DECAY OF SOLUTIONS TO A WATER WAVE MODEL WITH A NONLOCAL VISCOUS DISPERSIVE TERM

M. CHEN[1], S. DUMONT[2], L. DUPAIGNE[2] AND O. GOUBET[2]

[1] *Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA.*
[2] *LAMFA CNRS UMR 6140 Université de Picardie Jules Verne, 33, rue Saint-Leu, 80039 Amiens, France.*

Abstract. In this article, we investigate a water wave model with a nonlocal viscous term

$$u_t + u_x + \beta u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds + uu_x = \nu u_{xx}.$$

The wellposedness of the equation and the decay rate of solutions are investigated theoretically and numerically.

1. Introduction

1.1. **Dias-Dutykh models.** Modeling the effect of viscosity with asymptotic models for water waves is a challenging issue. In the recent work [5], D. Dutykh and F. Dias have introduced a system which models water waves in a fluid layer of finite depth under the influence of viscous effects. The model is a generalization of the ones introduced by P. Liu and A. Orfila [8] and J. Bona, M. Chen and J-C. Saut [3]: it contains the same nonlocal viscosity term as in [8] and it has the flexibility of taking the horizontal velocity at various water levels as in [3]. The derivation holds in the linear 3D case. One of the corresponding two-dimensional nonlinear systems reads

$$\begin{aligned} \eta_t + u_x + (\eta u)_x + \frac{1}{3} u_{xxx} &= 2\nu \eta_{xx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_x(s)}{\sqrt{t-s}} ds, \\ u_t + \eta_x + uu_x &= 2\nu u_{xx}. \end{aligned} \tag{1.1}$$

Here η is the deviation of the free surface of the wave from its equilibrium state, u is the horizontal velocity and ν is the damping parameter.

The linear analysis of the nonlocal term shows that it has both dispersive and dissipative effects (see the discussion in the next section). This phenomenon has been first observed by T. Kakutani and K. Matsuuchi [7], whose

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model involves a nonlocal term in space and is equivalent to the model (1.1) for long waves (that is, the two models share the same dispersion relation for long waves). The Kakutani-Matsuuchi model itself is similar to the famous Ott-Sudan model [10], supplemented by a nonlocal dispersive term. In the present article, we restrict the analysis of (1.1) to one-way waves. With the usual one-way wave reduction (see e.g. [3] and [4] for details), namely setting $u_t + \eta_x = 0$ as a first order approximation of the system (1.1), one obtains the following dispersive dissipative equation

$$\eta_t + \eta_x + \frac{1}{6}\eta_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}}\int_0^t \frac{\eta_t(s)}{\sqrt{t-s}}ds + \frac{3}{2}\eta\eta_x = 2\nu\eta_{xx}. \tag{1.2}$$

For the sake of convenience, we set $u = \eta$ in the remaining of the article. Our aim is to analyze the decay rate of solutions $u(t,x)$ to

$$u_t + u_x + u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}}\int_0^t \frac{u_t(s)}{\sqrt{t-s}}ds + uu_x = \nu u_{xx}. \tag{1.3}$$

as $t \to +\infty$. Since the result will not depend on the numerical constants appearing in (1.2), they are normalized in (1.3). Before going any further, we note that in [4], the author investigates numerically the decay rate in L^∞ norm- of an approximation of a solitary wave for an analog of equation (1.3) that reads

$$u_t + u_x + u_{xxx} - \frac{\sqrt{\nu}}{\sqrt{\pi}}\int_0^t \frac{u_x(s)}{\sqrt{t-s}}ds + uu_x = \nu u_{xx}. \tag{1.4}$$

He obtains numerical evidence of linear decay, but the result is only an "impression", according to the author, since the solution was computed only for a very short time. We have not investigated equation (1.4), but we will prove rigorously the decay rate for (1.3) with $\beta = 0$ which is formally equivalent to (1.4) and investigate numerically the decay rates for other cases.

In order to understand the terms in (1.3), we will also investigate several simplified models which include the models with only nonlocal dissipative term (without νu_{xx} term)

$$u_t + u_x + u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}}\int_0^t \frac{u_t(s)}{\sqrt{t-s}}ds + uu_x = 0, \tag{1.5}$$

its equivalent BBM form

$$u_t + u_x - u_{txx} + \frac{\sqrt{\nu}}{\sqrt{\pi}}\int_0^t \frac{u_t(s)}{\sqrt{t-s}}ds + uu_x = 0, \tag{1.6}$$

and the viscous dominant equation (without u_{xxx} term)

$$u_t + u_x + \frac{\sqrt{\nu}}{\sqrt{\pi}}\int_0^t \frac{u_t(s)}{\sqrt{t-s}}ds + uu_x = \nu u_{xx}. \tag{1.7}$$

Each equation is indeed a valid approximation of (1.3) in a certain parameter regime and can be justified from the point of dispersion analysis as presented in the next section. Our investigation will be centered on the nonlocal term, which was derived earlier and believed to be more realistic, but fewer results are available due in part to the nonlocal complexity.

1.2. **Dispersion analysis.** In this section, we discuss the dispersion/dissipation relation for the linearized asymptotic models. Because we have to deal with a nonlocal term, the Laplace-Fourier analysis is more convenient to use. This analysis shows that the nonlocal term in (1.3) is both dispersive and dissipative. Moreover, the dissipation from the nonlocal term is approximately half-derivative, namely behaving like $\sqrt{|k|}$ in Fourier space.

To begin, we start with the KdV-Burgers equation

$$u_t + u_x + u_{xxx} = \nu u_{xx}. \tag{1.8}$$

By substituting the plane wave ansatz $u(t,x) = v(t)e^{ikx}$ with $v(0) = 0$ into the equation, we find

$$v_t + (i(k - k^3) + \nu k^2)v = 0.$$

We now perform the Laplace transform in time

$$\widetilde{v}(\tau) = \mathcal{L}(v)(\tau) = \int_0^\infty v(t)e^{-t\tau}dt.$$

We seek for a plane wave ansatz $v(t)$ that has decay $O(e^{-at})$ for some $a \geq 0$ at $t = +\infty$, then the Laplace transform is defined for $a + \tau \geq 0$. This provides us with the relation $\tau + i(k - k^3) + \nu k^2 = 0$. The real part of τ gives the dissipation rate $-\nu k^2$ while the imaginary part gives the dispersion relation $\omega = -\Im\tau = k - k^3$.

We now turn to the linear version of (1.3), namely

$$u_t + u_x + u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}}\int_0^t \frac{u_t(s)}{\sqrt{t-s}}ds = \nu u_{xx}. \tag{1.9}$$

Plugging the plane-wave ansatz in, we obtain

$$v_t + i(k - k^3)v + \frac{\sqrt{\nu}}{\sqrt{\pi}}\int_0^t \frac{v_t(s)}{\sqrt{t-s}}ds + \nu k^2 v = 0.$$

Assume $v_{|t=0} = 0$ and apply the Laplace transform in time yield

$$\tau\widetilde{v} + i(k - k^3)\widetilde{v} + \sqrt{\nu\tau}\widetilde{v} + \nu k^2\widetilde{v} = 0.$$

Using the change of unknown $\tau = z^2$ the relation between τ and k reads

$$z^2 + \sqrt{\nu}z + i(k - k^3) + \nu k^2 = 0. \tag{1.10}$$

The above equation has two solutions given by

$$2z = -\sqrt{\nu} \pm \sqrt{\nu - 4ik + 4ik^3 - 4\nu k^2}.$$

Therefore,

$$-z^2 = -\tau = -\frac{\nu}{2} \pm \frac{1}{2}\sqrt{\nu}\sqrt{\nu - 4ik + 4ik^3 - 4\nu k^2} + i(k - k^3) + \nu k^2.$$

The leading order approximation is the linear wave equation, namely $\tau+ik=0$. So, by restricting to the regime $\nu << k << 1$, we see that

$$\sqrt{\nu-4ik+4ik^3-4\nu k^2}=\sqrt{-4ik}\sqrt{1+o(1)}=2e^{-i\mathrm{sgn}k\frac{\pi}{4}}\sqrt{|k|}+o(|k|^{\frac{1}{2}}).$$

Therefore

$$-z^2=-\tau=-\frac{\nu}{2}\pm e^{-i\mathrm{sgn}k\frac{\pi}{4}}\sqrt{\nu|k|}+i(k-k^3)+o(\sqrt{\nu|k|}).$$

We have

$$-\tau^{\pm}=i(k-k^3\mp\mathrm{sgn}k\frac{\sqrt{\nu|k|}}{\sqrt{2}})\pm\frac{\sqrt{\nu|k|}}{\sqrt{2}}+o(\sqrt{\nu|k|}). \tag{1.11}$$

We first note only one of the above two modes is stable, namely the τ^+. The $-\Im\tau$ has two nonlinear parts: $-k^3$, the geometric dispersion coming from the term u_{xxx} and $-\mathrm{sgn}k\frac{\sqrt{\nu|k|}}{\sqrt{2}}$, the dispersion due the nonlocal viscous effect. The integral term also provides diffusion which is half derivative with symbol $\sqrt{|k|}$ in Fourier space !

Therefore, in the case of $\nu << k << 1$, the local dissipation term is of smaller order than the non-local dissipative term. By neglecting the lower order term, one obtains the equation (1.5) or (1.6), with dispersion relation (1.11) in the leading orders. Furthermore, when the geometric dispersion is stronger than the viscous dispersion, and when we neglect the dissipation, then we are back to some KdV-type equation. When there is a balance between these two terms, namely $\nu\sim 2k^5$, we have (1.5) or (1.6). Finally, when the viscous dispersion is stronger, we can neglect u_{xxx} in the equation.

Remark 1.1. *The above result is qualitatively exactly equivalent to those of Kakutani and Maatsuchi* [7] *and Liu and Orfila* [8].

It is worth to note that the first order approximation to equation (1.9) is $u_t+u_x=0$. This amounts to assuming that to leading order, there is a balance between the space and the time frequency of the wave: $k\sim\omega$. Second order terms that come from geometric dispersion and nonlinearity are given by $u_{xxx}+uu_x$. In the classical analysis, it is assumed that there exists a balance between the height h of the wave and the space frequency k of the wave, of the form $h\sim k^2$. As we have seen, the quantity $-\nu u_{xx}$ should be thought of as a viscous dissipative term, while the term $\frac{\sqrt{\nu}}{\sqrt{\pi}}\int_0^t\frac{u_t(s)}{\sqrt{t-s}}ds$, acts as a viscous dispersive dissipative term.

1.3. **Statement of the results.** In this subsection and section 2, our theoretical analysis is restricted to the viscous dominant equation (1.7). For the sake of convenience, we also set the parameter ν to be 1. We now begin with the linear theory.

Theorem 1.2. *Consider the equation*

$$u_t+u_x+\frac{1}{\sqrt{\pi}}\int_0^t\frac{u_t(s)}{\sqrt{t-s}}ds=u_{xx}, \tag{1.12}$$

supplemented with initial condition $u_0 \in L^2(\mathbb{R})$. There exists a unique global solution $u \in C(\mathbb{R}_+; L^2_x(\mathbb{R})) \cap C^1(\mathbb{R}_+; H^{-2}_x(\mathbb{R}))$ of (1.12). In addition, we have the following partial smoothing effect, with $\mathbb{R}^*_+ = \mathbb{R}_+ \backslash \{0\}$,

$$u \in C(\mathbb{R}^*_+; H^2_x(\mathbb{R}))$$

and the representation formula

$$u(t,x) = \left[K\left(t,\cdot\right) \star u_0\right](x)\,, \tag{1.13}$$

where $\star$ denotes the usual convolution product and

$$K(t,x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} e^{-x^-} \left(1 + \frac{1}{2}\int_0^{+\infty} e^{-\frac{\mu^2}{4t} - \frac{\mu|x|}{2t} - \frac{\mu}{2}}\, d\mu\right) \tag{1.14}$$

with $x^- = \max(-x, 0)$.

Remark 1.3. *Unlike the standard heat equation, the solution of (1.7) can not be smoother than $H^2_x(\mathbb{R})$ for $t > 0$, unless additional assumptions are made on the initial datum. This can be seen by differentiating K twice: $\partial^2 K/\partial x^2$ is the sum of a continuous function and a constant times the Dirac mass at the origin. In particular, if $u_0 \notin H^1(\mathbb{R})$, then $u(t,\cdot) \notin H^3_x(\mathbb{R})$ for any $t > 0$.*

Turning to the nonlinear problem (1.7), we obtain the following global existence and decay result, for small initial datum.

Theorem 1.4. *Consider (1.7) supplemented with initial data $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. There exists $\epsilon > 0$, $C(u_0) > 0$ such that for all $\|u_0\|_{L^1(\mathbb{R})} < \epsilon$, there exists a unique global solution $u \in C(\mathbb{R}_+; L^2_x(\mathbb{R})) \cap C^1(\mathbb{R}_+; H^{-2}_x(\mathbb{R}))$. In addition, u satisfies*

$$t^{\frac{1}{2}} ||u(t)||_{L^\infty_x(\mathbb{R})} + t^{\frac{1}{4}} ||u(t)||_{L^2_x(\mathbb{R})} \leq C(u_0) \tag{1.15}$$

and u solves the fixed point equation

$$u(t,x) = K(t,\cdot) \star u_0 + N \circledast u^2, \tag{1.16}$$

where K is given by (1.14) and N by

$$N(t,x) = \frac{1}{2\sqrt{\pi t}} \partial_x \left[e^{-\frac{x^2}{4t} - x^-} \left(1 - \frac{1}{2}\int_0^{+\infty} e^{-\frac{\mu^2}{4t} - \frac{\mu}{2} - \frac{\mu|x|}{2t}}\, d\mu\right)\right], \tag{1.17}$$

with $\star$ denotes the usual convolution product in space and $\circledast$ the time-space convolution product defined by

$$v \circledast w\,(t,x) = \int_0^t \int_{\mathbb{R}} v(s,y) w(t-s, x-y)\, dy\, ds$$

whenever the integrals make sense.

It is worth to point out that the decay rate (1.15) coincides with that of the classical KdV-Burgers equation

$$u_t + u_{xxx} + u u_x = u_{xx}. \tag{1.18}$$

The proof of (1.15) for solutions of (1.18) is given in the seminal work [2]. We also refer readers who are interested in the KdV equation with nonlocal diffusion of the form $(-\Delta)^\alpha u$, $0 < \alpha < 1$ to [9], [12], [13] and to the references therein. The references [9] and [12] deal with the initial value problem, while [13] studies the asymptotic decay of solutions.

The remaining of the article is organized as follows. In Section 2 we prove the Theorem 1.2 and the main Theorem 1.4. For this purpose we provide a rigorous definition of solution for (1.7). In Section 3, we provide some numerical evidence for these decay estimates.

2. Proof of Theorem 1.2 and Theorem 1.4

2.1. **What is a solution to the integral-differential equation** (1.7)**?** We first consider a simple ordinary differential equation

$$v_t = f(t), \quad f \in C(\mathbb{R}_+),$$

supplemented with initial datum v_0 at $t = 0$. By the Fundamental Theorem of Calculus, there exists a unique function $v \in C^1(\mathbb{R}_+)$, characterized by the integral equation

$$v(t) = v_0 + \int_0^t f(s)ds, \qquad \text{for } t > 0. \tag{2.1}$$

Now, for $f \in C(\mathbb{R}_+)$, we consider the ordinary integro-differential equation

$$\begin{aligned} v_t + \frac{1}{\sqrt{\pi}} \int_0^t \frac{v_t(s)}{\sqrt{t-s}} ds &= f(t), \\ v(0) &= v_0. \end{aligned} \tag{2.2}$$

Define the operator $I : C(\mathbb{R}_+; \mathbb{R}) \to C(\mathbb{R}_+)$ for $u \in C(\mathbb{R}_+)$, $t > 0$ by

$$I(u)(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds, \tag{2.3}$$

direct computation implies that $I(Iv) = \int_0^t v(s)ds$. So the second term in the left hand side of (2.2) acts as a half-derivative $\partial_t^{\frac{1}{2}} v$, as it was show in Section 1. In order to derive the integral form of the solution to (2.2), we apply the Laplace transform to it. Since $\widetilde{v_t} = -v(0) + \tau \widetilde{v}$ and using integration by parts, we obtain

$$\begin{aligned}\mathcal{L}\left(\frac{1}{\sqrt{\pi}}\int_0^t \frac{v_t(s)}{\sqrt{t-s}}\,ds\right) &= \frac{1}{\sqrt{\pi}}\int_0^{+\infty} e^{-t\tau}\left(\int_0^t \frac{v_t(s)}{\sqrt{t-s}}ds\right)\,dt\\ &= \frac{1}{\sqrt{\pi}}\int_0^{+\infty} v_t\left(\int_s^{+\infty} \frac{e^{-t\tau}}{\sqrt{t-s}}dt\right)\,ds\\ &= \frac{1}{\sqrt{\pi}}\int_0^{+\infty} v_t e^{-s\tau}\left(\int_0^{+\infty} \frac{e^{-u\tau}}{\sqrt{u}}du\right)\,ds\\ &= \frac{1}{\sqrt{\tau}}\int_0^{+\infty} v_t e^{-s\tau}\,ds = \frac{1}{\sqrt{\tau}}(-v(0)+\tau\widetilde{v}).\end{aligned}$$

Plugging this information in (2.2), we obtain

$$\widetilde{v}(\tau) = \frac{v_0}{\tau} + \frac{1}{\tau+\sqrt{\tau}}\widetilde{f}(\tau). \tag{2.4}$$

We now state a result which proof is left as an exercise for the reader.

Lemma 2.1. *For $t \geq 0$ and let*

$$N_0(t) = \frac{1}{\sqrt{\pi}}e^t\int_t^{+\infty} \frac{e^{-s}}{\sqrt{s}}ds. \tag{2.5}$$

Then $\widetilde{N}_0(\tau) = \frac{1}{\tau+\sqrt{\tau}}$.

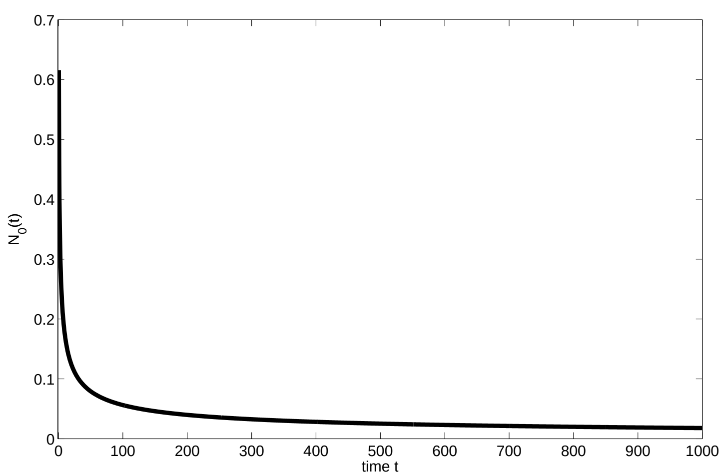

FIGURE 1. Curve of $N_0(t)$ with respect to t.

Taking inverse Laplace transform on (2.4), we obtain the solution of (2.2) has the integral form

$$v(t) = v_0 + \int_0^t N_0(t-s)f(s)ds. \tag{2.6}$$

The above calculations are somewhat formal, but one can easily check that given $f \in C(\mathbb{R}_+)$, the function v given by (2.6) belongs to $C^1(\mathbb{R}_+)$ (since N_0 belongs to $W^{1,1}_{\mathrm{loc}}$) and solves the equation. In addition, such a solution is unique, as we demonstrate now.

Proposition 2.2. *Let $\lambda \in \mathbb{C}$. Given $f \in C(\mathbb{R}_+)$, $v_0 \in \mathbb{R}$, the equation*

$$v_t + \frac{1}{\sqrt{\pi}} \int_0^t \frac{v_t(s)}{\sqrt{t-s}} ds + \lambda v = f$$

admits at most one solution $v \in C^1(\mathbb{R}_+)$ such that $v(0) = v_0$.

Proof. Indeed, if $v, \widetilde{v}$ are two solutions, then $w = v - \widetilde{v}$ solves

$$w_t + \frac{1}{\sqrt{\pi}} \int_0^t \frac{w_t(s)}{\sqrt{t-s}} ds + \lambda w = 0. \tag{2.7}$$

Letting $m(t) = \sup_{s \in [0,t]} |w_t(s)|$, it follows that

$$m(t) \le \frac{2}{\sqrt{\pi}} \sqrt{t}\, m(t) + |\lambda| \sup_{s \in [0,t]} |w(s)| \le \left(\frac{2}{\sqrt{\pi}} \sqrt{t} + |\lambda| t \right) m(t).$$

Therefore, $m(T) = 0$ for any given $T > 0$ such that $\frac{2}{\sqrt{\pi}} \sqrt{T} + |\lambda| T < 1$. By a direct inductive argument, one easily deduces that $m(kT) = 0$ for all $k \in \mathbb{N}$, so that w must be constant. Since $w(0) = 0$, uniqueness follows. □

Remark 2.3. *We warn the reader that unlike the case of a standard ODE, the solution v of* (2.2) *is never of class C^2 at the origin when $f \in C^1(\mathbb{R}_+)$, unless $f(0) = 0$. If this were the case, then we would have $v_t = f(0) + \mathcal{O}(t)$ as $t \to 0$. Whence by* (2.2), *$\frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{f(0)+\mathcal{O}(s)}{\sqrt{t-s}} ds = \mathcal{O}(t)$, a contradiction.*

Remark 2.4. *The integral form* (2.6) *is useful to construct numerical schemes for equation* (2.2)*, just as (2.1) is useful for solving numerically a standard ODE.*

For integro-differential equations involving derivatives in x, as in the case of (1.7), the situation is more complicated. We refer to [1] where the author studies partial diffusion processes of the form

$$\frac{1}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds = u_{xx}, \tag{2.8}$$

and to [11] for the evolution equation with delay

$$u_t = \frac{1}{\sqrt{\pi}} \int_0^t \frac{u_{xx}(s)}{\sqrt{t-s}} ds. \tag{2.9}$$

Consider now a m-accretive unbounded linear operator $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ where A generates a semi-group of contraction on $L^2(\mathbb{R})$ (denoted e^{-tA}). To solve the equation

$$\begin{aligned} u_t + Au &= f, \\ u(0) &= 0, \end{aligned} \tag{2.10}$$

it is convenient to seek for a fixed point of the Duhamel's form of the equation that reads

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)ds. \tag{2.11}$$

We aim at providing the analog of the Duhamel's form for (1.7), using the Laplace-Fourier transform.

2.2. **Proof of Theorem 1.2; the linear homogeneous problem.** We begin by proving that the solution of (1.12), the linearization of (1.7), is unique. If $v, \widetilde{v}$ are two solutions, let $w = v - \widetilde{v}$ and apply the Fourier transform in space to w. Then, for almost every $\xi \in \mathbb{R}$, $t \to \widehat{w}(t,\xi) \in C^1(\mathbb{R}^+)$ solves (2.7) with parameter $\lambda = \xi^2 + i\xi$. An application of Proposition 2.2 yields the uniqueness result of the solution.

We proceed to the existence of a solution of (1.12). Given $u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}))$, consider its Laplace-Fourier transform $\widehat{u}$ defined for $\tau > 0$ and $\xi \in \mathbb{R}$ by

$$\widehat{u}(\tau,\xi) = \mathcal{L}_t\mathcal{F}_x u = \int_0^{+\infty} e^{-t\tau} \int_{\mathbb{R}} e^{-ix\xi} u(t,x)dx\, dt.$$

With an abuse of notation, we also write the usual Fourier transform of u_0 as $\widehat{u}_0 = \mathcal{F}_x u_0$. Apply the Laplace-Fourier transform to (1.12):

$$-\widehat{u}_0 + \tau\widehat{u} + i\xi\widehat{u} + \frac{1}{\sqrt{\tau}}(-\widehat{u}_0 + \tau\widehat{u}) = \xi^2\widehat{u}.$$

Solving for $\widehat{u}$, we obtain

$$\widehat{u}(\tau,\xi) = \widehat{K}(\tau,\xi)\widehat{u}_0, \tag{2.12}$$

with

$$\widehat{K}(\tau,\xi) = \left(1 + \frac{1}{\sqrt{\tau}}\right) \frac{1}{(\sqrt{\tau} + 1/2)^2 + (\xi + i/2)^2}. \tag{2.13}$$

Lemma 2.5. *$\widehat{K}$ is the Laplace-Fourier transform of*

$$K(t,x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} e^{-x^-} \left(1 + \frac{1}{2}\int_0^{+\infty} e^{-\frac{\mu^2}{4t} - \frac{\mu|x|}{2t} - \frac{\mu}{2}}\, d\mu\right) \tag{2.14}$$

where $x^- = \frac{|x|-x}{2}$, the negative part of $x \in \mathbb{R}$.

Proof. We claim that $\widehat{K}(\tau,\xi)$ is the Fourier transform (in the x variable) of

$$\widetilde{K}(\tau,x) = \frac{1 + 1/\sqrt{\tau}}{1 + 2\sqrt{\tau}} e^{-|x|\sqrt{\tau}} e^{-x^-}. \tag{2.15}$$

Indeed,

$$\begin{aligned}\int_{\mathbb{R}} e^{-ix\xi}e^{-|x|\sqrt{\tau}}e^{-x^-}\,dx &= \int_{-\infty}^{0} e^{-ix\xi}e^{-|x|(\sqrt{\tau}+1)}\,dx + \int_{0}^{+\infty} e^{-ix\xi}e^{-x\sqrt{\tau}}\,dx\\ &= \int_{0}^{+\infty} e^{ix\xi}e^{-x(\sqrt{\tau}+1)}\,dx + \int_{0}^{+\infty} e^{-ix\xi}e^{-x\sqrt{\tau}}\,dx\\ &= \frac{1}{\sqrt{\tau}+1-i\xi} + \frac{1}{\sqrt{\tau}+i\xi}\\ &= \frac{\sqrt{\tau}+i\xi+\sqrt{\tau}+1-i\xi}{\tau+i\xi\sqrt{\tau}+\sqrt{\tau}+i\xi-i\xi\sqrt{\tau}+\xi^2}\\ &= \frac{2\sqrt{\tau}+1}{\tau+\sqrt{\tau}+i\xi+\xi^2} = \frac{2\sqrt{\tau}+1}{(\sqrt{\tau}+1/2)^2+(\xi+i/2)^2}.\end{aligned}$$

Next, we claim that $\widetilde{K}$ is the Laplace transform of $K(t,x)$ given by (2.14). To see this, we first recall that if $\widetilde{f}(\tau)$ is the Laplace transform of $f \in L^\infty(\mathbb{R}_+;\mathbb{R})$, then $\widetilde{f}(\sqrt{\tau})$ is the Laplace transform of

$$\int_0^{+\infty} \frac{1}{\sqrt{\pi t}}\frac{\mu}{2t}e^{-\frac{\mu^2}{4t}}f(\mu)\,d\mu. \tag{2.16}$$

See e.g. Formula 23, Tables of Laplace transforms, General Formulas in [6]. Setting $\sigma=\sqrt{\tau}$, our task reduces to finding $f(t,x)$ such that

$$\widetilde{f}(\sigma,x) = \left(1+\frac{1}{\sigma}\right)\left(\frac{1}{1+2\sigma}\right)e^{-|x|\sigma}e^{-x^-}.$$

Now,

$$\left(1+\frac{1}{\sigma}\right)\left(\frac{1}{1+2\sigma}\right) = \frac{1}{\sigma} - \frac{1}{1+2\sigma}.$$

Denote by χ_A the characteristic function of a set $A\subset\mathbb{R}$. Then,

$$\mathcal{L}\left(\chi_{[t\geq|x|]}\right)(\sigma) = \frac{e^{-\sigma|x|}}{\sigma}$$

and

$$\mathcal{L}\left(\frac{1}{2}e^{-t/2}\chi_{[t\geq|x|]}\right)(\sigma) = \frac{e^{-|x|/2}e^{-\sigma|x|}}{2(\sigma+1/2)}.$$

By identification, it follows that

$$f(t,x) = e^{-x^-}\left[1-\frac{1}{2}e^{\frac{|x|-t}{2}}\right]\chi_{[t\geq|x|]}$$

and (2.14) follows by using (2.16). □

Define u by

$$u(t,x) = K(t,\cdot)\star u_0, \tag{2.17}$$

where K is given by (2.14) and $\star$ denotes the usual convolution product. We claim that $u\in C(\mathbb{R}^+;L^2_x(\mathbb{R}))\cap C^1(\mathbb{R}^+;H^{-2}_x(\mathbb{R}))$. This follows easily from the Dominated Convergence Theorem and the following estimates

Lemma 2.6. *There exists a constant $C>0$ such that for all $t>0$,*

$$\|K(t,\cdot)\|_{L^\infty_x(\mathbb{R})}\le \frac{C}{\sqrt{t}},\quad \|K(t,\cdot)\|_{L^1_x(\mathbb{R})}\le C \quad \text{and} \quad \|K(t,\cdot)\|_{L^2_x(\mathbb{R})}\le \frac{C}{t^{1/4}},$$

where K is given by (2.14)*.*

Proof. We simply observe that

$$0\le K(t,x)\le \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}\left(1+\frac{1}{2}\int_0^{+\infty}e^{-\frac{\mu}{2}}\,d\mu\right)=\frac{1}{\sqrt{\pi t}}e^{-\frac{x^2}{4t}}.$$

The L^1 and L^∞ estimates directly follow. We then use the Cauchy-Schwarz inequality $\|K(t,\cdot)\|_{L^2_x(\mathbb{R})}\le \|K(t,\cdot)\|^{1/2}_{L^1_x(\mathbb{R})}\|K(t,\cdot)\|^{1/2}_{L^\infty_x(\mathbb{R})}$ to finish the proof. □

2.3. **The linear inhomogeneous problem.** In this subsection we address the following problem: to solve

$$u_t+u_x-u_{xx}+\frac{1}{\sqrt{\pi}}\int_0^t\frac{u_t(s)}{\sqrt{t-s}}ds=-\frac{1}{2}f_x. \tag{2.18}$$

for a suitable right hand side f and with an initial data $u(x,0)=0$. In the next section, we will solve the nonlinear problem by a fixed point argument, pretending that $f=u^2$. We now state and prove

Proposition 2.7. *For any f which belongs to $C(\mathbb{R}_+;L^1_x(\mathbb{R}))$, there exists a unique function $u\in C^0(\mathbb{R}_+;L^2_x(\mathbb{R}))\cap C^1(\mathbb{R}_+;H^{-2}_x(\mathbb{R}))$ that solves (2.18).*

Proof. To begin with, the uniqueness result has already been established in the proof of Proposition 1.2. We now seek for a function u that solves (2.18) in its Duhamel's form. Working as in the homogeneous case, we discover that u must solve

$$\widehat{u}(\tau,\xi)=\widehat{N}(\tau,\xi)\widehat{f}(\tau,\xi), \tag{2.19}$$

where $\widehat{N}$ is given by

$$\widehat{N}(\tau,\xi)=\frac{1}{2}\frac{i\xi}{(\sqrt{\tau}+1/2)^2+(\xi+i/2)^2}. \tag{2.20}$$

We start out by computing the inverse Laplace-Fourier transform of $\widehat{N}$:

Lemma 2.8. *$\widehat{N}$ is the Laplace-Fourier transform of*

$$N(t,x)=\frac{1}{2\sqrt{\pi t}}\partial_x\left[e^{-\frac{x^2}{4t}-x^-}\left(1-\frac{1}{2}\int_0^{+\infty}e^{-\frac{\mu^2}{4t}-\frac{\mu}{2}-\frac{\mu|x|}{2t}}\,d\mu\right)\right]. \tag{2.21}$$

Proof. Due to the computations in the proof of Lemma 2.5 above, we know that the inverse Fourier transform (in space) of $\widehat{N}$ is given by

$$\widetilde{N}(\tau,x)=\partial_x\left(\frac{1}{1+2\sqrt{\tau}}e^{-|x|\sqrt{\tau}}e^{-x^-}\right). \tag{2.22}$$

Using once again (2.16), we then have

$$\sqrt{\pi}N(t,x) = \frac{1}{2}\partial_x \left[e^{\frac{x}{2}} \int_{|x|}^{+\infty} \frac{\lambda}{2t^{3/2}} e^{-\frac{\lambda^2}{4t}} e^{-\frac{\lambda}{2}}\, d\lambda \right],$$

whence

$$\begin{aligned} 2\sqrt{\pi t}\, N(t,x) &= \partial_x \left[e^{\frac{x}{2}} \int_{|x|}^{+\infty} -\partial_\lambda \left(e^{-\frac{\lambda^2}{4t}} \right) e^{-\frac{\lambda}{2}}\, d\lambda \right] \\ &= \partial_x \left[e^{\frac{x}{2}} \left(-\frac{1}{2} \int_{|x|}^{+\infty} e^{-\frac{\lambda^2}{4t}} e^{-\frac{\lambda}{2}}\, d\lambda - \left[e^{-\frac{\lambda^2}{4t}} e^{-\frac{\lambda}{2}} \right]_{|x|}^{+\infty} \right) \right] \\ &= \partial_x \left[e^{\frac{x}{2}} \left(\frac{1}{2} e^{-\frac{x^2}{4t}} e^{-\frac{|x|}{2}} \int_0^{+\infty} e^{-\frac{\mu^2}{4t}} e^{-\frac{\mu}{2}} e^{-\frac{\mu|x|}{2t}}\, d\mu \, + e^{-\frac{x^2}{4t}} e^{-\frac{|x|}{2}} \right) \right] \\ &= \partial_x \left[e^{-\frac{x^2}{4t}} e^{-x^-} \left(1 - \frac{1}{2} \int_0^{+\infty} e^{-\frac{\mu^2}{4t}} e^{-\frac{\mu}{2}} e^{-\frac{\mu|x|}{2t}}\, d\mu \right) \right]. \end{aligned}$$

□

We now proceed to some estimates on this kernel that will be used subsequently.

Lemma 2.9. *There exists a constant $C > 0$ such that for all $t > 0$,*

$$\|N(t,\cdot)\|_{L^\infty_x(\mathbb{R})} \le \frac{C}{t}, \quad \|N(t,\cdot)\|_{L^1_x(\mathbb{R})} \le \frac{C}{\sqrt{t}} \quad \text{and} \quad \|N(t,\cdot)\|_{L^2_x(\mathbb{R})} \le \frac{C}{t^{3/4}},$$

where N is given by (2.21).

By (2.21), it follows that for $x > 0$,

$$\begin{aligned} N(t,x) &= \frac{1}{2\sqrt{\pi t}} \partial_x \left[e^{-\frac{x^2}{4t}} (1 - a(t,x)) \right] \\ &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \left[-\frac{x}{2t}(1 - a(t,x)) - \frac{\partial a}{\partial x}(t,x) \right], \end{aligned} \tag{2.23}$$

where

$$a(t,x) = \frac{1}{2} \int_0^{+\infty} e^{-\frac{\mu^2}{4t} - \frac{\mu}{2} - \frac{\mu x}{2t}}\, d\mu.$$

Clearly,

$$0 \le a(t,x) \le \frac{1}{2} \int_0^{+\infty} e^{-\mu/2}\, d\mu = 1, \tag{2.24}$$

while

$$\begin{aligned} 2\left|\frac{\partial a}{\partial x}\right|(t,x) &= \int_0^{+\infty} e^{-\frac{\mu^2}{4t}} e^{-\frac{\mu}{2}} e^{-\frac{\mu x}{2t}} \frac{\mu}{2t}\, d\mu \\ &\le \frac{1}{\sqrt{t}} \int_0^{+\infty} \frac{\mu}{2\sqrt{t}} e^{-\frac{\mu^2}{4t}} e^{-\frac{\mu}{2}}\, d\mu \\ &\le \frac{C}{\sqrt{t}} \int_0^{+\infty} e^{-\frac{\mu}{2}}\, d\mu \le \frac{C}{\sqrt{t}}. \end{aligned} \tag{2.25}$$

Collecting these estimates and plugging them in (2.23), we obtain for $x>0$

$$|N(t,x)| \le \frac{C}{t} e^{-\frac{x^2}{4t}} \left(\frac{x}{\sqrt{t}} + 1 \right) \le \frac{C}{t}. \tag{2.26}$$

Next, we estimate $N(t,x)$ for $x<0$. In this case,

$$\begin{aligned} N(t,x) &= \frac{1}{2\sqrt{\pi t}} \partial_x \left[e^{-\frac{x^2}{4t}+x} \left(1 - a(t,-x)\right) \right] \\ &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}+x} \left[\left(1 - \frac{x}{2t}\right) \left(1 - a(t,-x)\right) + \frac{\partial a}{\partial x}(t,-x) \right]. \end{aligned} \tag{2.27}$$

Using (2.24) and (2.25), we deduce that if $t \le 1$ and $x<0$, then

$$\begin{aligned} |N(t,x)| &\le \frac{C}{\sqrt{t}} e^{x - \frac{x^2}{4t}} \left[1 + \frac{|x|}{t} + \frac{1}{\sqrt{t}} \right] \\ &\le \frac{C}{t} e^{-\frac{x^2}{4t}} \left[\frac{|x|}{\sqrt{t}} + 1 \right] \le \frac{C}{t}. \end{aligned} \tag{2.28}$$

For $t>1$, we observe that

$$\begin{aligned} 2\left(1 - a(t,|x|)\right) &= \int_0^{+\infty} e^{-\frac{\mu}{2}} \left(1 - e^{-\frac{\mu^2}{4t} - \frac{|x|\mu}{2t}} \right) \, d\mu \\ \le \int_0^{+\infty} e^{-\frac{\mu}{2}} &\left(\frac{\mu^2}{4t} + \frac{|x|\,\mu}{2t} \right) \, d\mu \le C \left(\frac{1}{t} + \frac{|x|}{t} \right). \end{aligned}$$

Using the above inequality and (2.25) in (2.27), we obtain for $t>1$, $x<0$,

$$\begin{aligned} |N(t,x)| &\le \frac{C}{\sqrt{t}} e^{x - \frac{x^2}{4t}} \left[\left(\frac{1+|x|}{t} \right)^2 + \frac{1}{\sqrt{t}} \right] \\ &\le \frac{C}{t} e^{-\frac{x^2}{4t}} \left[1 + \frac{|x|}{\sqrt{t}} + \frac{|x|^2}{t^{3/2}} \right] \le \frac{C}{t}. \end{aligned} \tag{2.29}$$

(2.26), (2.28) and (2.29) provide the desired L^∞_x estimate. For the L^1_x bound, we write $\|N(t,\cdot)\|_{L^1_x(\mathbb{R})} = \int_0^{+\infty} |N(t,x)| \, dx + \int_{-\infty}^0 |N(t,x)| \, dx$ and estimate each term separately. By (2.26), we obtain on the one hand

$$\begin{aligned} \int_0^{+\infty} |N(t,x)| \, dx &\le \frac{C}{t} \int_0^{+\infty} e^{-\frac{x^2}{4t}} \left(\frac{x}{\sqrt{t}} + 1 \right) \, dx \\ &\le \frac{C}{\sqrt{t}} \int_0^{+\infty} e^{-y^2} (y+1) \, dy \le \frac{C}{\sqrt{t}}. \end{aligned} \tag{2.30}$$

On the other hand, using (2.28), it follows that for $t \le 1$,

$$\begin{aligned} \int_{-\infty}^0 |N(t,x)| \, dx &\le \frac{C}{t} \int_{-\infty}^0 e^{-\frac{x^2}{4t}} \left(\frac{|x|}{\sqrt{t}} + 1 \right) \, dx \\ &\le \frac{C}{\sqrt{t}} \int_0^{+\infty} e^{-y^2} (y+1) \, dy \le \frac{C}{\sqrt{t}}, \end{aligned} \tag{2.31}$$

while for $t > 1$, by (2.29)

$$\begin{aligned}\int_{-\infty}^{0} |N(t,x)|\, dx &\le \frac{C}{t}\int_{-\infty}^{0} e^{-\frac{x^2}{4t}}\left[1+\frac{|x|}{\sqrt{t}}+\frac{|x|^2}{t^{3/2}}\right]\, dx \\ &\le \frac{C}{\sqrt{t}}\int_{0}^{+\infty} e^{-y^2}\left(1+y+\frac{y}{\sqrt{t}}\right)\, dy \le \frac{C}{\sqrt{t}}.\end{aligned} \tag{2.32}$$

The L^1 estimate follows and the L^2 estimate is obtained using the Cauchy-Schwarz inequality. □

We now complete the proof of Proposition 2.7. We pretend that u defined as

$$u = N \circledast f\,(t,x) = \int_0^t \int_{\mathbb{R}} N(s,y) f(t-s, x-y)\, dy\, ds, \tag{2.33}$$

is the solution for (2.18). Let us prove that $u(t,.)$ is a continuous mapping from $\mathbb{R}_+$ to $L^2_x(\mathbb{R})$. Due to the estimate in Lemma 2.9, we have, for $t \to t_0$,

$$\begin{aligned}&||u(t,.) - u(t_0,.)||_{L^2_x} \\ &\le \int_0^{t_0} ||N(s,.)||_{L^2_x}(||f(t-s,.) - f(t_0-s,.)||_{L^1_x})ds \\ &+|\int_{t_0}^{t} ||N(s,.)||_{L^2_x}(||f(t-s,.)||_{L^1_x}ds| \\ &\le \int_0^{t_0} \frac{c}{s^{\frac{3}{4}}}(||f(t-s,.) - f(t_0-s,.)||_{L^1_x})ds + o(1).\end{aligned} \tag{2.34}$$

Since the kernel $s^{-\frac{3}{4}}$ is integrable, then it is an exercise to pass to the limit. We now check that u_t is a continuous function which takes values in $H^{-2}_x(\mathbb{R})$. Introducing, for any given ξ, $v(t) = \widehat{u}(t,\xi)$ the Fourier transform of u in x, we observe that v is solution for

$$v_t + \frac{1}{\sqrt{\pi}}\int_0^t \frac{v_t(s)}{\sqrt{t-s}}ds = \widehat{f}(t,\xi) + (\xi^2 - i\xi)\widehat{u}(t,\xi). \tag{2.35}$$

We now apply the formula (2.6) which provides us with a C^1 function v. Moreover u_t is a continuous function which takes values in $H^{-2}_x(\mathbb{R})$, since

$$\int_{\mathbb{R}} (1+|\xi|^2)^{-2}|\widehat{f}(t,\xi) + (\xi^2 - i\xi)\widehat{u}(t,\xi)|^2 d\xi < +\infty.$$

The proof of Proposition 2.7 is then complete. □

2.4. **Solving the nonlinear equation.** We finally address the nonlinear model (1.7) when $\nu = 1$. We seek u as the superposition of a solution of an homogeneous linear problem and a fixed point for the solution of the linear inhomogeneous problem; eventually, we solve together these two problems. Working as in the linear case, we discover that u must solve

$$u = K \star u_0 + N \circledast u^2, \tag{2.36}$$

where K and N are as above. Loosely speaking, the kernel K and N satisfy the same L^p estimates that the operators involved in the Burgers equation

$$u_t - u_{xx} + uu_x = 0, \tag{2.37}$$

that reads in its Duhamel's form

$$u = e^{t\Delta}u_0 - \frac{1}{2}\int_0^t \partial_x(e^{(t-s)\Delta})(u^2(s))ds. \tag{2.38}$$

It is the standard to prove by the fixed point theorem that for a given u_0 in $L^2_x(\mathbb{R})$ there exists a unique (local in time) solution $u(t) \in C(0,T;L^2_x(\mathbb{R}))$. We use Lemma 2.6 and Lemma 2.9 to obtain that

$$||K \star u_0||_{L^2_x} \leq c||u_0||_{L^2_x}, \tag{2.39}$$

and

$$||N \star u^2||_{L^2_x} \leq \frac{C}{t^{\frac{3}{4}}}||u^2||_{L^1_x}, \tag{2.40}$$

and then, integrating in time,

$$||N \circledast u^2||_{L^\infty(0,T;L^2_x)} \leq CT^{\frac{1}{4}}||u||^2_{L^\infty(0,T;L^1_x)}. \tag{2.41}$$

Choosing $R_0 = 2c||u_0||_{L^2_x}$, we then perform a fixed point in the ball of radius R_0 in $C([0,T];L^2_x)$ for $TR_0^4 \sim \frac{1}{2}$.

2.5. **Proof of the main Theorem.** We now prove that this local in time solution extends to a global one if we assume that u_0 is small enough in $L^1_x(\mathbb{R})$, exactly as for the Burgers equation.

Let X denote the Banach space of functions $v \in C(\mathbb{R}_+;L^2_x(\mathbb{R}))$ such that

$$\|v\|_X := \sup_{t>0} t^{1/4}\|v(t,\cdot)\|_{L^2_x(\mathbb{R})} < +\infty$$

and let $\mathcal{N} : X \to X$ defined by

$$\mathcal{N}u(t,x) = K(t,\cdot) \star u_0 + N \circledast u^2.$$

Then, any fixed point of $\mathcal{N}$ is a solution of (1.7) (in which $\nu = 1$) and it suffices to apply the Fixed Point Theorem in a suitable ball $\mathcal{B}_R(v_0) \subset X$. To this end, we note that by Lemma 2.6, we have

$$\|K(t,\cdot) \star u_0\|_{L^2_x(\mathbb{R})} \leq \|K(t,\cdot)\|_{L^2_x(\mathbb{R})}\|u_0\|_{L^1_x(\mathbb{R})} \leq \frac{C}{t^{1/4}}\|u_0\|_{L^1_x(\mathbb{R})}. \tag{2.42}$$

Hence, $v_0 := K(t,\cdot) \star u_0 \in X$. By Lemma 2.9, we have for $u \in X$,

$$\begin{aligned}
\|N \circledast u^2(t)\|_{L^2_x(\mathbb{R})} &= \left\| \int_0^t N(t-s,\cdot) \star u^2(s,\cdot)\, ds \right\|_{L^2_x(\mathbb{R})} \\
&\le \int_0^t \|N(t-s)\|_{L^2_x(\mathbb{R})} \|u^2\|_{L^1_x(\mathbb{R})}\, ds \\
&\le C \int_0^t \frac{1}{(t-s)^{3/4}} \|u\|^2_{L^2_x(\mathbb{R})}\, ds \qquad (2.43) \\
&\le C \int_0^t \frac{1}{s^{1/2}(t-s)^{3/4}} \left(s^{1/4} \|u(s,\cdot)\|_{L^2_x(\mathbb{R})} \right)^2\, ds \\
&\le \frac{C}{t^{1/4}} \|u\|^2_X.
\end{aligned}$$

Gathering (2.42) and (2.43), we obtain

$$\|\mathcal{N}u\|_X \le C \left(\|u\|^2_X + \|u_0\|_{L^1_x(\mathbb{R})} \right).$$

Choosing now $R > 0$, $\|u_0\|_{L^1_x(\mathbb{R})}$ so small that $R \ge C(R^2 + \|u_0\|_{L^1_x(\mathbb{R})})$, we deduce that $\mathcal{N}$ maps the ball $\mathcal{B}_R \subset X$ into itself. It remains to prove that $\mathcal{N}$ is contractive in $\mathcal{B}_R$. Let $u, v \in \mathcal{B}_R$. Then,

$$\begin{aligned}
\|(N \circledast u^2 - N \circledast v^2)(t)\|_{L^2_x(\mathbb{R})} &= \left\| \int_0^t N(t-s,\cdot) \star \left(u^2(s,\cdot) - v^2(s,\cdot) \right)\, ds \right\|_{L^2_x(\mathbb{R})} \\
&\le \int_0^t \|N(t-s)\|_{L^2_x(\mathbb{R})} \|u^2 - v^2\|_{L^1_x(\mathbb{R})}\, ds \\
&\le C \int_0^t \frac{1}{(t-s)^{3/4}} \|u+v\|_{L^2_x(\mathbb{R})} \|u-v\|_{L^2_x(\mathbb{R})}\, ds \\
&\le C \int_0^t \frac{1}{(t-s)^{3/4}} s^{-1/2} \|u+v\|_X \|u-v\|_X\, ds \\
&\le \frac{C}{t^{1/4}} \|u+v\|_X \|u-v\|_X.
\end{aligned}$$

Hence,

$$\|\mathcal{N}u - \mathcal{N}v\|_X \le CR\|u-v\|_X$$

i.e. $\mathcal{N}$ is a contraction in B_R for small $R > 0$.

We now prove the $L^\infty_x(\mathbb{R})$ estimate given in Theorem 1.4. On the one hand, due to the L^∞ estimate in Lemma 2.6

$$||K \star u_0||_{L^\infty_x(\mathbb{R})} \le \frac{c}{t^{\frac{1}{2}}} ||u_0||_{L^1_x(\mathbb{R})}. \qquad (2.44)$$

On the other hand, due to Lemma 2.9

$$\begin{aligned}
||N(t-s) \star u^2(s)||_{L^\infty_x(\mathbb{R})} &\le ||N(t-s)||_{L^2_x(\mathbb{R})} ||u^2(s)||_{L^2_x(\mathbb{R})} \\
&\le \frac{c}{(t-s)^{\frac{3}{4}}} ||u(s)||_{L^2_x(\mathbb{R})} ||u(s)||_{L^\infty_x(\mathbb{R})}. \qquad (2.45)
\end{aligned}$$

Introducing $M_2(t) = \sup_{s\le t}(s^{\frac14}||u(s)||_{L^2_x(\mathbb{R})})$ and $M_\infty(t) = \sup_{s\le t}(s^{\frac12}||u(s)||_{L^\infty_x(\mathbb{R})})$, we then have

$$t^{\frac12}||u(t)||_{L^2_x(\mathbb{R})} \le c||u_0||_{L^1_x(\mathbb{R})} + (\int_0^t \frac{ct^{\frac12}ds}{(t-s)^{\frac34}s^{\frac34}})M_2(t)M_\infty(t). \tag{2.46}$$

Since $M_2(t) \le 2||u_0||_{L^1_x(\mathbb{R})}$ and since u_0 is small in $L^1_x(\mathbb{R})$, then the last term in the right hand side of (2.46) is bounded by above by $\frac12 M_\infty(t)$ and moves to the left hand side. This completes the proof of the Theorem. □

Remark 2.10. *An open issue is to remove the smallness assumption on u_0.*

3. Numerical computation

3.1. **The scheme.** Numerical simulations are performed on the equation

$$u_t + \frac{\sqrt{\nu}}{\sqrt{\pi}}\int_0^t \frac{u_t(s)}{\sqrt{t-s}}ds = f = \alpha u_{xx} - u_x - \beta u_{xxx} - \gamma u u_x. \tag{3.1}$$

Here we have introduced parameters α, β and γ that will vary with the computations which allow us to observe the effect of each term, namely the viscous diffusion, the geometric dispersion and the nonlinearity.

We consider a large interval of $\mathbb{R}$ and we work with periodic boundary conditions in space. The space approximation of the solutions was performed by standard Fourier methods. Since we perform the numerics with an initial data that provides us with a wave that moves to the right boundary, we expect our computations to be physically relevant until this wave reaches the right boundary.

We now explain how to advance in time for the equation (3.1) (or its space approximation with Fourier series). The procedure is based on the integral form (2.6) of the solution. We then have

$$u(t) = u_0 + \int_0^t N_0(\nu(t-s))f(s)ds, \tag{3.2}$$

with N_0 defined in (2.5), ν appears in the kernel and $f = \alpha u_{xx} - u_x - \beta u_{xxx} - \gamma u u_x$. Introduce now a time step δ and set $t_n = n\delta$,

$$\begin{aligned} u(t_{n+1}) - u(t_n) = &\int_0^{t_n} (N_0(\nu(t_{n+1}-s)) - N_0(\nu(t_n - s)))f(s)ds \\ &+ \int_{t_n}^{t_{n+1}} N_0(\nu(t_{n+1}-s))f(s)ds. \end{aligned} \tag{3.3}$$

The first term in the r.h.s of (3.3) is approximated by the following quadrature formula, with $t_{k+\frac12} = (k+\frac12)\delta$,

$$\begin{aligned} &\int_0^{t_n} (N_0(\nu(t_{n+1}-s)) - N_0(\nu(t_n-s)))f(s)ds \approx \\ &\frac{\delta}{2}\sum_{k=0}^{n-1}\Big(N_0(\nu(t_{n+1}-t_{k+\frac12})) - N_0(\nu(t_n - t_{k+\frac12}))\Big)\left(f(t_{k+1}) + f(t_k)\right). \end{aligned} \tag{3.4}$$

Straightforwardly, this method is order 2 in time. The linear part $f_{lin} = \alpha u_{xx} - u_x - \beta u_{xxx}$ of the last term is handled in a similar manner, namely

$$\int_{t_n}^{t_{n+1}} N_0(\nu(t_{n+1}-s))f_{lin}(s)ds \approx \frac{\delta}{2}N_0(\frac{\nu\delta}{2})\left(f_{lin}(t_{n+1}) + f_{lin}(t_n)\right).$$

and the nonlinear part $f_{nl} = -\gamma u u_x$ is treated explicitly, i.e.

$$\int_{t_n}^{t_{n+1}} N_0(\nu(t_{n+1}-s))f_{nl}(s)ds \approx \delta N_0(\frac{\nu\delta}{2})f_{nl}(t_n). \tag{3.5}$$

This is only order 1 in time, but since it matters only on an interval of width δ, this does not affect the order of the scheme. Let u^n be the approximate value of $u(n\delta)$, the scheme reads,

$$\begin{aligned} &\frac{u^{n+1}-u^n}{\delta} = \\ &\frac{1}{2}\sum_{k=0}^{n-1}\left(N_0(\nu t_{n-k+\frac{1}{2}}) - N_0(\nu t_{n-k-\frac{1}{2}})\right)\left(f(t_{k+1}) + f(t_k)\right) \\ &+ N_0(\frac{\nu\delta}{2})\left(\frac{1}{2}f_{lin}(t_{n+1}) + \frac{1}{2}f_{lin}(t_n) + f_{nl}(t_n)\right). \end{aligned} \tag{3.6}$$

The scheme is semi-implicit, since the nonlinearity is computed explicitly.

3.2. **The numerical results.** In all the computations presented below, the initial data is $u_0(x) = 0.32*\text{sech}^2(0.4*(x-x_0))$, where x_0 is the middle of the interval. This initial datum provides a small amplitude and long wave KdV soliton for $\alpha = \nu = 0$, $\beta = 1$ and $\gamma = 6$. For the numerics, the stepsizes are: $h = 0.2$ (space step discretization) and $\delta = 0.2$ (time step discretization).

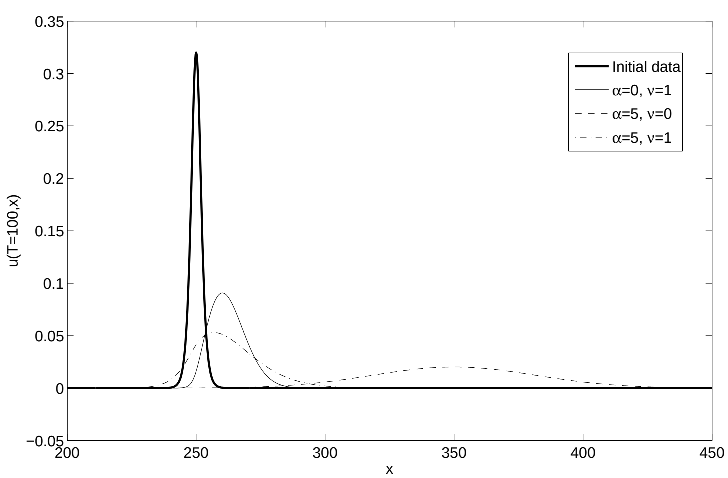

FIGURE 2. Solutions at time $T = 100$ for different viscosity (ν equal to 0 and 1, α equal to 0 and 5 and $\gamma = \beta = 0$).

In Figure 2, we observe the effects of the local and non-local viscous terms in the linear case ($\gamma = 0$) for $(\alpha, \nu) = (0, 1)$, $(\alpha, \nu) = (5, 0)$ and $(\alpha, \nu) = (5, 1)$, which correspond to cases with only local viscous term, only non-local viscous term and with both terms. The solutions are plotted at time $T = 100$ and the effects of these two terms are quite different!

We first note that when $(\alpha, \nu) = (0, 0)$, the solution of the linear wave equation is a travelling wave with speed 1. So the solution at $T = 100$ would be the same shape of wave, but centered at 350. Comparing that with the case that $(\alpha, \nu) = (0, 1)$, we see the local viscous term slows the wave down significantly and also at the same time, enlarge the wave length. On the other hand, by comparing the cases $(\alpha, \nu) = (0, 0)$ and $(\alpha, \nu) = (5, 0)$, we see the non-local viscous term also enlarge the wave length, but keeps the speed of the wave the same. When both viscous terms are involved in the simulation, the wave profile is more close to the case with only local viscous term.

We now plan to observe numerically the results of the main theorem and obtain some quantitative insight on the decay of the solutions. Furthermore, we will investigate the cases where theoretical results are not available. For this purpose, we study the decay of the solutions for the L^∞ and the L^2 norm of (3.1), on the interval $(0, 1000)$, when the viscosity coefficients $(\alpha, \nu) = (0, 0.1), (0.1, 0)$ and $(0.1, 0.1)$, $\beta = 0$ and γ equal to 0 (linear) and 1 (non linear).

Since the expected decay is of the form $O(t^a)$, Figure 3 (resp. Figure 4) shows the $\dfrac{\log \frac{\|u(t+\delta,\cdot)\|_{L^\infty_x}}{\|u(t,\cdot)\|_{L^\infty_x}}}{\log \frac{t+\delta}{t}}$ (resp. $\dfrac{\log \frac{\|u(t+\delta,\cdot)\|_{L^2_x}}{\|u(t,\cdot)\|_{L^2_x}}}{\log \frac{t+\delta}{t}}$) versus the time t for the linear problem (γ equal to 0).

From Fig. 3, one observe the local dissipative term produce a bigger decay rate when compared with the nonlocal dissipative term. The decay rates in all three cases appear to approach 0.5, but the convergence rate is quite small. It is worth to note that our theoretical result does not cover the second case, namely the case with $(\alpha, \nu) = (0.1, 0)$. The Fig. 4 is for L^2-norm, instead of L^∞-norm and the results are similar. Similar computations are performed with $\gamma = 1$ (the nonlinear case).

Norm	$\alpha = 0, \nu = 0.1$		$\alpha = 0.1, \nu = 0$		$\alpha = 0.1, \nu = 0.1$	
	$\gamma = 0$	$\gamma = 1$	$\gamma = 0$	$\gamma = 1$	$\gamma = 0$	$\gamma = 1$
L^∞	-0.51	-0.51	-0.49	-0.49	-0.51	-0.50
L^2	-0.27	-0.28	-0.24	-0.25	-0.27	-0.28

TABLE 1. Decay rate of the solution $u(t, \cdot)$ versus the time (ν and α equal to 0 and 0.1, $\beta = 0$, $\gamma = 0$ and 1).

We also computed the decay rate a by a least square method, for each of the two norms using the data from $[T - 200, T]$. The results are given in

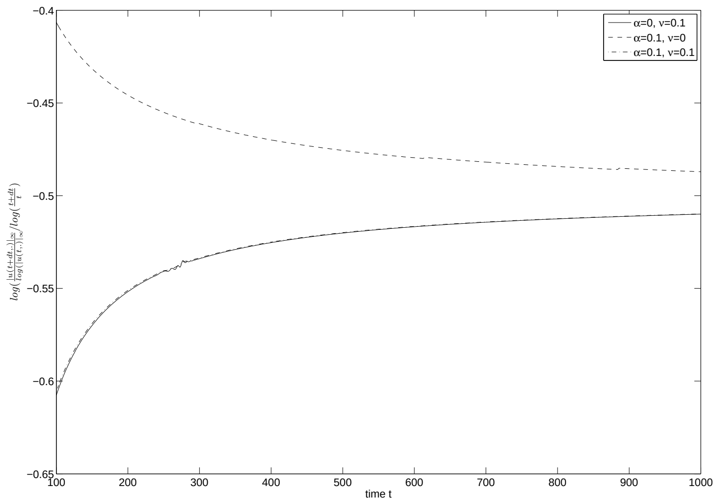

FIGURE 3. Decay of the solution $\frac{\log\frac{\|u(t+\delta,\cdot)\|_{L^\infty_x}}{\|u(t,\cdot)\|_{L^\infty_x}}}{\log\frac{t+\delta}{t}}$ versus the time $(\alpha,\nu)=(0,0.1),(0.1,0),(0.1,0.1)$, $\beta=0$, $\gamma=0$). The first and the third curves overlap each other.

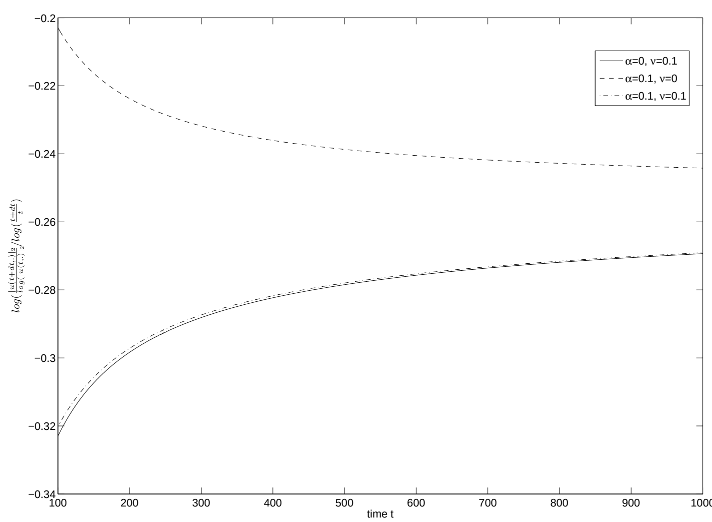

FIGURE 4. Decay of the solution $\frac{\log\frac{\|u(t+\delta,\cdot)\|_{L^2_x}}{\|u(t,\cdot)\|_{L^2_x}}}{\log\frac{t+\delta}{t}}$ versus the time t (ν and α equal to 0 and 0.1, $\beta=0$, $\gamma=0$). The first and the third curves overlap each other.

table 1. These results match the theoretical results given in Theorem 1.4 for the cases where theoretical results are available. We can also observe that there is no significant difference between the linear and the non linear case. Moreover, the decay of the solution is the same when $\alpha = 0.1$ or $\nu = 0.1$.

In the final sequence of computations, we plan to study the different effects of dispersion and diffusion. We consider now the full equation (1.3), where the geometric dispersive term u_{xxx} plays a role. When there is no viscosity ($\alpha = \nu = 0$), the exact solution of the problem is the soliton $u(t,x) = u_0(x - 1.64 * t)$. In Figure 5, we compare solutions from (3.1) with different set of coefficients. The solutions with $(\alpha, \nu, \beta, \gamma) = (0, 0, 6, 1)$-the KdV equation, $(\alpha, \nu, \beta, \gamma) = (0.1, 0, 6, 1), (0, 0.1, 6, 1), (0.1, 0.1, 6, 1)$ and the exact KdV are plotted. Again, the local dissipative term slows the wave down. The local dispersive term might contribute to the appearance of the double hump.

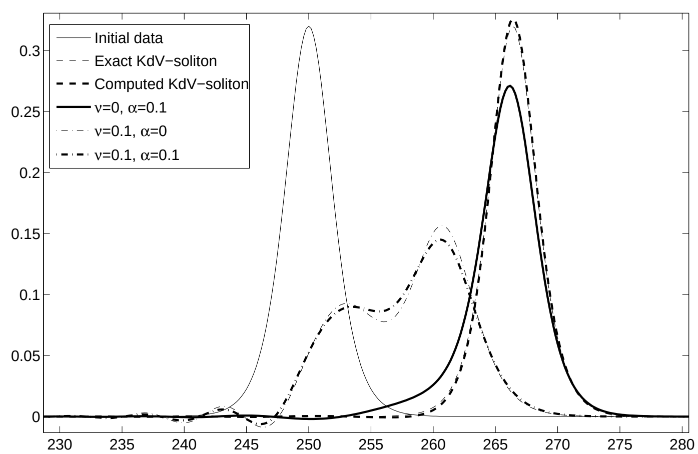

FIGURE 5. Solution at time $T = 10$ (ν and α equal to 0 and 0.1, $\beta = 6$, $\gamma = 1$).

We now investigate numerically the decay rate of the solutions when the local dispersion u_{xxx} term is present, cases where the theoretical result in Section 2 do not cover. Figure 6 (resp. Figure 7) shows the $\frac{\log \frac{\|u(t+\delta,\cdot)\|_{L^\infty_x}}{\|u(t,\cdot)\|_{L^\infty_x}}}{\log \frac{t+\delta}{t}}$ (resp. $\frac{\log \frac{\|u(t+\delta,\cdot)\|_{L^2_x}}{\|u(t,\cdot)\|_{L^2_x}}}{\log \frac{t+\delta}{t}}$) versus the time t.

Eventually, we are led with the use of least square method to the results given in Table 2.

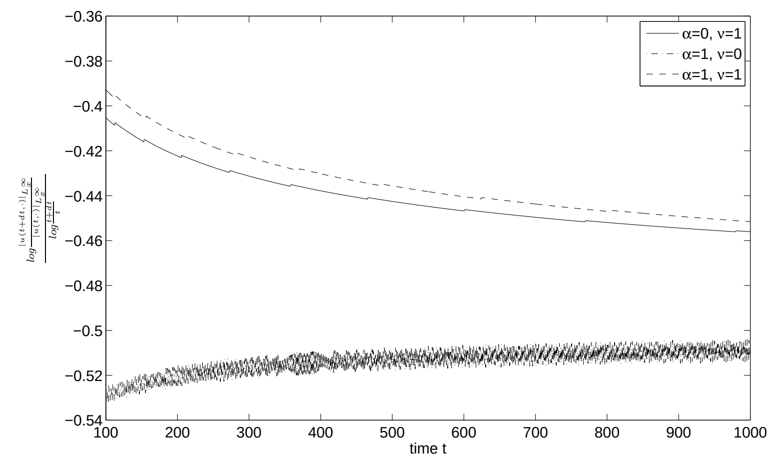

FIGURE 6. Decay of the solution $\frac{\log \frac{\|u(t+\delta,\cdot)\|_{L^\infty_x}}{\|u(t,\cdot)\|_{L^\infty_x}}}{\log \frac{t+\delta}{t}}$ versus the time t (ν and α equal to 0 and 1, $\beta = 6$, $\gamma = 1$).

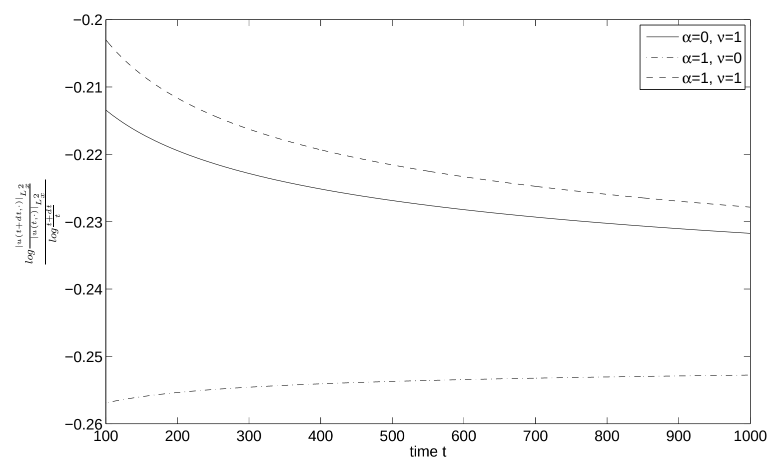

FIGURE 7. Decay of the solution $\frac{\log \frac{\|u(t+\delta,\cdot)\|_{L^2_x}}{\|u(t,\cdot)\|_{L^2_x}}}{\log \frac{t+\delta}{t}}$ versus the time t (ν and α equal to 0 and 1, $\beta = 6$, $\gamma = 1$).

Norm	$\alpha = 0,\ \nu = 1$	$\alpha = 1,\ \nu = 0$	$\alpha = 1,\ \nu = 1$
L^∞	-0.46	-0.51	-0.45
L^2	-0.23	-0.25	-0.23

TABLE 2. Decay rate of the solution $u(t,\cdot)$ versus the time (ν and α equal to 0 and 1, $\beta = 6$, $\gamma = 1$).

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