left(\epsilon^{\frac{m-3}{6}}\|u_\epsilon\|_{s+m-3}\|u_\epsilon\|_{s+m-2}\epsilon^{\frac{m-2}{6}}\right)\epsilon^{\frac{m-(m-3+m-2)}{6}}$$
$$\leq C + C\epsilon^{\frac{5-m}{6}} \leq C$$

if $m \leq 5$.                    $\square$

**Proposition 3.3.11.** *Let $\{u_\epsilon\}$ be the solutions of $(P_\epsilon)$, then $\{u_\epsilon\}$ is Cauchy in $C(0, T; H^s)$, for $g \in H^s$ with $s \geq 3$.*

*Proof.* Let $u = u_\epsilon$ and $v = u_\delta$, where $\delta \leq \epsilon$ say. It is enough to show that we can make $\|u - v\|_{H^s}$ as small as we like, independently of $t \in [0, T]$, by choosing $\epsilon$ small enough. Let $w = u - v$, then

$$w_t + (uw + \frac{1}{2}w^2)_x + w_{xxx} - \delta w_{xxt} = (\epsilon - \delta)u_{xxt} \qquad (3.49)$$

with $w(x, 0) = g_\epsilon(x) - g_\delta(x) = h(x)$, say. Multiply (3.49) by $w_{(2j)}$ and integrate over $\mathbb{R}$ with respect to $x$, and play, to get

$$\int_{\mathbb{R}} (w_{(j)}^2 + \delta w_{(j+1)}^2) \, dx = \int_{\mathbb{R}} (h_{(j)}^2 + \delta h_{(j+1)}^2) \, dx$$
$$- 2 \int_0^t \int_{\mathbb{R}} \left( (uw + \frac{1}{2}w^2)_{(j+1)} - (\epsilon - \delta)u_{t,(j+2)} \right) w_{(j)} \, dx \, d\tau. \qquad (3.50)$$

Denote $V_j^2(t) = \int_{\mathbb{R}} (w_{(j)}^2 + \delta w_{(j+1)}^2) \, dx$, let's just write out the details for $s = 3$. First for $j = 0$, the master relation looks like

$$\int_{\mathbb{R}} (w^2 + \delta w_x^2) \, dx$$
$$= \int_{\mathbb{R}} (h^2 + \delta h_x^2) \, dx - 2 \int_0^t \int_{\mathbb{R}} ((uw)_x + ww_x - (\epsilon - \delta)u_{xxt}) \, w \, dx d\tau$$
$$\leq \int_{\mathbb{R}} (h^2 + \delta h_x^2) \, dx + 2 \int_0^t (\|w_x + \frac{1}{2}u_x\|_\infty \int_{\mathbb{R}} w^2 \, dx + \epsilon \|u_x xt\|_{L^2} \|w\|_{L^2}) d\tau. \qquad (3.51)$$

From Corolarry 3.3.10, $(\|w_x + \frac{1}{2}u_x\|_\infty \leq C(\|g\|_3)$ and $\epsilon^{\frac{1}{3}}\|u_x xt\|_{L^2} \leq c(\|g\|_3)$, Then,

$$V_0^2(t) \leq V_0^2(0) + 2 \int_0^t [c_1 V_0^2(\tau) + c_2 \epsilon^{\frac{2}{3}} V_0(\tau)] \, d\tau.$$

Since $V_0'(t) \leq c_1 V_0(t) + c_2 \epsilon^{\frac{2}{3}}$, applying Gronwall inequality,

$$\|w\| \leq V_0(t) \leq V_0(0)e^{c_1 T} + \frac{\epsilon^{\frac{2}{3}} c_2}{c_1}(e^{c_1 T} - 1),$$

where

$$V_0(0) = \left[ \int_{\mathbb{R}} (g_\delta - g_\epsilon)^2 + \delta(g'_\delta - g'_\epsilon)^2 dx \right]^{\frac{1}{2}}$$

$$\leq \|g - g_\delta\|_{H^1} + \|g - g_\epsilon\|_{H^1}$$

$$\leq C\epsilon^{\frac{1}{3}} \qquad \text{for} \quad \epsilon \leq 1.$$

Hence $\{u_\epsilon\}$ is indeed Cauchy in $C(0, T; L^2)$ and we have the estimate

$$\|u_\epsilon - u_\delta\|_{L^2} \leq C\epsilon^{\frac{1}{3}} \qquad \text{for} \quad \delta \leq \epsilon \quad \text{and} \quad \epsilon \quad \text{sufficiently small.}$$

Next for $j = 1$, from (3.50), we derive

$$V_1^2(t) = V_1^2(0) - 2 \int_0^t \int_{\mathbb{R}} \left( \frac{1}{2} w_x^2 + \frac{3}{2} u_x \right) w_x^2 \, dx d\tau$$

$$-2 \int_0^t \int_{\mathbb{R}} [u_{xx} w - (\epsilon - \delta) u_{xxxt}] \, w_x \, dx d\tau,$$

where $|w_x|_\infty$, $|u_x|_\infty$, $|u_{xx}|_\infty$ and $\epsilon^{\frac{1}{2}} \|u_{xxxt}\|$ are all bounded, independently of $\epsilon$ sufficiently small. Hence,

$$V_1^2(t) \leq V_1^2(0) + 2 \int_0^t c_1 V_1^2(\tau) \, d\tau + 2 \int_0^t \left[ c_2 \|w\| V_1(\tau) + c_3 \epsilon^{\frac{1}{2}} V_1(\tau) \right] \, d\tau,$$

$$V_1(t) \leq V_1(0) e^{c_1 T} + C \frac{\epsilon^{\frac{1}{2}} + \|w\|_{C(0,T;L^2)}}{c_1} \left( e^{c_1 T} - 1 \right)$$

where

$$V_1(0) \leq \|g - g_\epsilon\|_1 + \|g - g_\delta\|_1 + \delta^{\frac{1}{2}} \|g - g_\epsilon\|_2 + \delta^{\frac{1}{2}} \|g - g_\delta\|_2$$

$$\leq C\epsilon^{\frac{1}{3}}.$$

Hence,

$$\|w_x\| \leq V_1(t) \leq C\epsilon^{\frac{1}{3}}. \tag{3.52}$$

Use the relation (3.50) for $j = 2$, we have:

$$\int_{\mathbb{R}} \left( w_{xx}^2 + \delta w_{xxx}^2 \right) \, dx$$

$$= \int_{\mathbb{R}} \left( h_{xx}^2 + \delta h_{xxx}^2 \right) \, dx - 2 \int_0^t \left( (uw + \frac{1}{2} w^2)_{xxx} w_{xx} - (\epsilon - \delta) u_{xxxxt} w_{xx} \right) \, dx \, d\tau$$

$$= 2 \int_0^t \int_{\mathbb{R}} \left( -\frac{5}{2} (u_x + w_x) w_{xx}^2 - 3 u_{xx} w_x w_{xx} - u_{xxx} w w_{xx} \right) \, dx \, d\tau$$

$$- (\epsilon - \delta) \int_0^t \int_{\mathbb{R}} u_{xxxxt} w_{xx} \, dx \, d\tau,$$

in which we know that, for $0 \leq t \leq T$, $|u_x||w_x| \leq C$, $|u_{xx}| \leq C$, $\|u_{xxx}\| \leq C$, $\|u_{xxxt}\| \leq \frac{C}{\epsilon^{\frac{2}{3}}}$, $\|w_x\| \leq C\epsilon^{\frac{1}{3}}$, $|w| \leq C\epsilon^{\frac{1}{3}}$, where $C's$ stand for constants dependent on $T$ and on $\|g\|_{H^3}$. Hence,

$$\int_{\mathbb{R}} \left(w_{xx}^2 + \epsilon w_{xxx}^2\right) dx \leq \int_{\mathbb{R}} (h_{xx}^2 + \delta h_{xxx}^2) dx + 2 \int_0^t \int_{\mathbb{R}} (Cw_{xx}^2 + C\epsilon^{\frac{1}{3}}|w_{xx}|) dx$$

$$+ 2\epsilon \int_0^t \|w_{xx}\| \|u_{xxxxt}\| dx$$

$$\leq \int_{\mathbb{R}} (h_{xx}^2 + \delta h_{xxx}^2) dx + 2 \int_0^t (C\|w_{xx}\|^2 + C\epsilon^{\frac{1}{3}}\|w_{xx}\|) dx,$$

where

$$V_2^2(0) = \int_0^t (h_{xx}^2 + \delta h_{xxx}^2) dx \leq C\epsilon^{\frac{1}{3}}.$$

Thus,

$$V_2^2(t) \leq C\epsilon^{\frac{1}{3}} + 2C \int_0^t [V_2^2(\tau) + \epsilon^{\frac{1}{3}} V_2(\tau)] d\tau,$$

from which it follows that

$$\|w_{xx}\| \leq V_2(t) \leq C\epsilon^{\frac{1}{6}} \qquad \text{for} \quad 0 \leq t \leq T. \tag{3.53}$$

To finish off, let's take $j = 3$ in the master relation (3.40), we have

$$V_3^2(t) = V_3^2(0) + 2 \int_0^t \int_{\mathbb{R}} \left( (uw + \frac{1}{2}w^2)_{xxxx} w_{xxx} - (\epsilon - \delta) u_{xxxxxt} w_{xxx} \right) dx \, d\tau$$

where,

$$\|u_{xxxxxt}\| \leq \frac{C}{\epsilon^{\frac{5}{6}}}.$$

Hence,

$$2(\epsilon - \delta) \int_0^t \int_{\mathbb{R}} u_{xxxxxt} w_{xxx} \, dx \leq 2\epsilon C\epsilon^{-\frac{5}{6}} \int_0^t \|w_{xxx}\| \, d\tau$$

$$\leq 2C\epsilon^{\frac{1}{6}} \int_0^t V_3(\tau) \, d\tau,$$

and the other term under the integral is estimated as following:

$$2 \int_0^t \int_{\mathbb{R}} [(uw + \frac{1}{2}w^2)_{xxxx} w_{xxx}] \, dx \, d\tau$$

$$= 2 \int_0^t \int_{\mathbb{R}} \left( \frac{7}{2}(u_x + w_x)w_{xxx}^2 - 4w_x u_{xxx} w_{xxx} - 6u_{xx} w_{xx} w_{xxx} - u_{xxxx} ww_{xxx} \right) dx \, d\tau$$

$$\leq 2C \int_0^t \left( \|w_{xxx}\|^2 + \|w_{xxx}\|(\epsilon^{\frac{1}{6}} + \epsilon^{\frac{1}{6}} + \epsilon^{-\frac{1}{6}}\epsilon^{\frac{1}{3}}) \right) d\tau$$

$$\leq 2C \int_0^t \left( V_3^2(\tau) + \epsilon^{\frac{1}{6}} V_3(\tau) \right) d\tau.$$

$$V_3^2(0) = \int_{-\infty}^{\infty} (h_{xxx}^2 + \delta h_{xxxx}^2) \, dx = o(1),$$

so it follows that

$$V_3^2(t) \leq o(1) + C \int_0^t (V_3^2(\tau) + \epsilon^{\frac{1}{6}} V(\tau) \, d\tau.$$

Applying the Gronwall's lemma,

$$\|w_{xxx}\| \leq V_3(t) \leq 0(1) \qquad \text{as } \epsilon \downarrow 0.$$

Adding $\|w\|, \|w_x\|, \|w_{xx}\|$ and $\|w_{xxxx}\|$ all up leads to

$$\|w\|_{H^3} = 0(1) \qquad \text{as } \epsilon \downarrow 0,$$

uniformly on $0 \leq t \leq T$.

**Corollary 3.3.12.** $\{\partial_t u_\epsilon\}_{\epsilon > 0}$ is Cauchy in $C(0,T;H^{s-3})$ as $\epsilon \to 0$.

**Theorem 3.3.13.** Let $g \in H^s, s \geq 3$. Then there exists a unique solution $u$ in $C(0,T;H^s)$, for all $T > 0$, to the KdV equation posed with initial data $g$.

*Proof.* Uniqueness is a simple Gronwall estimate. Existence is likewise easy. Let $\{u_\epsilon\}$ be associated solutions of problem $\{P_\epsilon\}$. Then there exists $u \in C(0,T;H^s)$, for each $T > 0$, such that

$$u_\epsilon \to u \qquad \text{in} \quad C(0,T;H^s),$$
$$\partial_t u_\epsilon \to v \qquad \text{in} \quad C(0,T;H^{s-3}),$$
$$\partial_x(u_\epsilon^2) \to \partial_x(u^2) \qquad \text{in} \quad C(0,T;H^{s-1}),$$
$$\partial_{xxx} u_\epsilon \to \partial_{xxx} u \qquad \text{in} \quad C(0,T;H^{s-3}),$$
$$\epsilon^{\frac{1}{3}} \partial_x^2 \partial_t u_\epsilon \quad \text{is bounded in} \quad C(0,T;H^{s-3}),$$
$$\text{so,} \qquad \epsilon \partial_x^2 \partial_t u_\epsilon \to 0 \qquad \text{in} \quad C(0,T;H^{s-3}).$$

We'd like to know that $v = u_t$. This is easily established. Let $\phi \in C_0^\infty(\mathbb{R} \times [0, T])$, then

$$\int_0^T \int_{\mathbb{R}} u_\epsilon \phi_t \, dx \, d\tau \to \int_0^T \int_{\mathbb{R}} u \phi_t \, dx \, d\tau \qquad \text{as } \epsilon \downarrow 0;$$

on the other hand,

$$\int_0^T \int_{\mathbb{R}} u_\epsilon \phi_t \, dx \, d\tau = -\int_{\mathbb{R}} \int_0^T (\partial_t u_\epsilon) \phi \, dx \, d\tau \to \int_0^T \int_{\mathbb{R}} v \phi \, dx \, d\tau \qquad \text{as } \epsilon \downarrow 0,$$

this shows that $u$ is weakly differentiable and that $u_t = v$. Since $v \in C(0, T; H^{s-3})$, we get that $u$ is strongly differentiable and that

$$u \in C^1(0, T; H^{s-3}).$$

This establishes the existence and uniqueness, and the continuous dependence follows using the fact that the 0 bounds were uniform on compacts. $\square$

**Remark 3.3.14.** $g \to C(0, T; H^k)$ *is sharp.*

We now proceed to make use of the method of proof used for existence of smooth solutions of KdV. Here we shall be interested in the regime that is of physical interest, namely small waves of long wave length.

Let's go back to dimensionless unscaled variables, where $u$ and its derivatives are all of order 1.

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{1}{6}\beta\eta_{xxt} = O(\alpha^2, \alpha\beta, \beta^2)$$
$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} = O(\alpha^2, \alpha\beta, \beta^2) \qquad (3.54)$$
$$\eta(x, 0) = g(x),$$

where the stokes number $S = \frac{\alpha}{\beta} = \frac{al^2}{h^3}$. Inherent in keeping both $\alpha$ and $\beta$ terms on an equal footing is that $S \backsim 1$.

Now, over what time scales do we expect the model to be valid? Consider

$$\eta_t + \eta_x = \epsilon, \quad \eta(x, 0) = g(x).$$

The solution of this equation is

$$\eta(x, t) = g(x - t) + \epsilon t.$$

Thus the long-term effect of a small perturbation is, in general, to grow linearly in time. Hence presuming $S \backsim 1$, the effect of the small nonlinear term and small dispersive term is to grow, over a time-scale of order $\frac{1}{\alpha} \backsim \frac{1}{\beta}$, to order 1. Thus these terms can have a significant effect on the shape of the wave profile on a time-scale of order $\frac{1}{\alpha}$. Equally, the neglected terms of order $\alpha^2 \backsim \beta^2$, can grow to an order-one contribution of a time scale of order $\frac{1}{\alpha^2} \backsim \frac{1}{\beta^2}$. Thus we have the following situation:

$t \backsim \frac{1}{\alpha}$, nonlinear and dispersive effects can affect the basic wave profile.

$t \backsim \frac{1}{\alpha^2}$, neglected can have accumulated to order of the basic wave, so the model may no longer be reliable.

Now let's consider the following pair of problems: take $\alpha = \beta$, $S = 1$.

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{1}{6}\beta\eta_{xxt} = 0,$$

$$\xi_t + \xi_x + \frac{3}{2}\alpha\xi\xi_x + \frac{1}{6}\beta\xi_{xxx} = 0, \qquad\qquad (3.55)$$

$$\eta(x,0) = \xi(x,0) = g(x), \qquad \text{an order one initial profile.}$$

Over what time scales are $\eta$ and $\xi$ close together. By close together, we shall mean that

$$|\eta(x,t) - \xi(x,t)| \le C\alpha.$$

This is the resolution of either $\eta$ or $\xi$, so this result would mean practically that we couldn't tell the two apart.

Let $\tilde{u}(x,t) = \frac{3}{2}\alpha\eta\left(\sqrt{\frac{\alpha}{6}}x, \sqrt{\frac{\alpha}{6}}t\right)$, and $\tilde{v}(x,t) = \frac{3}{2}\alpha\xi\left(\sqrt{\frac{\alpha}{6}}x, \sqrt{\frac{\alpha}{6}}t\right)$. Note that

$$g_\alpha = \alpha g(\alpha^{\frac{1}{2}}x),$$

Now, for $u$ and $v$, we have

$$\tilde{u}_t + \tilde{u}_x + \tilde{u}\tilde{u}_x - \tilde{u}_{xxt} = 0,$$

$$\tilde{v}_t + \tilde{v}_x + \tilde{v}\tilde{v}_x + \tilde{v}_{xxx} = 0, \qquad\qquad (3.56)$$

$$\tilde{u}(x,0) = \tilde{v}(x,0) = \alpha g(\alpha^{\frac{1}{2}}x)$$

and we want $|\tilde{u} - \tilde{v}| \le C\alpha^2$ as $\alpha \downarrow 0$.

Now the parameter is hidden in the initial data. Let's get out an appropriate magnifying glass, and follow the wave better. So let

$$u(x,t) = \alpha^{-1}\tilde{u}(\alpha^{-\frac{1}{2}}x + \alpha^{-\frac{3}{2}}t, \alpha^{-\frac{3}{2}}t),$$

and

$$v(x,t) = \alpha^{-1}\tilde{v}(\alpha^{-\frac{1}{2}}x + \alpha^{-\frac{3}{2}}t, \alpha^{-\frac{3}{2}}t),$$

then it is easy to verify that

$$\begin{cases} u_t + u_x + u_{xxx} - \alpha u_{xxt} = 0, \\ v_t + v_x + v_{xxx} = 0, \\ u(x,0) = v(x,0) = g(x). \end{cases}$$

This is just the problem we've just handled!! We know that, as $\alpha \downarrow 0$, $u \to v$, we need more precise information than that. Let $w = u - v$, so that $u = w + v$, then

$$\begin{cases} w_t + ww_x + w_{xxx} - \epsilon w_{xxt} = \alpha v_{xxt} - (vw)_x \\ w(x,0) = 0. \end{cases}$$

We may now proceed to estimate $\|\partial_x^j w\|_{L^2}$ for various values of $j = 0, 1, 2, \ldots$. In order to keep this sharp, we need to know more about solutions of KdeV.

Let $u$ be an $H^\infty$ solution of KdeV. It can be shown that the KdeV equation possesses an infinite collection of polynomial invariants. There are various ways to see this. There is a simple but _ad hoc_ method which can establish this. Let's just state the result, for now, and then come back to its proof later.

**Theorem 3.3.15.** _Let $g \in H^{k+s}$, where $k \geq 0$. Let $\alpha > 0$ and let $\eta^\alpha$ and $\xi^\alpha$ be the unique solutions of_

$$\eta_t + \eta_x + \alpha\eta\eta_x - \alpha\eta_{xxt} = 0,$$

_and_

$$\xi_t + \xi_x + \alpha\xi\xi_x + \alpha\xi_{xxx} = 0,$$

_respectively. Then there are order-one constants $C_j$ and $D_j$, $j = 0, 1, 2, \ldots$ such that_

$$\|\eta_{(j)}^\alpha - \xi_{(j)}^\alpha\|_{L^2} \leq C_j \alpha^{\frac{7}{4}+\frac{i}{2}}(\alpha^{\frac{3}{2}}t),$$

_and_

$$\|\eta_{(j)}^\alpha - \xi_{(j)}^\alpha\|_{L^\infty} \leq D_j \alpha^{2+\frac{i}{2}}(\alpha^{\frac{3}{2}}t)$$

_at least for $0 \leq t \leq \alpha^{-\frac{3}{2}}$. The constants $C_j$ and $D_j$ are not dependent on $\alpha$ for $\alpha$ in a bounded domain $[0, \alpha_0]$._

<u>*Conjecture*</u>: these bounds hold right up to $t \backsim \alpha^{-\frac{5}{2}}$, thus the models have diverged at the break-down time.

Numerical evidence supports this conjecture. But one must be careful in discounting the models beyond $t \backsim \alpha^{-\frac{5}{2}}$, for reasons that will appear later.

## 3.4   The Quarter-Plane Problem

The pure initial-value problem that has been treated is, in some aspects, not as useful as a model problem as an initial-value and boundary-value problem to be treated presently. The fact is that the pure initial value problem is not at all well-suited to comparison with initial data.

A convenient experimental set-up is as follows. In a channel, a disturbance is created at one end, which subsequently propagates down the channel. At a certain point $x_0$ in the channel the disturbance is recorded as it passes by. One wishes from this data, to predict what the wave will look like downstream.
This leads naturally to the initial and boundary value problems

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{1}{6}\beta\eta_{xxt} = 0,$$

$$\xi_t + \xi_x + \frac{3}{2}\alpha\xi\xi_x + \frac{1}{6}\beta\xi_{xxx} = 0,$$

with

$$\eta(x,0) = \xi(x,0) = f(x), \qquad \text{for} \quad x \geq x_0,$$
$$\xi(x_0,t) = \eta(x_0,t) = g(t), \qquad \text{for} \quad x \geq x_0.$$

Without loss of generosity we take $x_0 = 0$, and we scale out the $\frac{1}{6}\beta$ and the $\frac{3}{2}\alpha$. Then we have

$$
\begin{aligned}
& u_t + u_x + uu_x - u_{xxt} = 0, \\
& u(x,0) = f(x), \qquad u(0,t) = g(t), \qquad \text{for} \quad x \geq x_0, t \geq 0,
\end{aligned}
\tag{3.57}
$$

where we require $f(0) = g(0)$ for consistency. This the problem posed in a quarter plane. Then the question is: is it a well-posed problem?
The answer is "yes" as we now show. The method is the same as for the initial-value problem, so we shall just outline the situation.
Note that we still have $H^1(\mathbb{R}^+) \hookrightarrow C_b(\mathbb{R})$ with $\|f\|_{H^1} \geq \|f\|_{C_b}$.

*Proof.* For

$$f^2(x) = -2 \int_x^\infty f(\xi) f'(\xi)\, d\xi \leq \int_0^\infty [f^2(\xi) + f'^2(\xi)]\, d\xi = \|f\|_{H^1}^2.$$

Hence,

$$\|f\|_\infty \leq \|f\|_{H^1}.$$

Let $g$ be the function on $\mathbb{R}$ obtained by reflecting $f$ about the origin, then $g \in H^1(\mathbb{R})$, so $g$ is continuous and asymptotically null at $\pm\infty$. Thus $f$ is continuous a.e and $f \to 0$ at $+\infty$, and the above bound hold. $\square$

Let's now convert the differential equation to an integral equation, as for the pure initial value problem,

$$(1 - \partial_x^2) u_t = -\partial_x (u + \frac{1}{2} u^2).$$

Again, regard this as an ordinary differential equation for $u_t$, we can solve for $u_t$ as:

$$u_t(x,t) = -\frac{1}{2} \int_0^\infty e^{-|x-\xi|} \partial_\xi (u + \frac{1}{2} u^2)\, d\xi + \frac{1}{2} \int_0^\infty e^{-(x+\xi)} \partial_\xi (u + \frac{1}{2} u^2)\, d\xi + g'(t) e^{-x},$$

$$(3.58)$$

using the fact that $u_t(0,t) = g'(t)$, a formal integration by parts, followed by integration over $[0,t]$ yields

$$u(x,t) = f(x) + (g(t) - g(0)) e^{-x} + \int_0^t \int_0^\infty K(x,\xi)(u(\xi,\tau) + \frac{1}{2} u^2(\xi,\tau))\, d\xi\, d\tau,$$

$$(3.59)$$

where $K(x,\xi) = \frac{1}{2} sgn(x - \xi) e^{-|x-\xi|} + \frac{1}{2} e^{-(x+\xi)}$.

**Lemma 3.4.1.** *Let $f \in C_b(\mathbb{R}^+)$ and $g \in C(0,T)$. then there exists $S$ with $0 < S \leq T$, dependent on $\|f\|_{C_b}$ and $\|g\|_{C(0,T)}$ such that there is a solution of the integral equation.*

*Proof.* Write (3.59) as $u = Au = g(x) + e^{-x}(h(t) - h(0)) + B(u)$ say. View this as a mapping of the space $C(0, S; C_b(\mathbb{R}^+))$ into itself. We argue that, by thinking $R$ large and $S$ small, $A$ is a contraction mapping of $\mathcal{B}_R(0) \subset C(0, S; C_b(\mathbb{R}^+))$ into itself. The crucial estimate is

$$\|Au - Av\|_{C(0,S;C_b)} = \|Bu - Bv\|_{C(0,S;C_b)}.$$

$$\leq S \left(1 + \frac{1}{2} [\|u\|_{C(0,S;C_b)} + \|v\|_{C(0,S;C_b)}]\right) \|u - v\|_{C(C_b)}$$

This is the fact that $\sup_{x \geq 0} \int_0^\infty |K(x, \xi)| \, d\xi = 1$. From this there follows the estimate:

$$\|Au\|_{C(C_b)} \leq \|f\|_{C_b} + 2\|g\|_{C(0,T)} + S\|u\|_C(1 + \frac{1}{2}\|u\|_C).$$

Since

$$\|Au\|_C \leq \|Au - A(0)\|_C + \|A(0)\|_C$$

and now proceed as before, we get, for $u, v \in B_R(0)$,

$$\|Au\|_C \leq a + SR(1 + R),$$

$$\|Au - Av\|_C \leq S(1 + R)\|u - v\|_C.$$

Choose $S(1 + R) = \frac{1}{2}$ or $S + \frac{1}{2(1+R)}$, then choose $R = 2a$. The lemma is proved. $\square$

**Theorem 3.4.2.** *Suppose $f \in C_b^2(\mathbb{R}^+)$ and $g \in C^1(0, T)$, then any solution of the integral solution (3.59) has*

$$\partial_t^i \partial_x^j u \in C(0, T; C_b) \qquad \text{for } 0 \leq i \leq 1, 0 \leq j \leq 2,$$

*moreover, $u$ is a classical solution of the problem (3.57).*

*Proof.* More or less the same procedure as before for the pure initial-value problem can derive the result. $\square$

**Corollary 3.4.3.** *If $f \in C_b^l(\mathbb{R}^+)$ and $g \in C^k(0, T)$, $k \geq 1, l \geq 2$, then any solution $u$ of the integral equation in $C(0, T; C_b)$ has*

$$\partial_t^i \partial_x^j u \in C(0, T; C_b(\mathbb{R}^+)) \qquad \text{for } 0 \leq i \leq k, 0 \leq j \leq l.$$

**Remark 3.4.4.** *There is at most one bounded continuous solution of this integral equation. For if we had two solutions $u, v \in C(0, T; C_b)$, then by choosing $R$ large enough, both $u, v \in B_R(0) \subset C(0, T; C_b)$, then, without loss of generosity, we can assume $v \neq u$ on any interval $[0, t]$ for $t \geq 0$. But then, for a sufficiently small interval, $A$ is contractive in $B_R(0)$, so $u = v$ in a small interval, a contradiction.*

**Lemma 3.4.5.** *If $f \in C_b^l$ and $g \in C^k(0, T)$, $l \geq 2$, $k \geq 1$ and suppose $f, f', \ldots f^{(p)}$ are null at $+\infty$ for some $p \geq 0$, but $p \geq l$, then if $u \in C(0, T; C_b)$ is the solution of the integral equation (3.59), then $\partial_t^i \partial_x^j u$ is null at $+\infty$, for $0 \leq i \leq k, 0 \leq j \leq p$, uniformly for $0 \leq t \leq T$.*

*Proof.* as same as before. □

**Lemma 3.4.6.** *If $f \in C_b^2(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$ and $g \in C^1(0,T)$, then the classical solution of the initial-value problem that exists on $[0,S]$ for $S$ small enough satisfies an estimate of the form:*

$$\int_0^\infty [u^2(x,t) + u_x^2(x,t)]\, dx + \frac{1}{2}\int_0^t u_{xt}^2(0,\tau)\, d\tau \le \int_0^\infty [f^2(x) + f_x^2(x)]\, dx + C(t),$$

*where $C(t)$ depends only on $g$ and $g'$ on $[0,T]$.*

*Proof.* Multiply (3.57) by $u$ and integrate over $\mathbb{R}^+$ with respect to spatial variable $x$ :

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty u^2 - \frac{1}{2}g^2(t) - \frac{1}{3}g^3(t) + g(t)u_{xt}(0,t) + \frac{1}{2}\frac{d}{dt}\int_0^\infty u_x^2 = 0,$$

or

$$\frac{d}{dt}\int_0^\infty (u^2 + u_x^2)\, dx = g^2(t) + g^3(t) - g(t)u_{xt}(0,t), \qquad (3.60)$$

the term $u_{xt}(0,t)$ is not nice, so we need to work on it. Multiply (3.57) by $u^2$ and integrate over $\mathbb{R}^+$ :

$$\frac{1}{3}\int_0^\infty u^3(x,t)\, dx - \frac{1}{3}g^3(t) - \frac{1}{4}g^4(t) + g^2(t)u_{xt}(0,t) + 2\int_0^\infty uu_xu_{xt} = 0, \quad (3.61)$$

that looks less then useful. Multiply (3.57) by $u_{xt}$ and integrate over $\mathbb{R}^+$ :

$$-\frac{1}{2}[g'(t)]^2 + \frac{1}{2}\frac{d}{dt}\int_0^\infty u_x^2 + \int_0^\infty uu_xu_{xt} + \frac{1}{2}u_{xt}^2(0,t) = 0,$$

or

$$\frac{d}{dt}\int_0^\infty u_x^2 = -2\int_0^\infty uu_xu_{xt} + g'(t)^2 - u_{xt}^2(0,t). \qquad (3.62)$$

Now combine (**??**) through (3.62) by (**??**)-(3.61)+(3.62) to come to

$$\frac{d}{dt}\int_0^\infty u^2 + 2u_x^2 - \frac{1}{3}u^3$$

$$= g'(t)^2 - u_{xt}^2(0,t) - \frac{1}{3}g^3(t) - \frac{1}{4}g^4(t) + g^2(t)u_{xt}(0,t) \qquad (3.63)$$

$$+ g^2(t) + \frac{2}{3}g^3(t) - g(t)u_{xt}(0,t).$$

Integrate (3.63) up in time [0,t]:

$$\int_0^\infty u^2 + 2u_x^2 = \frac{1}{3}\int_0^\infty u^3 + \int_0^\infty f^2 + 2f_x^2 - \frac{1}{3}f^3 + \int_0^t (\partial - values)\, d\tau$$

$$\leq V + \frac{1}{3}\|u\|_{H^1}^3 + \int_0^t (\partial - values)\, d\tau,$$

where $V = \int_0^\infty f^2 + 2f_x^2 - \frac{1}{3}f^3$ is bounded since $f$ starts life in $H^1$.

Integrate (??) in time over $[0,t]$ :

$$\int_0^\infty u^2 + u_x^2 = \int_0^\infty f^2 + f_x^2 + \int_0^t [g^2 + \frac{2}{3}g^3 + gu_{xt}]\, d\tau.$$

Hence, *

$$\int_0^\infty u^2 + 2u_x^2$$

$$\leq V + \frac{1}{3}(W + \int_0^t [\partial - value)\, d\tau)^{\frac{3}{2}} + \int_0^t (\partial - value)\, d\tau$$

$$\leq V + (W + \int_0^t [g^2(\tau) + \frac{2}{3}g^3(\tau)]\, d\tau)^{\frac{3}{2}} + (\int_0^t |g(\tau)u_{xt}(0,\tau)|\, d\tau)^{\frac{3}{2}}$$

$$+ \int_0^t [g'(\tau)^2 + \frac{1}{3}g^3(\tau) - \frac{1}{4}g^4(\tau)]\, d\tau + \int_0^t [g^2(\tau) - g(\tau))u_{xt})(0,t\tau) - u_{xt}^2(0,\tau)]\, d\tau$$

$$\leq V + c_1(t) + c_2(t) + (\int_0^t g^2(\tau)\, d\tau)^{\frac{3}{4}}(\int_0^t u_{xt}^2(0,\tau)\, d\tau)^{\frac{3}{4}}$$

$$+ \int_0^t (g^2(\tau) - g(\tau))^2\, d\tau + \frac{1}{4}\int_0^t u_{xt}^2(0,\tau)\, d\tau - \iint_0^t u_{xt}^2(0,\tau)\, d\tau$$

$$\leq V + c_1(t) + c_2(t) + c_3(t) + \frac{1}{4}\int_0^t u_{xt}^2\, d\tau + c(\int_0^t g^2(\tau)\, d\tau)^3 - \frac{3}{4}\int_0^t u_{xt}^2(0,\tau 0\, d\tau,$$

$$\int_{\mathbb{R}^+} u^2 + u_x^2 \leq V + C(t) - \frac{1}{2}\int_0^t u_{xt}^2(0,t)\, d\tau,$$

where $C(t)$ depends only on $g$ and $g'$ over $[0,t]$.                    □

This $H^1$ bound is enough to pass to a global solution by way of iterating the contraction mapping argument. The results of continuous dependence may now be derived much as before, as well as further regularity results. We

won't go into this details now.

As far as the same initial-boundary-value problem for KdV, the results are less conclusive. First note that, formally, the solution is uniquely specified by $f$ and $g$, for if $u$ and $v$ are two solutions and $w = u - v$, then

$$\begin{cases} w_t + w_x + \dfrac{1}{2}[(u+v)w]_x + w_{xxx} = 0, \\ w(x,0) = 0, w(0,t) = 0. \end{cases} \qquad (3.64)$$

Multiply (3.64) by $w$ and integrate over $\mathbb{R}$,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}+} &+ \frac{1}{2} w^2\big|_0^\infty + (u+v)w^2\big|_0^\infty + ww_{xx}\big|_0^\infty - \int_{\mathbb{R}+} w_x w_{xx} \\ &= \int_{\mathbb{R}+} (u+v)ww_x \qquad (3.65) \\ &= \frac{1}{2}(u+v)w^2\big|_0^\infty - \frac{1}{2}\int_{\mathbb{R}+}(u_x+v_x)w^2, \end{aligned}$$

or

$$\frac{d}{dt}\int_{\mathbb{R}+} w^2(x,t)\,dx + w_x^2(0,t) \leq |u_x + v_x|_\infty \int_{\mathbb{R}+} w^2.$$

Assuming $u_x$ and $v_x$ are bounded, then Gronwall lemma finishes it off. N.B. the sign of the term $u_{xxx}$ was important here.

**Remark 3.4.7.** *Look at the equation*

$$\begin{cases} u_t + u_x - u_{xxx} = 0, \\ u(x,0) = f(x), u(0,t) = g(t), \end{cases}$$

*just a change of sign in front of $u_{xxx}$ term in linearized KdV equation, but the uniqueness of solution is no more true, even when requiring the solution to decay at $\pm\infty$. Eg. take the Laplace transform in t, with 0 initial and 0 boundary condition, and denote $v = \mathcal{L}u$,*

$$sv + v_x - v_{xxx} = 0,$$

*or*

$$v''' - v' - sv = 0, v(0,s) = \sqcap(x=0) = 0,$$

*where $s$ is a parameter of course, $s \geq 0$. To solve this constant coefficient problem, find the character equation:*

$$r^3 - r - s = 0,$$

*let $r, r_1, r_2$ be its roots, then the solution, in general, is*

$$a(s)e^{rx} + b(s)e^{r_1 x} + c(s)e^{r_2 x},$$

*according to the property of this cubic equation, one root is positive, $r \geq 0$, say, other two are non-positive, $r_1 \leq 0, r_2 \leq 0$, but not both 0.*

Suppose that the solution $a(s)e^{rx} + b(s)e^{r_1 x} + c(s)e^{r_2 x}$ decays at $x = +\infty$, then $a(s) = 0$. The condition $v(0, s) = 0$ means that $b(s) + c(s) = 0$, or $b(s) = -c(s)$, hence a non-zero solution is $b(s)(e^{r_1 x} - e^{r_2 x})$ with $r_1 = r_1(s) < 0$, $r_2 = r_2(s) < 0$, for $s$ small anyway. So let $b(s)$ be concentrated near $s = 0$, with compact support in $[0, s_0]$, then we have a perfectly good non-trivial solution, where inverse Laplace transform is the desired non-trivial solution of the linear problem. It is easy to derive that: 1) there is a family of strong solutions in $C(0, T; H^3(\mathbb{R}^+))$, 2) there is a global weak solution in $L^\infty(0, T; H^1(\mathbb{R}^+))$, and there it stands.

**Remark 3.4.8.** *Continuous dependence is different from stability.*

- *For continuous dependence: Given $T > 0$, given $\epsilon > 0$, there is a $\delta$, such that if $\|g - h\| \leq \delta$, then $\|u - v\| \leq \epsilon$ for $0 \leq t \leq T$;*

- *A solution $u$ is stable means that for given $\epsilon > 0$, there is a $\delta$, such that if $\|g - h\| \leq \delta$, then $\|u - v\| \leq \epsilon$ for any $t$.*

*In general, a solitary wave*

$$\phi_c(x, t) = 3c \, sech^2(\sqrt{\frac{c}{1 + c}}(x - (c + 1)t))$$

*is not stable in the above sense because ...*

# Chapter 4

# Solitary waves, their existence, stability and instability

The solitary wave, a special type of traveling waves, was first observed by John Scott Russsel in 1844 on a surface of canal. Airy and Stokes began to study this phenomenon, they derived governing equations

$$\eta_t + \eta_x + \epsilon\eta\eta_x = 0$$

and

$$\eta_t + \eta_x + \epsilon\eta_{xxx} = 0$$

respectively in 1845 and 1849, but none of them has solitary wave. Boussinesq (1871, 1872, 1877) and Rayleigh (1876) began with two dimensional Euler equation, and Korteweg and DeVries (1895) derived the KdV equation

$$\eta_t + \eta_x + \epsilon\eta\eta_x + \eta_{xxx} = 0,$$

which does have solitary wave. More general model equations can be written as

$$\begin{cases} u_t + f(u)_x + Lu_x = 0, \\ u(x,0) = g(x), \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is some nonlinear function in $C^\infty(\mathbb{R})$ space with $f(0) = 0$ and $L$ is an operator defined by Fourier symbol as $\widehat{Lu}(\xi) = \alpha(\xi)\hat{u}(\xi)$. The general natural questions to ask are 1) is there solitary wave? 2) is the solitary wave solution stable? 3) is the initial value problem well posed.

Experimental results show that the solitary wave is a very stable form, so we turn first to the question of the stability of the solitary-wave solutions.

## 4.1   Exact traveling wave solutions

We first consider the solutions of permanent form for the KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

We are looking for solutions of the form

$$u(x, t) = \phi(\xi) = \phi(x - ct)$$

where $c$ is a real constant, representing the speed of the traveling wave. Substitude the form and one obtains an ODE

$$-c\phi' + \phi\phi' + \phi''' = 0$$

Integrating once,

$$\phi'' = -\frac{1}{2}\phi^2 + c\phi + A$$

For fixed $A$ and $c$, the phase diagram of this equation can be used to get a general idea of the solutions. Let $x = \phi$ and $y = \phi'$, then

$$x' = y$$

$$y' = -\frac{1}{2}x^2 + cx + A$$

## 4.2   Stability of the Solitary-waves

Consider the model equation

$$\begin{aligned} u_t + f(u)_x - (Lu)_x &= 0, \\ u(x, 0) &= \psi(x), \end{aligned} \tag{4.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is some nonlinear function in $C^\infty(\mathbb{R})$ space with $f(0) = 0$ and $L$ is an operator defined by Fourier symbol as $\widehat{Lu}(\xi) = \alpha(\xi)\hat{u}(\xi)$. Suppose $\phi = \phi_C = \phi(x - Ct)$ is a solitary wave solution, then it satisfies the ordinary differentiable equation

$$-C\phi' + (f(\phi))' - (L\phi)' = 0,$$

or

$$(C + L)\phi = f(\phi),$$

where $C \geq 0$ is the velocity of wave propagation.

To establish stability, we would like to have the definition first. Naturally, if the initial data $\psi$ of (4.1) is closed to the solitary wave $\phi$, then we expect that the solution of (4.1) $u$ is closed to $\phi$ for all time, i.e. for any small $\epsilon > 0$ given, there is always $\delta > 0$ such that if $\|\psi - \phi\| < \delta$ then $\|u(\cdot, t) - \phi_C(\cdot - Ct)\| \leq \epsilon$ for all time. However, we cannot expect a result like this, because the speeds of propagation of $\phi_C$ and $u$ may be different. More precisely, suppose we let $\psi = \phi_D$ where $D \neq C$, then if $D \to C$, $\phi_D \to \phi_C$ in any of the norms we have been using. On the other hand we know $u$ explicitly as

$$u(x, t) = \phi_D(x - Dt),$$

hence,

$$\|u - \phi_C\| = \|\phi_D - \phi_C\|$$

is always a positive constant, no matter how close $D$ is to $C$, and for $t$ large enough ($t >> \frac{1}{|C-D|}$), $\phi_C(x-Ct)$ and $\phi_D(x-Dt)$ will have essentially disjoint supports and so the norm of the difference will not be small. In fact, for the kind of spatially homogeneous norms we have been using

$$\lim_{t \to \infty} \|\phi_C(x - Ct) - \phi_D(x - Dt)\| = \|\phi_C\| + \|\phi_D\|.$$

Thus the result we'd like is too strong.

Similar happens to asymptotic stability:

$$\|\psi - \phi_C\| << 1 \implies \text{ there exists } d \text{ near to } C s.t. \sup_{t \geq T} \|u - \phi_D\| << 1.$$

(???)

Two natural possible definitions of stability failed. Now let's try a stability in "shape" or "orbit" and just forget about speed altogether. This might lead us to define a new measure of distance, eg. let $f, g$ be elements in a Banach space $X = X(\mathbb{R})$, let

$$d(f, g) = \inf_{y \in \mathbb{R}} \|f(\cdot) - g(\cdot + y)\|_X.$$

then this "distance" is the closest approach of $f$ and $g$ under the translation group in $\mathbb{R}$, more precisely, $d$ is defined on the quotient space $X/\tau$, where $\tau$ is the translation group.

**Definition 4.2.1.** *A function $d : A \times A \to \mathbb{R}$ is a metric on $A$ if*

$$A1. \quad d(a, b) \geq 0$$
$$A2. \quad d(a, b) = 0 \text{ if and only if } a = b$$
$$A3. \quad d(a, b) = d(b, a)$$
$$A4. \quad d(a, b) \leq d(a, c) + d(c, b)$$

*The function $d$ is a pseudo-metric if it fulfils the same properties as a metric except relaxes the definition to allow the distance between two different points to be zero.*

**Lemma 4.2.2.** *The operator $d$ defined above is a pseudo-metric if the norm in Banach space $X = X(\mathbb{R})$ is invariant under translation.*

*Proof.* We are assuming that if $f \in X$ then $\tau_y f \in X$ and $\|f\| = \|\tau_y f\|$ for any $y \in X$, where $(\tau_x f)(y) = f(x + y)$.

Define $X/\tau$ or $[f] = \{g : g = \tau_y f \text{ for some } y \in \mathbb{R}\}$, let

$$d([f], [g]) = \inf_{y \in \mathbb{R}} \|f - \tau_y g\|,$$

then we claim
i) $d$ is symmetric since,

$$d([f], [g]) = \inf_{y \in \mathbb{R}} \|f - \tau_y g\| = \inf_{y \in \mathbb{R}} \|\tau_{-y}(f - \tau_y g)\|$$
$$= \inf_{y \in \mathbb{R}} \|\tau_{-y} f - g\| = \inf_{y \in \mathbb{R}} \|g - \tau_y f\| = d([g], [f]).$$

ii) $d$ has triangle inequality property since, for any $[f], [g], [h] \in X/\tau$,

$$\|f - \tau_y g\| \leq \|f - \tau_x\| + \|\tau_x h - \tau_y g\|,$$

and taking the inferior over $y$, we see that

$$d([f], [g]) \leq \|f - \tau_x h\| + d([\tau_x h], [g])$$
$$= \|f - \tau_x h\| + d([h], [g]),$$

taking the inferior over $x$, we have,

$$d([f], [g]) \leq d([f], [h]) + d([h], [g]).$$

Therefore, the lemma is proved.                              $\square$

This pseudo-metric measures the distance between functions in a broad meaning. Now the stability of solitary waves can be redefined.

**Definition 4.2.3.** *Solitary wave solution $\phi_C$ of (4.1) is said to be stable, if for any $\epsilon > 0$ given there is always $\delta > 0$ such that if $\|\psi - \phi_C\| < \delta$ then $\inf_{y \in \mathbb{R}} \|u(\cdot, t) - \phi_C(\cdot + y)\| < \epsilon$.*

**Lemma 4.2.4.** *Let $u$ be a "nice" solution of (4.1), then $\int_{-\infty}^{\infty} u(x)\, dx$, $V(u) = \int_{-\infty}^{\infty} u^2(x, t)\, dx$ and $M(u) = \int_{-\infty}^{\infty} [\frac{1}{2} u L u(x, t) - F(u)(x)]\, dx$ are time independent, where $F'(x) = f(x)$ and $F(0) = 0$.*

*Proof.* : We already checked this early by multiplying (4.1) by $u$ and $f(u) - Lu$ respectively then integrate over $\mathbb{R}$. $\qquad\square$

*Intermediate hypothesis:* For the time being, we assume $\phi$ is the fixed solitary wave solution of KdV and $\psi$ the perturbed initial data, and we assume additionally that $V(\phi) = V(\psi)$. This restriction will be removed later.

Define $\Lambda(u) = M(u) + CV(u)$, $h(x, t) = u(x + a, t) - \phi(x - Ct)$, where $C$ is the phase speed of the solitary wave we are interested in. Denote $\phi = \phi_C$, then $\phi$ satisfies equation:

$$(C + L)\phi = f(\phi). \tag{4.2}$$

Let $T(u) = \Lambda(u) - \Lambda(\phi)$, then $T : \psi \to \mathbb{R}$. Compute the difference:

$$\Lambda(u) - \Lambda(\phi) = \Lambda(\phi + h) - \Lambda(\phi) = \Lambda'(\phi)h + \frac{1}{2}\left(\Lambda''(\phi)h, h\right) + O(\|h\|^3),$$

or

$$\int \frac{1}{2}(\phi + h)L(\phi + h)dx - F(\phi + h) - \int \frac{1}{2}\phi L\phi dx + F(\phi)$$
$$+ \frac{C}{2}\int (\phi + h)^2 dx - \frac{C}{2}\int \phi^2 dx$$
$$= \int hL\phi dx + \int hLh dx - \int f(\phi)h dx + \frac{1}{2}f'(\phi)h^2 + O(\|h\|^3)$$
$$+ C\int \phi h dx + \frac{C}{2}\int h^2 dx$$

Let $\Lambda'(\phi) = 0$, then

$$m\|h\|^2 + a\|h\|^3 \geq \Lambda(u) - \Lambda(\phi) = \frac{1}{2}\left(\Lambda''(\phi)h, h\right) + o(\|h\|^3) \geq \lambda\|h\|^2 - b\|h\|^3.$$

If $\Lambda''$ is positive definite. then

$$\delta \geq m\|h\|^2 + a\|h\|^3 \geq \Lambda(u) - \Lambda(\phi) \geq \lambda\|h\|^2 - b\|h\|^3,$$

for all $t$.

As $\Lambda$ is constant in time, this means that if, at $t = 0$, $\|h\|_{H^1}$ is small, then $\Lambda$ will keep small, just by the choice of the initial small data $\psi$.

Define $\mathcal{L} = \mathcal{L}_C = \Lambda''(a) = L + C - f'(a)$, then $\mathcal{L}(\phi') = 0$ since $(L+C)\phi' - f'(\phi)\phi' = 0$, $\mathcal{L}$ has zero-eigenvalue with $\phi'$ as corresponding eigenfunction.

Consider the equation (**??**), denote $\dot\phi = \frac{d\phi}{dC}$, differentiate it with respect to $C$, we have

$$(L + C)\dot\phi - f'(\phi)\dot\phi + \phi = 0,$$

i.e.

$$\mathcal{L}(\dot\phi) = -\phi,$$

$$\left(\mathcal{L}(\dot\phi), \phi\right) = (\phi, \dot\phi) = -\frac{1}{2}\frac{d}{dC}(\phi, \phi).$$

Assumptions:

1. i)$\frac{d}{dC}(\phi, \phi) > 0$.

   **Remark 4.2.5.** *then $\mathcal{L}$ is self-adjoint unbounded operator on $L_2(\mathbb{R},)$ and $\left(\mathcal{L}(\dot\phi), \dot\phi\right) = -\frac{1}{2}\frac{d}{dC}(\phi, \phi)$.*

2. ii)spectra $(\mathcal{L}) = \{0, C > 0....\}$(See April 8 note)

3. iii) 0 is simple eigenvalue.

4. iv) $\alpha(\xi) \geq \mu|\xi|$ at least for $|\xi|$ large for some $\mu > 0$. There is only one negative eigenvalue, simple with eigenfunction $\mathcal{X}$.

Remember, $\Lambda(c) = M(\phi_C) + CV(\phi_C)$, so $\Lambda'(C) = (M' + CV')(\phi_C) + V(\phi_C) = V(\phi_C)$, and $\Lambda''(C) = \frac{d}{dC}V(\phi_C) = \frac{d}{dC}(\phi, \phi)$.

Denote $X = \left\{f \in L_2 : \int(1 + \alpha(\xi))|\hat{f}|^2\, d\xi < \infty\right\}$, then $L : X \to X^*$.

**Lemma 4.2.6.** *Suppose $\psi \in X$ is such that $0 = (\psi, \phi') = (\psi, \phi)$, let*
$\eta = \inf\{< \mathcal{L}f, f >: f \in X, (f, \phi) = (f, \phi') = 0, \|f\| = 1\}$. $\eta > 0$.

**Remark 4.2.7.** $\eta \leq 0$ *refers Lemma 5.1 of Bona et al. (1987)*

*Proof.* Step 1, suppose $\eta = 0$, then there exists a sequence $\{q_n\} \subset X$ such that $(q_n, \phi) = (q_n, \phi') = 0, (q_n, q_n) = 1$ for
$n \geq 1$ and $< \mathcal{L}q_n, q_n > \to 0$ as $n \to \infty$.

Without loss of generosity, take $q_n$ infinitely smooth, then

$$0 \leq (Lq_n, q_n) = \int_{-\infty}^{\infty} \alpha(\xi)\hat{q}_n(\xi)\hat{q}_n^*(\xi)\, d\xi = (\mathcal{L}q_n, q_n) - C(q_n, q_n) + \int_{-\infty}^{\infty} f'(\phi)q_n^2(x)\, dx,$$

$\{q_n\}$ is bounded in $X$, hence there exists a subsequence $\{q_{n_k}\}$, still called $\{q_n\}$ such that,

$$q_n \to q_* \text{ weakly in } X.$$

Recall assumption iv), $\alpha(\xi) > \mu|\xi|$ for some $\mu > 0$, so $X \subset H^{\frac{1}{2}}(\mathbb{R}) \subset L_p$, $\{q_n\}$ is bounded in $H^{\frac{1}{2}}(-M, M)$ for any finite number $M$ and hence there is a convergent subsequence of $\{q_n\}$ in $L_2(-M, M)$ and $L_4(-M, M)$. By taking a suitable subsequence, we can suppose that

$$q_n \to q_* \text{ weakly in } X \text{ and } q_n \to G \text{ pointwise a.e. in } \mathbb{R}.$$

To prove $q_* = G$, let $\rho \in \mathcal{D}(\mathbb{R}) = C_0^{\infty}(\mathbb{R})$, and in $L_2(-M, M), L_4(-M, M)$,

$$\int_{-\infty}^{\infty} [q_n(x) - G(x)]\rho(x)\, dx \leq \|\rho\|_{L_2}\|q_n - G\|_{L_2(supp(\rho))}.$$

so $q_n \to G$ in the sense of distribution, $q_n \to G$ in $\mathcal{D}'$. Therefore $G = q_*$, moreover $q_n^2 \to q_*^2$.
Now

$$0 = \lim_{n \to \infty}(\phi, q_n) = (\phi, q_* = \lim_{n \to \infty}(\phi', q_n) = (\phi', q_*),$$

so,

$$(Lq_*, q_*) \leq \liminf_{n \to \infty}(Lq_n, q_n),$$

$$\|q_*\|^2 = (q_*, q_* \leq \liminf_{n \to \infty}(q_n, q_n) = 1.$$

One more condition, $\lim_{n\to\infty}(f'(\phi), q_n^2) = (f'(\phi), q_*^2)$,

$$0 = \lim n \to \infty (\mathcal{L}q_n, q_n) = (Lq_n, q_n) + C(q_n, q_n) - (f'(\phi), q_n^2)$$

$$\geq (Lq_*, q_*) + C - \int f'(\phi)q_*^2\, dx$$

$$\geq C - \int f'(\phi)q_*^2\, dx,$$

therefore, $q_* \neq 0$.

Let's normalize $q_*$ by $f_* = \frac{q_*}{\|q_*\|}$, then

$$(\mathcal{L}f_*, f_*) = \frac{1}{\|q_*\|}^2 (\mathrm{II}_*, q_*) \leq 0.$$

Making use of $\frac{d}{dC}\|\phi\|^2 \geq 0$ and $(L+C)\dot\phi - f(\phi)\dot\phi = -\phi$ leads to

$$(\mathcal{L}\dot\phi, \dot\phi) < 0.$$

Suppose the negative eigenvalue $\lambda$ of $\mathcal{L}$ has corresponding eigenfunction $\mathcal{X}$, and 0 has corresponding eigenfunction $\phi'$, decompose $\dot\phi = a\mathcal{X} + b\phi' + P_0$, substitute it into $(\mathcal{L}\dot\phi, \dot\phi) < 0$, we have:

$$0 > ((a\mathcal{X} + b\phi' + P_0), a\mathcal{X} + b\phi' + P_0)$$
$$= a^2(\mathcal{L}\mathcal{X}, \mathcal{X}) + b^2(\mathcal{L}\phi', \phi') + (\mathcal{L}P_0, P_0)$$
$$- \lambda a^2\|\mathcal{X}\| + (\mathcal{L}P_0, P_0),$$

so,

$$(\mathcal{L}P_0, P_0) \leq \lambda a^2.$$

Remember $(\{_*, f_*) \leq 0$ and $f_*\lfloor\phi$, so $f_* = c\mathcal{X} + P$, and

$$0 = (\phi, f_*) = -(\mathcal{L}\dot\phi, f_*) = -ac^2\lambda + (\mathcal{L}P_0, P),$$

$$(\{_*, f_*) = -c^2\lambda + (\mathcal{P}, P)$$
$$- c^2\lambda + \frac{(\mathcal{L}P, P_0)^2}{\|(\mathcal{P}_0, P_0)\|}$$
$$> -c^2\lambda + \frac{ac\lambda)^2}{a^2\lambda^2}$$
$$= 0.$$

Contradiction. The lemma is proved.                                   □

**Corollary 4.2.8.** $(\mathcal{L}y, y) \geq \eta\|y\|^2$ *for any* $y \in X, y \lfloor \phi, y \lfloor \phi_x$.

Remember, the stability we are dealing with is orbital stability, solitary wave $\phi(x - Ct)$ of $u_t + f(u)_x - Lu_x = 0$ at time $t$ is just translate of the initial data, orbit $\phi = \{\tau_r\phi : r \in \mathbb{R}\}$. So to prove the solitary wave is stable, we need to prove that given $\epsilon > 0$, there is always $\delta > 0$ such that

$$\|\phi - \psi\|_X \delta \implies u \in \mathcal{U}_\epsilon = \{z : d(z, \phi) < \epsilon\}.$$

**Lemma 4.2.9.** *There exist* $\epsilon > 0$, $C > 0$ *and a unique* $C^1$ *mapping* $\alpha : U_\epsilon \to \mathbb{R}$ *such that for* $u \in U_\epsilon$,

1. *i)* $(u(\cdot + \alpha), \phi_x) = 0$,

2. *ii)* $\alpha\,(u(\cdot + r)) = \alpha(\tau_r u) = \alpha(u) - r$,

3. *iii)* $\alpha'(u) = \dfrac{\phi_x(\cdot - \alpha(u))}{\int_{-\infty}^{\infty} u(x)\psi_{xx}(x - \alpha(u))\,dx}$.

**Remark 4.2.10.** *To study stability of the solitary wave, we wish there exists some.*

*Proof.* Consider the functional

$$G : L_2 \times \mathbb{R} \to \mathbb{R} \text{ defined by}$$

$$G(u, \alpha) = \int_{-\infty}^{\infty} u(x + \alpha)\phi_x(x)\,dx,$$

then

$$G(u, \alpha)|_{(u=\phi, \alpha=0)} = 0$$

and

$$\frac{\partial G}{\partial \alpha}\Big|_{(u=\phi, \alpha=0)} = \int_{-\infty}^{\infty} u_x(x + \alpha)\phi_x(x)\,dx|_{(u=\phi, \alpha=0)} = \int_{-\infty}^{\infty} \phi_x(x)^2\,dx \neq 0.$$

By the implicit function theorem, there is a neighborhood $B_\epsilon(\phi)$ and a unique $C^1$ functional $\alpha : B_\epsilon(\phi) \to \mathbb{R}$ satisfying

i) $(u(\cdot + \alpha), \phi_x) = 0$,

ii) by translation invariance $u(\cdot + \alpha(u)) = u(\cdot + r + (\alpha(u) - r)) = \tau_r u(\cdot + \alpha(u) - r)$, and by uniqueness, $\alpha(u) - r = \alpha(\tau_r(u))$,

iii) Finally, i) can be rewritten by a variable change

$$0 = \int_{-\infty}^{\infty} u(x)\phi_x(x - \alpha(u))\, dx.$$

Differentiating it with respect to $u$ leads

$$0 = \phi_x(\cdot - \alpha(u)) - \int_{-\infty}^{\infty} u(x)\partial_x^2 \phi(x - \alpha(u))\, dx\, \alpha'(u),$$

so,

$$\alpha'(u) = \frac{\phi_x(\cdot - \alpha(u))}{\int_{-\infty}^{\infty} u(x)\phi_{xx}(x - \alpha(u))\, dx}.$$

The proof is complete.                                                           $\square$

**Lemma 4.2.11.** *Let $\phi = \phi_c$ be a fixed solitary wave, then there exist constants $C > 0$ and $\epsilon > 0$ such that*

$$\Lambda(u) - \Lambda(\phi) \geq C\|u(\cdot + \alpha(u), t) - \phi\|^2$$

*for all $u \in U_\epsilon$ such that $\|u\|_{L_2} = \|\phi\|_{L_2}$.*

*Proof.* Let $h(x, t) = u(x + \alpha(u(x, t)), t) - \phi(x)$, so $h$ is orthogonal to $\phi_x$. $u$ can be written in form $u = (1 + a)\phi + y$, where $y$ is orthogonal to $\phi_x$, and $\phi$, $h = a\phi + y$. By translation invariance and Taylor's expansion theorem,

$$V(\phi) = V(u) = V(u(\cdot + \alpha(u))) = V(\phi) + 2(\phi, h) + O(\|h\|^2),$$

$$a(\phi, \phi) = \frac{1}{2}\|h\|^2,$$

i.e $a = O(\|h\|^2)$.

$$\Lambda(u) = \Lambda(u(\cdot + \alpha(u)))$$
$$= \Lambda(\phi) + \frac{1}{2}(\mathcal{L}h, h) + O(\|h\|^2)$$
$$\Lambda(\phi) + \frac{1}{2}(\mathcal{L}(a\phi + y), a\phi + y) + O(\|h\|^2)$$
$$\geq \Lambda(\phi) + O(a^2) + O(a\|y\|) + \eta\|u\|^2 + o(\|h\|^2),$$
$$\|y\| = \|h - a\phi\|$$
$$\geq \|h\| - a\|\phi\|$$
$$= \|h\| - O(\|h\|^2)$$
$$\geq \frac{1}{2}\|h\|.$$

Then,

$$\Lambda(u) \geq \Lambda(\phi) + \frac{1}{2}\eta\|h\|^2 + o(\|h\|^2)$$
$$\geq \Lambda(\phi) + \frac{\eta}{4}\|h\|^2,$$

so,

$$\Lambda(u) - \Lambda(\phi) \geq C\|h\|^2$$

for $\|h\|$ small. $\square$

Therefore,

$$E(u) - E(\phi) \geq C\|h\|^2$$

for $\|h\|$ small.

**Theorem 4.2.12.** *The solitary wave* $\phi = \phi_c$ *is stable if and only if* $\Lambda''(c) > 0.$

$\square$

# Bibliography

[1] J. L. BONA AND R. SMITH, *The initial-value problem for the Korteweg-de Vries equation*, Philos. Trans. Royal Soc. London, Ser. A, 278 (1975), pp. 555–601.

[2] J. V. BOUSSINESQ, *Théorie de l'intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire*, C. R. Acad. Sci. Paris, 72 (1871), pp. 755–759.

[3] ——, *Théorie générale des mouvements qui sont propagés dans un canal rectangulaire horizontal*, C. R. Acad. Sci. Paris, 73 (1871), pp. 256–260.

[4] ——, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J. Math. Pures Appl., 17 (1872), pp. 55–108.

[5] L. HÖRMANDER, *The analysis of linear partial differential operators. I*, Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.

[6] G. H. KEULEGAN AND G. W. PATTERSON, *Mathematical theory of irrotational translation waves*, J. Research Nat. Bur. Standards, 24 (1940), pp. 47–101.

[7] D. J. KORTEWEG AND G. DE VRIES, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Phil. Mag., 39 (1895), pp. 422–443.

[8] M. D. KRUSKAL AND N. J. NORMAN, *Exact invariants for a class of nonlinear wave equations*, J. Mathematical Phys., 7 (1966), pp. 1256–1267.

[9] H. LAMB, *Hydrodynamics*, 6th ed. Cambridge University Press, 1932.

[10] J. S. RUSSELL, *Report on waves*, Rept. Fourteenth Meeting of the British Association for the Advancement of Science, (1844), pp. 311–390.

[11] G. G. STOKES, *On the theories of the internal friction of fluids in motion*, Camb. Phil. Trans., viii (1845).