Asymmetrical periodic travelling wave patterns of two-dimensional Boussinesq systems

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Abstract

We consider a Boussinesq system which describes three-dimensional water waves in fluid layers with a depth small with respect to the wave length. We prove the existence of a large family of bifurcating bi-periodic patterns of travelling waves, which are *non symmetric with respect to the direction of propagation*. Up to now, the existence of bifurcating asymmetrical bi-periodic travelling wave is still an open problem for the Euler equation (potential flow, without surface tension).

Here the lattice of wave vectors is spanned by two vectors $\mathbf{k}_1, \mathbf{k}_2$ of non equal lengths, and the direction of propagation of the waves is close to the critical one (solution of the dispersion equation). The wave pattern may be understood at main order as the superposition of two plane waves of different amplitudes, respectively propagating along directions $\mathbf{k}_1$ and $\mathbf{k}_2$.

Our class of non symmetric waves bifurcates from a 3-dimensional set of parameters which come from the components of the two basic wave vectors, constrained by the dispersion equation. Here we are able to escape from the *small divisor problem* in restricting the study with *one rationality condition* relating the bifurcation set and the direction of propagation close to the critical direction. However, we need to solve a problem of lack of smoothness with respect to the propagation direction, of the pseudo-inverse of the linearized operator. The rationality condition influences mildly the domain of existence of the bifurcating waves. This theory also applies when the lattice is built with wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$ of equal lengths with the bisector direction as the critical propagation direction. In such a case, the parameter set is two-dimensional and there is still one rationality condition for the bifurcating asymmetrical waves which propagate in a direction making a small angle with the bisector of $\mathbf{k}_1$, $\mathbf{k}_2$.

Examples of wave patterns for $\mathbf{k}_1$, $\mathbf{k}_2$ of equal or different length, with various amplitude ratios along the two basic wave vectors, and with

various angles between the traveling direction and the critical direction, are shown in the last section of the paper.

1 Introduction

We consider the following Boussinesq system

$$\begin{aligned} \eta_t + \nabla\cdot\mathbf{v} + \nabla\cdot(\eta\mathbf{v}) - \frac{1}{6}\Delta\eta_t &= 0, \\ \mathbf{v}_t + \nabla\eta + \frac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) - \frac{1}{6}\Delta\mathbf{v}_t &= 0, \end{aligned} \tag{1}$$

which was put forward by Bona, Colin, Lannes [2], and describe small-amplitude and long wavelength (the depth is small with respect to wave length) gravity waves of an ideal, incompressible liquid. Here the horizontal coordinate $\mathbf{x}$ and time t are scaled by h_0 and $\sqrt{h_0/g}$, with g being the acceleration of gravity and h_0 being the average water depth. The elevation of waves $\eta(\mathbf{x},t)$ and the horizontal velocity $\mathbf{v}(\mathbf{x},t)$ at the level of $\sqrt{2/3}h_0$ of the depth of the undisturbed fluid, are scaled by h_0 and $\sqrt{gh_0}$ respectively. The derivation of (1) is similar to its one-dimensional version, which is given in detail in [1].

We are interested in travelling waves of constant velocity $\mathbf{c}$ which have a periodic horizontal pattern in $\mathbf{x}\in\mathbb{R}^2$. In the paper [6] we considered diamond patterns Γ spanned by wave vectors $\mathbf{k}_1,\mathbf{k}_2$ having the same length and proved the existence of symmetric solutions, propagating in the direction of the bisector of the wave vectors, bifurcating from 0, for which the amplitudes ε_1 and ε_2 along the basic wave vectors are equal. On the system above we managed to apply a Lyapunov-Schmidt method, impossible to manage on the physical problem ruled by the full Euler equations without surface tension, due to a small divisor problem (see [7]).

In the present work we consider asymmetrical waves as experimentally shown by Hammack et al in [5]. *Assuming the presence of surface tension*, asymmetrical waves were theoretically obtained with the full Euler equation by Craig and Nicholls in [3] (numerically sketched on page 631) in using Lyapunov Schmidt reduction, and by Groves and Haragus in [4], with the use of spatial dynamics theory. As in [3] and [4] these waves may result from a choice of pattern Γ spanned by two wave vectors $\mathbf{k}_1,\mathbf{k}_2$ having different lengths. They may also result from a pattern Γ spanned by two wave vectors $\mathbf{k}_1,\mathbf{k}_2$ having the same length, but with *different amplitudes* $\varepsilon_1,\varepsilon_2$ along these basic waves vectors. Our main result is given in Theorem 3. In both cases, to avoid a small divisor problem, we restrict the study with *one rationality condition* between the bifurcation set and the direction of propagation w close to the critical direction (given by the dispersion relation). *This condition influences mildly the domain of existence* of the bifurcating waves in allowing an existence domain of the order $(\ln s)^{-1}$, where s is the denominator of a rational number close to the ratio of lengths of the projections of $\mathbf{k}_1$ and $\mathbf{k}_2$ along the critical propagation direction. In the case when the waves propagate in the critical direction our restriction only bears on the rationality of the ratio of the projections of $\mathbf{k}_1$ and $\mathbf{k}_2$ on this direction. This theory also applies when the lattice is built with wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$ of equal lengths with the bisector direction as the critical propagation direction. In such a case, the parameter set is two-dimensional and there is still one rationality condition for the bifurcating asymmetrical waves which propagate in a direction making a small angle w with the bisector of $\mathbf{k}_1$, $\mathbf{k}_2$.

Despite the *non smoothness in the wave angle parameter* w of the linear pseudo-inverse operator (see Lemma 2) , we are able to use a Lyapunov-Schmidt method for solving the bifurcation problem. This corresponds in solving a still open situation on the physical problem originally ruled by Euler equations, without surface tension. We show in the last section several patterns of traveling asymmetrical waves computed with the explicit expression of the elevation for the terms of order 1 and 2 in amplitudes $(\varepsilon_1, \varepsilon_2)$.

2 Position of the problem

We are looking for solutions of system (1) under the form of 2-dimensional travelling waves, i.e. η and $\mathbf{v}$ are functions of $\mathbf{x} - \mathbf{c}t$, where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, and $\mathbf{c}$ is the velocity of the travelling wave, which plays the role of a parameter. For these solutions, the system (1) reads as

$$\begin{aligned} \nabla \cdot (\mathbf{v} + \eta \mathbf{v}) - \mathbf{c} \cdot \nabla(\eta - \frac{1}{6}\Delta\eta) = 0, \\ \nabla(\eta + \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})) - \mathbf{c} \cdot \nabla(\mathbf{v} - \frac{1}{6}\Delta\mathbf{v}) = \mathbf{0}, \end{aligned} \tag{2}$$

where we assume that $curl(\mathbf{v}) = 0$, as it is shown to be consistent in [6]. We consider in what follows *periodic solutions* with Fourier expansions of the form

$$\begin{aligned} \eta(\mathbf{x}) &= \sum_{\mathbf{k}\in\Gamma} \eta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \\ \mathbf{v}(\mathbf{x}) &= \sum_{\mathbf{k}\in\Gamma} \mathbf{v}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \tag{3}$$

where Γ is a lattice of the plane defined by two noncolinear vectors $\mathbf{k}_1, \mathbf{k}_2$. This means that

$$\mathbf{k} \in \boldsymbol{\Gamma} : \mathbf{k} = (k_1, k_2) = n_1\mathbf{k}_1 + n_2\mathbf{k}_2, \quad n_1, n_2 \in \mathbb{Z}, \tag{4}$$

and since the unknown $(\eta, \mathbf{v})$ is such that $curl(\mathbf{v}) = 0$, we have

$$\mathbf{v}_{\mathbf{k}} \times \mathbf{k} = \mathbf{0}.$$

For simplicity, we require $\mathbf{v_0} = \mathbf{0}$, $\eta_{\mathbf{0}} = 0$, so the averages of the elevation η and of the horizontal velocity are set to be zero. One might treat the nonzero case as for the symmetric doubly periodic wave pattern (c.f. [6]), but this introduces 3 additional parameters which appear to not change the results qualitatively.

Let us define the basis $\{\mathbf{k}_1, \mathbf{k}_2\}$ of the lattice Γ :

$$\mathbf{k}_1 = l_1(1, \tau_1), \quad \mathbf{k}_2 = l_2(1, -\tau_2), \quad l_j, \tau_j > 0, \quad j = 1, 2$$

where $\tau_j = \tan\theta_j$. We then have for $\mathbf{k} = (k_1, k_2) = n_1\mathbf{k}_1 + n_2\mathbf{k}_2$

$$k_1 = n_1 l_1 + n_2 l_2, \quad k_2 = n_1\tau_1 l_1 - n_2\tau_2 l_2. \tag{5}$$

The lattice Γ *makes a diamond pattern if we choose* $\mathbf{k}_1, \mathbf{k}_2$ *symmetric with respect to the* x_1 *axis*, making an angle $\pm\theta$ with this axis. In such a case, we have

$$\begin{array}{rcl} l_1 &=& l_2 \stackrel{def}{=} l, \\ \tau_1 &=& \tau_2 \stackrel{def}{=} \tau, \\ \theta_1 &=& \theta_2 \stackrel{def}{=} \theta. \end{array}$$

Now we define the Sobolev space

$$H^p_{\natural\natural} \stackrel{def}{=} \{u = \sum_{\mathbf{k}\in\Gamma} u_k e^{i\mathbf{k}\cdot\mathbf{x}} \in H^p\{\mathbb{R}^2/\Gamma'\},$$

where Γ' is the lattice of periods defined by

$$\Gamma' = \{n_1\boldsymbol{\lambda}_1 + n_2\boldsymbol{\lambda}_2 \in \mathbb{R}^2; \boldsymbol{\lambda}_j \cdot \mathbf{k}_n = 2\pi\delta_{jn}, \ j, n \in \{1, 2\}, (n_1, n_2) \in \mathbb{Z}^2\}. \tag{6}$$

Observe that any $u \in H^p_{\natural\natural}$ is invariant under the shift

$$\sigma : \mathbf{x} \mapsto \mathbf{x} + \boldsymbol{\lambda}_j.$$

The scalar product in $H^p_{\natural\natural}$ is the usual scalar product of the Sobolev space H^p on a periodic domain (parallelogram built with $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$). We notice that l_j *has to be chosen small enough* for the consistence of the Boussinesq model, where the horizontal wave lengths $|\boldsymbol{\lambda}_j|$ should be large with respect to 1 (which is the depth at rest of the fluid layer). The basic function space in our study is

$$G_p \stackrel{def}{=} \{U = (\eta, \mathbf{v}) \in H^p_{\natural\natural}\}^3 \cap \{curl(\mathbf{v}) = \mathbf{0}\} \cap \{\eta_{\mathbf{0}} = 0, \ \mathbf{v_0} = \mathbf{0}\},$$

and we reformulate the system (2) in the form

$$\mathcal{L}_{\mathbf{c}}U + \mathcal{G}\mathcal{N}(U, U) = \mathbf{0}, \tag{7}$$

where,

$$\mathcal{L}_{\mathbf{c}}U = \begin{pmatrix} \nabla\cdot\mathbf{v} - \mathbf{c}\cdot\nabla(\eta - \frac{1}{6}\Delta\eta) \\ \nabla\eta - \mathbf{c}\cdot\nabla(\mathbf{v} - \frac{1}{6}\Delta\mathbf{v}) \end{pmatrix}, \tag{8}$$

$$\mathcal{N}(U, U) = (\frac{1}{2}(\mathbf{v}\cdot\mathbf{v}), \eta\mathbf{v}), \quad \mathcal{G}(g, \mathbf{f}) = (\nabla\cdot\mathbf{f}, \nabla g).$$

It is clear that the linear maps

$$\begin{aligned}\mathcal{L}_{\mathbf{c}} &: G_p \to G_{p-3},\ p \geq 3 \\ \mathcal{G} &: G_p \to G_{p-1},\ p \geq 1\end{aligned}$$

are bounded, and that the quadratic map

$$\mathcal{N} : G_p \to G_p, \quad p \geq 2$$

is bounded ($p \geq 2$ is there for having the product of two functions of $H^p_{\natural\natural}$ in $H^p_{\natural\natural}$). Moreover, with the Hermitian scalar product of $\{H^0_{\natural\natural}\}^3$, we have after integration by parts, that for any U_1 and $U_2 \in G_p$, $p \geq 3$ (second identity valid for $p \geq 1$)

$$\begin{aligned}\langle \mathcal{L}_{\mathbf{c}} U_1, U_2 \rangle_{H^0} &= - \langle U_1, \mathcal{L}_{\mathbf{c}} U_2 \rangle_{H^0}, \\ \langle \mathcal{G} U_1, U_2 \rangle_{H^0} &= - \langle U_1, \mathcal{G} U_2 \rangle_{H^0}.\end{aligned} \tag{9}$$

The system (7) possesses important symmetries. By defining the bounded linear operators $\mathcal{T}_{\mathbf{v}}$, $\mathcal{S}_0$ as follows

$$\begin{aligned}(\mathcal{T}_{\mathbf{y}} U)(\mathbf{x}) &= U(\mathbf{x} + \mathbf{y}), \\ (\mathcal{S}_0 U)(\mathbf{x}) &= (\eta(-\mathbf{x}), \mathbf{v}(-\mathbf{x})),\end{aligned}$$

it is clear that we have the following commutation properties

$$\begin{aligned}\mathcal{T}_{\mathbf{y}} \mathcal{L}_{\mathbf{c}} =& \mathcal{L}_{\mathbf{c}} \mathcal{T}_{\mathbf{y}}, \quad \mathcal{T}_{\mathbf{y}} \mathcal{N}(U,U) = \mathcal{N}(\mathcal{T}_{\mathbf{y}} U, \mathcal{T}_{\mathbf{y}} U),\ \mathcal{T}_{\mathbf{y}} \mathcal{G} = \mathcal{G} \mathcal{T}_{\mathbf{y}}, \\ \mathcal{S}_0 \mathcal{L}_{\mathbf{c}} =& - \mathcal{L}_{\mathbf{c}} \mathcal{S}_0, \quad \mathcal{S}_0 \mathcal{N}(U,U) = \mathcal{N}(\mathcal{S}_0 U, \mathcal{S}_0 U),\ \mathcal{S}_0 \mathcal{G} = -\mathcal{G} \mathcal{S}_0,\end{aligned} \tag{10}$$

the first set of properties results from the invariance of the original system under the translations of the plane, while the second set comes from the reversibility of the original system.

In the case *when the lattice* Γ *has a diamond structure*, the wave vectors $\mathbf{k}_1, \mathbf{k}_2$ being symmetric with respect to the x_1- axis, then we have an additional symmetry: let us define the symmetry S_1 by

$$(\mathcal{S}_1 U)(\mathbf{x}) = (\eta(\hat{\mathbf{x}}), \hat{\mathbf{v}}(\hat{\mathbf{x}})),$$

where $\hat{\mathbf{x}}$ is the symmetric vector of $\mathbf{x}$ with respect to the x_1- axis: $\hat{\mathbf{x}} = (x_1, -x_2)$. It is clear that in the case when the velocity $\mathbf{c}$ of the wave is colinear to the x_1- axis, we have the following additional commutation properties

$$\mathcal{S}_1 \mathcal{L}_{\mathbf{c}} = \mathcal{L}_{\mathbf{c}} \mathcal{S}_1, \quad \mathcal{S}_1 \mathcal{N}(U,U) = \mathcal{N}(\mathcal{S}_1 U, \mathcal{S}_1 U),\ \mathcal{S}_1 \mathcal{G} = \mathcal{G} \mathcal{S}_1. \tag{11}$$

3 Study of the linearized operator

We start with the study of the linearized system

$$\mathcal{L}_{\mathbf{c}} U = P, \tag{12}$$

where

$$U = (\eta, \mathbf{v}), \;\; P = (q, \mathbf{p}) \in G_l, \quad l \geq 0.$$

The vector function $\mathbf{p}$ and scalar function q are periodic with Fourier series

$$\begin{aligned} \mathbf{p}(\mathbf{x}, t) &= \sum_{\mathbf{k} \in \Gamma} \mathbf{p}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \;\; \mathbf{p}_{\mathbf{0}} = 0, \;\; \mathbf{p}_{\mathbf{k}} \times \mathbf{k} = 0, \\ q(\mathbf{x}, t) &= \sum_{\mathbf{k} \in \Gamma} q_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad q_{\mathbf{0}} = 0, \end{aligned} \tag{13}$$

and we get for $\mathbf{k} \in \Gamma$

$$\begin{aligned} -(1 + \frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\eta_{\mathbf{k}} + \mathbf{k}\cdot\mathbf{v}_{\mathbf{k}} &= -\,iq_{\mathbf{k}}, \\ \mathbf{k}\eta_{\mathbf{k}} - (1 + \frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\mathbf{v}_{\mathbf{k}} &= -\,i\mathbf{p}_{\mathbf{k}}. \end{aligned} \tag{14}$$

Define

$$\Delta(\mathbf{k}, \mathbf{c}) = (1 + \frac{1}{6}|\mathbf{k}|^2)^2(\mathbf{c}\cdot\mathbf{k})^2 - |\mathbf{k}|^2, \tag{15}$$

the linearized operator $\mathcal{L}_{\mathbf{c}}$ has a nontrivial kernel in G_l if there exists a pair $(\mathbf{k}_0, \mathbf{c}_0)$ satisfying

$$\Delta(\mathbf{k}_0, \mathbf{c}_0) = 0 \text{ and } \mathbf{k}_0 \neq 0. \tag{16}$$

If $\Delta(\mathbf{k}, \mathbf{c}) \neq 0$, the solution of (14) reads

$$\eta_{\mathbf{k}} = i\,\frac{(1 + \frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})q_{\mathbf{k}} + \mathbf{k}\cdot\mathbf{p}_{\mathbf{k}}}{\Delta(\mathbf{k}, \mathbf{c})}, \tag{17}$$

$$\mathbf{v}_{\mathbf{k}} = i\,\frac{(1 + \frac{1}{6}|\mathbf{k}|^2)(\mathbf{c}\cdot\mathbf{k})\mathbf{p}_{\mathbf{k}} + q_{\mathbf{k}}\mathbf{k}}{\Delta(\mathbf{k}, \mathbf{c})}, \tag{18}$$

where we notice that

$$curl(\mathbf{v}_k e^{i\mathbf{k}\cdot\mathbf{x}}) = 0.$$

If $\mathbf{k} = 0$, $\mathbf{v}_{\mathbf{0}} = \eta_{\mathbf{0}} = 0$.

If $(\mathbf{k}\cdot\mathbf{c}) = 0$ and $\mathbf{k} \neq 0$, a special case of (17,18) leads to

$$\eta_{\mathbf{k}} = -i\,\frac{\mathbf{k}\cdot\mathbf{p}_{\mathbf{k}}}{|\mathbf{k}|^2}, \quad \mathbf{v}_{\mathbf{k}} = -i\,\mathbf{k}\frac{q_{\mathbf{k}}}{|\mathbf{k}|^2} \tag{19}$$

If $\Delta(\mathbf{k}, \mathbf{c}) = 0$, $\mathbf{k} \neq 0$, when $(\mathbf{p}_{\mathbf{k}}, q_{\mathbf{k}})$ satisfies the compatibility condition

$$sgn(\mathbf{k}\cdot\mathbf{c})\mathbf{k}\cdot\mathbf{p}_{\mathbf{k}} + |\mathbf{k}|q_{\mathbf{k}} = 0, \tag{20}$$

the solution reads

$$\begin{aligned} \eta_{\mathbf{k}} &= i\,sgn(\mathbf{k}\cdot\mathbf{c})\frac{q_{\mathbf{k}}}{|\mathbf{k}|} + |\mathbf{k}|\beta \\ \mathbf{v}_{\mathbf{k}} &= sgn(\mathbf{k}\cdot\mathbf{c})\mathbf{k}\beta \end{aligned} \tag{21}$$

where β is arbitrary in $\mathbb{C}$.

For having a chance to obtain bifurcating solutions we need to have a non-trivial kernel for the operator $\mathcal{L}_{\mathbf{c}}$ for critical values of the parameters. Hence we need to study the set of $\mathbf{k}$ in the plane, satisfying $\Delta(\mathbf{k}, \mathbf{c}) = 0$ for a given velocity $\mathbf{c}$, where $\mathbf{k}$ belongs to the lattice Γ. *We do not restrict the generality* in assuming that $\mathbf{c} = \mathbf{c}_0 = c_0(1, 0)$. The basic wave vectors $\mathbf{k}_1, \mathbf{k}_2$ need to be solutions of

$$\Delta(\mathbf{k}_j, \mathbf{c}_0) = 0, \quad j = 1, 2. \tag{22}$$

This means that

$$c_0^2 = \frac{1 + \tau_j^2}{\{1 + \frac{l_j^2}{6}(1 + \tau_j^2)\}^2}, \quad j = 1, 2 \tag{23}$$

i.e

$$\frac{1}{c_0^2} = \left(\cos\theta_1 + \frac{l_1^2}{6\cos\theta_1}\right)^2 = \left(\cos\theta_2 + \frac{l_2^2}{6\cos\theta_2}\right)^2, \quad 0 < \theta_j < \pi/2, \tag{24}$$

which leads to the relationship (automatically satisfied when we choose a diamond lattice Γ)

$$6(\cos\theta_1 - \cos\theta_2) = \frac{l_2^2}{\cos\theta_2} - \frac{l_1^2}{\cos\theta_1}, \tag{25}$$

which indicates that, for fixed angles θ_1, θ_2, the *point* (l_1, l_2) *(close to 0) needs to belong to a hyperlola in the plane.* The critical set in the 4-dimensional space $(\tau_1, \tau_2, l_1, l_2)$ is a 3-dimensional hypersurface (restricted to the quadrant $\tau_1, \tau_2, l_1, l_2 > 0$. When Γ is a diamond lattice, we only have two parameters (τ, l) for the critical set.

Replacing $\mathbf{k}$ by $n_1\mathbf{k}_1 + n_2\mathbf{k}_2$ in the equation $\Delta(\mathbf{k}, \mathbf{c}_0) = 0$, we obtain.

$$(1 + \frac{1}{6}|n_1\mathbf{k}_1 + n_2\mathbf{k}_2|^2)|\mathbf{c}\cdot(n_1\mathbf{k}_1 + n_2\mathbf{k}_2)| = |n_1\mathbf{k}_1 + n_2\mathbf{k}_2|, \tag{26}$$

or more precisely

$$\begin{aligned} 0 = &\left(1 + \frac{1}{6}\{(n_1l_1 + n_2l_2)^2 + (n_1\tau_1l_1 - n_2\tau_2l_2)^2\}\right)^2 c_0^2(n_1l_1 + n_2l_2)^2 + \\ &- \{(n_1l_1 + n_2l_2)^2 + (n_1\tau_1l_1 - n_2\tau_2l_2)^2\}, \end{aligned} \tag{27}$$

where we already know the solutions

$$(n_1, n_2) = (\pm1, 0), (0, \pm1).$$

It is now important to know how many solutions (n_1, n_2) does (26) have.

Let us assume that the scalars l_1 and l_2 are such that

$$\frac{l_1}{l_2} = \frac{r_0}{s_0} \in \mathbb{Q}^+, \tag{28}$$

where r_0, s_0 are mutually prime. This assumption allows to avoid $\mathbf{c}_0 \cdot \mathbf{k}$ to be small for large $|\mathbf{k}|$, when $\mathbf{c}_0 \cdot \mathbf{k} \neq 0$. Indeed, we have

$$\mathbf{c}_0 \cdot \mathbf{k} = c_0(n_1 l_1 + n_2 l_2) = \frac{c_0 l_2}{s_0}(n_1 r_0 + n_2 s_0),$$

i.e.

$$|\mathbf{c}_0 \cdot \mathbf{k}| \geq \frac{c_0 l_2}{s_0}$$

for any $(n_1, n_2) \neq 0$ in $\mathbb{Z}^2$ such that $\mathbf{c}_0 \cdot \mathbf{k} \neq 0$. It is then clear that, for $|\mathbf{k}| > K$ where

$$K = \frac{9 s_0}{c_0 l_2},$$

and for any $(n_1, n_2) \neq 0$ in $\mathbb{Z}^2$ (even when $\mathbf{c}_0 \cdot \mathbf{k} = 0$), we have the estimate

$$\left|(1 + \frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}_0 \cdot \mathbf{k}| - |\mathbf{k}|\right| > \frac{1}{2}|\mathbf{k}|,$$

which provides a lower bound for $|\Delta(\mathbf{k}, \mathbf{c}_0)|$. Notice that when Γ is a diamond lattice, we have $l_1 = l_2 = l$ and $s_0 = 1$.

In satisfying the identity (23), the critical set in the 4-dimensional space of parameters $(\tau_1, \tau_2, l_1, l_2)$ is a 3-dimensional hypersurface. Expressing c_0 in function of τ_1, l_1, for a fixed couple (n_1, n_2), the equation (27) represents a 2-dimensional sub-manifold, for instance (τ_1, τ_2) in function of (l_1, l_2). Then the intersection with a condition (28) provides a curve and if we are able to avoid, for all possible values of (n_1, n_2), to sit on these curves in the 3-dimensional critical hypersurface, except $(n_1, n_2) = (\pm 1, 0)$, $(n_1, n_2) = (0, \pm 1)$, then the *dimension of the kernel is just 4*. Now, we have (proved below with the estimate (43))

$$|\mathbf{k}| \geq d_1 (n_1^2 + n_2^2)^{1/2}.$$

This shows that $\mathbf{k} \in \Gamma$ such that $|\mathbf{k}| < K$, leads to

$$(n_1^2 + n_2^2)^{1/2} < \frac{K}{d_1},$$

where d_1 satisfies (43). In choosing a region in the parameter space where $d_1 > \delta > 0$, i.e. l_1 and l_2 outside a small ball, we only have a finite number of possible (n_1, n_2). Hence it is easy to avoid to sit on this finite number of curves on the 3-dimensional manifold given by (23).

More generally, if we do not consider the restriction (28), for the ratio l_1/l_2, the set of relations (27) is denumerable for all $(n_1, n_2) \in \mathbb{Z}^2$, which makes a denumerable set of 2-dimensional sub-manifolds of the 3-dimensional critical hypersurface. Then, there is a full measure set of choice of parameters $(\tau_1, \tau_2, l_1, l_2)$ in the 3-dimensional hypersurface, such that none of relations (27) is satisfied, except for $(n_1, n_2) = (\pm 1, 0)$, $(n_1, n_2) = (0, \pm 1)$. Finally, a general choice of parameters provides no solution of (26) except $\pm \mathbf{k}_1, \pm \mathbf{k}_2$.

Assume that (c_0, l_j, τ_j), $j = 1, 2$ are such that $\pm\mathbf{k}_j$, $j = 1, 2$ are the only non-trivial solutions in Γ of (26) (*the general case)* and let us define the eigenvectors $\xi_{\pm\mathbf{k}_j}$ by

$$\mathcal{L}_{\mathbf{c}_0}\xi_{\pm\mathbf{k}_j} = 0, \ \mathbf{c}_0 = (c_0, 0),$$
$$\xi_{\pm\mathbf{k}_j} = (\sqrt{1+\tau_j^2}, 1, (-1)^{j+1}\tau_j)e^{\pm i\mathbf{k}_j\cdot\mathbf{x}} \tag{29}$$

then we observe that with the Hermitian scalar product in $\{H^0_{\natural\natural}\}^3$ the compatibility condition (20) reads

$$\langle P, \xi_{\pm\mathbf{k}_j}\rangle_{H^0} = 0. \tag{30}$$

Moreover in operating the symmetries, we have

$$\mathcal{T}_{\mathbf{y}}\xi_{\pm\mathbf{k}_j} = \xi_{\pm\mathbf{k}_j}e^{\pm i\mathbf{k}_j\cdot\mathbf{y}}, \ \mathcal{S}_0\xi_{\pm\mathbf{k}_j} = \overline{\xi}_{\pm\mathbf{k}_j} = \xi_{\mp\mathbf{k}_j}. \tag{31}$$

In the case *when the lattice* Γ *has a diamond structure*, the wave vectors $\mathbf{k}_1, \mathbf{k}_2$ being symmetric with respect to the x_1- axis, we have in addition the following symmetry property

$$\mathcal{S}_1\xi_{\pm\mathbf{k}_1} = \xi_{\pm\mathbf{k}_2}. \tag{32}$$

The above calculations (see [6] for the complete proof of estimates) show that we are able to define an operator $\widetilde{\mathcal{L}}^{-1}_{\mathbf{c}_0}$, which is the pseudo-inverse of $\mathcal{L}_{\mathbf{c}_0}$, mapping G_p into G_{p+1} for any $p \geq 0$, solving (12) with $\mathbf{c}_0 = (c_0, 0)$, provided the compatibility condition (20) is satisfied, and such that

$$\begin{aligned} U &= \widetilde{\mathcal{L}}^{-1}_{\mathbf{c}_0}P, \\ \{\widetilde{\mathcal{L}}^{-1}_{\mathbf{c}_0}P\}_{\mathbf{k}} &= U_{\mathbf{k}} = (\eta_{\mathbf{k}}, \mathbf{v}_{\mathbf{k}}), \end{aligned}$$

where

- $\{\widetilde{\mathcal{L}}^{-1}_{\mathbf{c}_0}P\}_{\mathbf{k}} = (\eta_{\mathbf{k}}, \mathbf{v}_{\mathbf{k}})$ is given by (17), (18) for $\Delta(\mathbf{k}, \mathbf{c}_0) \neq 0$ i.e. for $\mathbf{k} \neq \pm\mathbf{k}_1, \pm\mathbf{k}_2$, and $\mathbf{k} \neq 0$,
- $\{\widetilde{\mathcal{L}}^{-1}_{\mathbf{c}_0}P\}_{\mathbf{0}} = 0$, for $\mathbf{k} = (0, 0)$,
- for $\mathbf{k} = \pm\mathbf{k}_j$ we set (see (21))

$$\{\widetilde{\mathcal{L}}^{-1}_{\mathbf{c}_0}P\}_{\pm\mathbf{k}_j} = (\pm\frac{i}{2}\frac{q_{\pm\mathbf{k}_j}}{|\mathbf{k}_j|}, -\frac{i}{2}\frac{\pm\mathbf{k}_j q_{\pm\mathbf{k}_j}}{|\mathbf{k}_j|^2}),$$

  so that $\widetilde{\mathcal{L}}^{-1}_{\mathbf{c}_0}P$ is orthogonal, in $\{H^0_{\natural\natural}\}^3$, to the four-dimensional space E =span$\{\xi_{\pm\mathbf{k}_j}; j = 1, 2\}$:

$$\langle\widetilde{\mathcal{L}}^{-1}_{\mathbf{c}_0}P, \xi_{\pm\mathbf{k}_j}\rangle_{H^0} = 0, \quad j = 1, 2.$$

Notice that *the pseudo-inverse operator* $\widetilde{\mathcal{L}}^{-1}_{\mathbf{c}_0}$ *is defined here, even for* $P = (q, \mathbf{p})$ *not satisfying the compatibility condition* (20).

Lemma 1. *Let* $\mathbf{c} = c_0(1,0)$, $\frac{l_1}{l_2} = \frac{r_0}{s_0} \in \mathbb{Q}^+$ *and* $(c_0, l_j, \tau_j), j = 1,2$ *satisfy (23) and such that* $\pm\mathbf{k}_j, j = 1,2$ *are the only solution in* Γ *of (26). Then, for any given*

$$P = (q, \mathbf{p}) \in G_p, \quad p \geq 0,$$

such that the compatibility conditions (30) hold, the general solution $U = (\eta, \mathbf{v}) \in G_{p+1}$ *of the system*

$$\mathcal{L}_{\mathbf{c}_0} U = P,$$

is given by

$$U = \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} P + A\xi_{\mathbf{k}_1} + \overline{A}\xi_{-\mathbf{k}_1} + B\xi_{\mathbf{k}_2} + \overline{B}\xi_{-\mathbf{k}_2}, \tag{33}$$

where

$$\xi_{\pm\mathbf{k}_j} = (\sqrt{1+\tau_j^2}, 1, (-1)^{j+1}\tau_j)e^{\pm i\mathbf{k}_j\cdot\mathbf{x}},$$

$A, B \in \mathbb{C}$, *and* $\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}$ *is the bounded linear operator:* $G_p \to G_{p+1} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$ *defined above, and we have*

$$||\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{G}||_{\mathcal{L}(G_p)} \leq c. \tag{34}$$

In what follows we need to consider the perturbed operator $\mathcal{L}_{c_0(1,w)} = \mathcal{L}_{\mathbf{c}_0} + w\mathcal{L}^{(1)}$ for w close to 0, where

$$\mathcal{L}^{(1)}U = -c_0\frac{\partial}{\partial x_2}(I - \frac{1}{6}\Delta)U.$$

Allowing $w \neq 0$ (which plays the role of a parameter) means that we intend to find travelling waves moving not exactly in the direction of the x_1- axis. We shall see that this is linked with the ratio of amplitudes $\varepsilon_1, \varepsilon_2$ of the wave along the basic wave vectors $\mathbf{k}_1, \mathbf{k}_2$. The perturbation $w\mathcal{L}^{(1)}$ appears to be singular as it leads to a *small divisor problem* when we invert $\mathcal{L}_{c_0(1,w)}$, (contrary to the inversion of $\mathcal{L}_{\mathbf{c}_0}$ with our assumption (28)). Indeed, the $\Delta(\mathbf{k}, \mathbf{c})$ in the denominators of (17, 18) may become very small for large $|\mathbf{k}|$. In what follows, we control the smallness of $\Delta(\mathbf{k}, \mathbf{c})$ in assuming a rationality condition. We show the following

Lemma 2. *Let* $\mathbf{c} = c_0(1,w)$, *and fix* $\delta \in (0,1)$, *then choose* $\delta < \tau_2 < \delta^{-1}$, $l_2 < \delta$ *and* $|w| \leq \frac{\delta}{5}$ *with*

$$w = \frac{\frac{r}{s} - \frac{l_1}{l_2}}{\tau_1\frac{l_1}{l_2} + \tau_2\frac{r}{s}}, \quad r, s \in \mathbb{N}. \tag{35}$$

Assume $(c_0, l_j, \tau_j), j = 1,2$ *satisfy (23) and such that* $\pm\mathbf{k}_j, j = 1,2$ *are the only solution in* Γ *of (26). Then, except for* τ_2 *in a small neighborhood of a finite set* $\tau_2^{(p)}(\tau_1, l_1, l_2)$ *of cardinal at most* $O(\ln s)$, *the linear operator* $\mathcal{L}_{\mathbf{c}}$ *has a bounded inverse such that*

$$||\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1}\mathcal{G}||_{\mathcal{L}(G_l)} \leq c(s), \quad l \geq 0, \tag{36}$$

and for any $q \geq 0$

$$\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1} = \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} + \sum_{1 \leq n \leq q} (-w)^n (\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} \mathcal{L}^{(1)})^n \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} + \mathcal{R}_q(w), \tag{37}$$

$$||\mathcal{R}_q(w)||_{\mathcal{L}(G_l, G_{l-2(q+1)+1})} \leq |w|^{q+1} \gamma^{q+1} c(s)$$

holds, where the linear operator $\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1}$ *is computed in* $\{\ker \mathcal{L}_{\mathbf{c}_0}\}_{H^0}^{\perp}$, $(\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} \mathcal{L}^{(1)})^n \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1} \in \mathcal{L}(G_l, G_{l-2n+1})$ *and* $\gamma > 0$ *is independent of* s*. The function* $c(s)$ *is increasing, bounded by* $\gamma \ln s$*.*

Remark: we observe that the dependancy in w of the operator $\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1}$ in $\mathcal{L}(G_l, G_{l+1})$ is wealky differentiable in 0. The formula (37) gives precisely the loss of regularity of the successive derivatives in w at the origin (the loss is 2 at each increasing order).

Proof: First, for any $\mathbf{k} = n_1 \mathbf{k}_1 + n_2 \mathbf{k}_2$, $n_j \in \mathbb{Z}$, we have in using (35)

$$\frac{l_1(1 + \tau_1 w)}{l_2(1 - \tau_2 w)} = \frac{r}{s} \in \mathbb{Q}^+.$$

Hence

$$\mathbf{c} \cdot \mathbf{k} = c_0 l_2 (1 - \tau_2 w)(n_1 \frac{r}{s} + n_2)$$

and

$$|\mathbf{c} \cdot \mathbf{k}| \geq \frac{c_0 d}{s} \text{ if } \mathbf{c} \cdot \mathbf{k} \neq 0, \tag{38}$$

where d is such that

$$d \leq l_2 |1 - \tau_2 w|.$$

In choosing w such that $|w| \leq \frac{\delta}{5}$ we can take

$$d = \frac{4 l_2}{5}.$$

Notice that if $\mathbf{c} \cdot \mathbf{k} = 0$, we have (19) then

$$|\eta_{\mathbf{k}}| + |\mathbf{v}_{\mathbf{k}}| \leq \frac{1}{|\mathbf{k}|} (|q_{\mathbf{k}}| + |\mathbf{p}_{\mathbf{k}}|). \tag{39}$$

Now, if $\mathbf{c} \cdot \mathbf{k} \neq 0$, we have

$$(1 + \frac{1}{6} |\mathbf{k}|^2) |\mathbf{c} \cdot \mathbf{k}| - |\mathbf{k}| \geq |\mathbf{k}| \{ \frac{|\mathbf{k}| c_0 d}{6s} - 1 \}$$

and for $|\mathbf{k}| \geq \frac{7s}{c_0 d}$ we obtain

$$(1 + \frac{1}{6} |\mathbf{k}|^2) |\mathbf{c} \cdot \mathbf{k}| - |\mathbf{k}| \geq \frac{|\mathbf{k}|}{6}.$$

We then observe that

$$\begin{aligned}\Delta(\mathbf{k},\mathbf{c}) &= \{(1+\frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}|-|\mathbf{k}|\}\{(1+\frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}|+|\mathbf{k}|\}\\ &\geq \frac{|\mathbf{k}|}{6}\{(1+\frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}|+|\mathbf{k}|\},\end{aligned}$$

and (17, 18) leads to the estimate

$$|\eta_{\mathbf{k}}|+|\mathbf{v}_{\mathbf{k}}| \leq \frac{6}{|\mathbf{k}|}(|q_{\mathbf{k}}|+|\mathbf{p}_{\mathbf{k}}|). \tag{40}$$

We also observe that if $|\mathbf{k}||\mathbf{c}\cdot\mathbf{k}|>7$, (40) holds.

It remains to study the region $\mathcal{R}$ of the plane (n_1,n_2) where

$$|\mathbf{k}|\leq\frac{7s}{c_0 d},\quad |\mathbf{k}||\mathbf{c}\cdot\mathbf{k}|\leq 7,\ \Delta(\mathbf{k},\mathbf{c})\neq 0,\ \text{and } \mathbf{c}\cdot\mathbf{k}\neq 0. \tag{41}$$

For estimating $\mathcal{R}$, let us notice that $|\mathbf{k}|^2=(n_1l_1+n_2l_2)^2+(n_1\tau_1l_1-n_2\tau_2l_2)^2$ is a positive definite quadratic form, hence

$$d_1^2(n_1^2+n_2^2)\leq|\mathbf{k}|^2\leq d_0^2(n_1^2+n_2^2)$$

where

$$\begin{aligned}d_1^2 =&\frac{1}{2}\left((1+\tau_1^2)l_1^2+(1+\tau_2^2)l_2^2\right)-\frac{1}{2}\sqrt{\Delta},\\ \Delta =&\left((1+\tau_1^2)l_1^2+(1+\tau_2^2)l_2^2\right)^2-4l_1^2l_2^2(\tau_1+\tau_2)^2\end{aligned} \tag{42}$$

and, since for $a>0$, $a-\sqrt{a^2-b^2}>b^2/2a$

$$d_1>\frac{l_1l_2(\tau_1+\tau_2)}{((1+\tau_1^2)l_1^2+(1+\tau_2^2)l_2^2)^{1/2}}. \tag{43}$$

Hence the region $\mathcal{R}$ is included in the region $\mathcal{A}$ defined by

$$\mathcal{A}=\left\{(n_1,n_2)\in\mathbb{Z}^2; n_1^2+n_2^2<\left(\frac{7s}{c_0dd_1}\right)^2, |n_2+\frac{r}{s}n_1|\leq\frac{7}{c_0dd_1\sqrt{n_1^2+n_2^2}}\right\}.$$

We can compute the area of $\mathcal{A}$ in the plane (n_1,n_2), in using polar coordinates

$$\begin{aligned}n_1 &= \rho\cos\theta,\ n_2=\rho\sin\theta,\\ \rho &\leq \min\left\{\left(\frac{7\cos\theta_0}{c_0dd_1}\right)^{1/2}|\sin(\theta-\theta_0)|^{-1/2}, \frac{7s}{c_0dd_1}\right\}\end{aligned}$$

where

$$\tan\theta_0=-r/s,\quad \theta_0\in(-\pi/2,0).$$

We then obtain by estimating $2\int_\phi^{\pi/2}\rho^2(\theta)d\theta + 4\phi\frac{7s}{c_0dd_1}$ for large s, where $\rho^2(\theta) = \left(\frac{7\cos\theta_0}{c_0dd_1}\right)|\sin\theta|^{-1}$ and $\sin\phi = \frac{\cos\theta_0}{s}$,

$$\begin{aligned} Area(\mathcal{A}) &= \frac{14\cos\theta_0}{c_0dd_1}\ln(\frac{1}{\tan\phi/2}) + \frac{28s}{c_0dd_1}\sin^{-1}(\frac{\cos\theta_0}{s}) \\ &\sim \frac{14\cos\theta_0}{c_0dd_1}\ln s. \end{aligned}$$

We notice that by construction, r/s is close to l_1/l_2, hence $\cos\theta_0$ is close to $\frac{l_2}{\sqrt{l_1^2+l_2^2}}$ and the following estimate holds

$$\frac{\cos\theta_0}{dd_1} \le \frac{5\left((1+\tau_1^2)l_1^2 + (1+\tau_2^2)l_2^2\right)^{1/2}}{4l_1l_2(l_1^2+l_2^2)^{1/2}(\tau_1+\tau_2)}.$$

For $\tau_2 < \delta^{-1}$ we have an estimate of c_0 (see (23)) independant of τ_2 (then depending on l_2 and δ), which shows that $Area(\mathcal{A}) \le \gamma_0(\ln s)$ with γ_0 independent of s. Hence the number of points (n_1, n_2) lying in $\mathcal{A}$ is of order $\ln s$.

It is useful in what follows to notice that for

$$|\mathbf{k}|^2 > \frac{7l_2d_0^2}{l_1c_0dd_1}$$

then

$$n_2k_2 < 0. \tag{44}$$

To see this, we look at the intersection of the curve (in polar coordinates) which bounds the region $\mathcal{A}$

$$\rho^2 = \frac{7\cos\theta_0}{c_0dd_1}|\sin(\theta-\theta_0)|^{-1}$$

with the n_1 axis ($\theta = 0$). The points of this curve such that $\theta_0 < \theta < 0$ are such that $n_1 > 0$, $n_2 < 0$. This shows that for points in the region of $\mathcal{A}$ such that

$$n_1^2 + n_2^2 > \frac{7}{c_0dd_1}\frac{s}{r}$$

n_1 and n_2 have opposite signs. Then for obtaining (44) we conclude in observing that r/s is close to l_1/l_2, and $k_2 = n_1l_1\tau_1 - n_2l_2\tau_2$ has the sign of n_1.

The equation

$$(1+\frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}| - |\mathbf{k}| = 0$$

is equivalent to writing (27) where we replace c_0^2 by its expression (23) in function of τ_2, l_2, which makes for every "bad" couple (n_1, n_2) a polynomial equation of degree 8 in τ_2. Hence we cannot have more than 8 roots $\tau_2 > 0$ for every "bad" couple (n_1, n_2). This makes a finite set of "bad" values for $\tau_2 = \tau_2^{(p)}(\tau_1, l_1, l_2)$ of cardinal $O(\ln s)$. We then need to exclude little neighborhoods of these roots for controlling the size of the inverse of $(1+\frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}| - |\mathbf{k}|$. Let us exclude $O(\ln s)$

neighborhoods of these specific values of τ_2 and for insuring that it remains most of good values for the $(\tau_2)'s$, we may choose, for each (n_1, n_2), neighborhoods of exclusions of size $O(\eta/\ln s)$ around every such root τ_2, with $\eta << 1$. Let us show that outside these neighborhoods we have

$$\left|(1+\frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}| - |\mathbf{k}|\right| \geq \frac{c|\mathbf{k}|}{\ln s}, \text{ for large } s. \tag{45}$$

To show this, it is sufficient to show that the derivative of $g(\tau_2)$ defined by

$$g(\tau_2) = (1+\frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}| - |\mathbf{k}|$$

with respect to τ_2 at any root τ_0 of (27) is such that $|g'(\tau_0)| > c|\mathbf{k}|$ for some c independent of s. Indeed, an elementary computation gives

$$\frac{\partial_{\tau_2}|\mathbf{c}\cdot\mathbf{k}|}{|\mathbf{c}\cdot\mathbf{k}|}|_{\tau_2=\tau_0} = -\frac{w}{1-\tau_0 w} + \tau_0\frac{6-l_2^2(1+\tau_0^2)}{(1+\tau_0^2)(6+l_2^2(1+\tau_0^2)}$$

hence

$$g'(\tau_0) = |\mathbf{k}|\left\{\frac{-n_2 l_2 k_2}{|\mathbf{k}|^2}\left(\frac{|\mathbf{k}|^2-6}{|\mathbf{k}|^2+6}\right) - \frac{w}{1-\tau_0 w} + \tau_0\frac{6-l_2^2(1+\tau_0^2)}{(1+\tau_0^2)(6+l_2^2(1+\tau_0^2)}\right\}.$$

For

$$|\mathbf{k}| > M, \quad M = \max\{\frac{7 l_2 d_0^2}{l_1 c_0 d d_1}, \sqrt{6}\}$$

the inequality (44) above shows that the first term on the right side is > 0. Moreover, for $\tau_2 < \delta^{-1}$, and $|w| < \delta/5$, we have

$$|\frac{w}{1-\tau_2 w}| < \frac{\delta}{4}$$

In taking l_2 small enough, such that

$$l_2 < 1, \quad l_2\tau_0 < 1$$

and since $\tau_2 < \delta^{-1}$, this is realized as soon as

$$l_2 < \delta < 1 \tag{46}$$

we obtain $l_2^2(1+\tau_0^2) < 2$, hence

$$\frac{6-l_2^2(1+\tau_0^2)}{(6+l_2^2(1+\tau_0^2)} > \frac{1}{2},$$

and it results that (recall that $\delta < \tau_2 < 1/\delta$)

$$\tau_0\frac{6-l_2^2(1+\tau_0^2)}{(1+\tau_0^2)(6+l_2^2(1+\tau_0^2)} > \frac{\delta}{2(1+\delta^2)}$$

which is independent of s. We notice that

$$4 > 2(1+\delta^2)$$

hence

$$g'(\tau_0) > |\mathbf{k}|\left\{\frac{\delta}{2(1+\delta^2)} - \frac{\delta}{4}\right\} = c|\mathbf{k}|, \quad c > 0.$$

In the region $\mathcal{R}$ where

$$|\mathbf{k}| \leq M,$$

the number of points of the plane (n_1, n_2) is bounded by a finite number independent of s. For avoiding the corresponding bad values of τ_2 near the corresponding roots, we just need to avoid a fixed (independent of s) small η neiborhood of this finite number of roots, since the minimal value of $|g'(\tau_0)|$ at these roots is independent of s.

This ends the proof of the fact that in choosing τ_2 outside a small open set of (δ, δ^{-1}) and for $|\mathbf{k}| \leq \frac{7s}{c_0 d}$ we obtain (45). Finally we find a constant $\gamma > 0$ independent of s such that

$$|\eta_{\mathbf{k}}| + |\mathbf{v}_{\mathbf{k}}| \leq \frac{\gamma \ln s}{|\mathbf{k}|}(|q_{\mathbf{k}}| + |\mathbf{p}_{\mathbf{k}}|). \tag{47}$$

Now collecting (39,40,47) we obtain an estimate valid for all $\mathbf{k}$ such that $\mathbf{k} \neq \pm\mathbf{k}_1, \pm\mathbf{k}_2$

$$|(1+\frac{1}{6}|\mathbf{k}|^2)|\mathbf{c}\cdot\mathbf{k}| - |\mathbf{k}|| \geq \frac{|\mathbf{k}|}{c(s)},$$

and the required estimate (36) follows for $\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1}\mathcal{G}$. The property (9) and

$$\mathcal{G}\xi_{\pm\mathbf{k}_j} = \pm i l_j \sqrt{1+\tau_j^2}\,\xi_{\pm\mathbf{k}_j} \tag{48}$$

imply that the subspace $\{\ker \mathcal{L}_{\mathbf{c}_0}\}_{H^0}^{\perp}$ is mapped into itself by $\mathcal{G}$. Notice that the dependancy in s of the bound of the linear operator $\widetilde{\mathcal{L}}_{\mathbf{c}}^{-1}\mathcal{G}$ is delicate to control, since the dangerous values of (n_1, n_2) (for which we may have roots of (27)) are large ones, and not so frequent in the set $\mathcal{A}$.

For obtaining (37) we first observe that the subspace $\{\ker \mathcal{L}_{\mathbf{c}_0}\}_{H^0}^{\perp}$ is stable under $\mathcal{L}^{(1)}$ since we have the property (9) and

$$\mathcal{L}^{(1)}\xi_{\pm\mathbf{k}_j} = \pm i(-1)^j l_j \tau_j \sqrt{1+\tau_j^2}\,\xi_{\pm\mathbf{k}_j}. \tag{49}$$

Then, for $F \in \{\ker \mathcal{L}_{\mathbf{c}_0}\}_{H^0}^{\perp}$, the equation

$$\mathcal{L}_{\mathbf{c}}U = (\mathcal{L}_{\mathbf{c}_0} + w\mathcal{L}^{(1)})U = F$$

leads to

$$\begin{aligned} U &= \widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}F + U_1, \\ \mathcal{L}_{\mathbf{c}}U_1 &= -w\mathcal{L}^{(1)}\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}F, \end{aligned}$$

which leads to (37) for $q=0$. Writing now

$$\begin{aligned} U_1 &= -w\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{L}^{(1)}\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}F + U_2 \\ \mathcal{L}_{\mathbf{c}}U_2 &= w^2\mathcal{L}^{(1)}\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{L}^{(1)}\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}F, \end{aligned}$$

leads to (37) for $q=1$. Then the result of (37) for any q follows.

4 Bifurcation equations

Let us set $\mathbf{c} = \frac{c_0}{1+\mu}(1,w)$, and rewrite equation (7) as

$$\mathcal{L}_{\mathbf{c}_0}U + \mu\mathcal{G}U + (1+\mu)\mathcal{G}\mathcal{N}(U,U) + w\mathcal{L}^{(1)}U = 0. \tag{50}$$

Then we decompose $U \in G_p$ as

$$U = X + V$$

where

$$\begin{aligned} X &= A\xi_{\mathbf{k}_1} + \overline{A}\xi_{-\mathbf{k}_1} + B\xi_{\mathbf{k}_2} + \overline{B}\xi_{-\mathbf{k}_2} \in E, \\ \langle V, \xi_{\pm\mathbf{k}_j}\rangle_{H^0} &= 0, \quad j=1,2. \end{aligned}$$

Observe that $E \subset G_p$ for all $p \geq 0$. The above decomposition is unique for any $p \geq 0$, hence the mapping $U \mapsto V$ defines a *projection* $\mathcal{Q}$ *from* G_p *to* $G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, which is orthogonal for $p=0$. Now, we may observe that

$$\begin{aligned} \mathcal{Q}\mathcal{G}X &= 0, \quad \mathcal{Q}\mathcal{G}V = \mathcal{G}V, \quad p \geq 1, \\ \mathcal{Q}\mathcal{L}^{(1)}X &= 0, \quad \mathcal{Q}\mathcal{L}^{(1)}V = \mathcal{L}^{(1)}V, \ p \geq 3. \end{aligned}$$

Assuming $U \in G_p$, $p \geq 3$, we then obtain from (50) the following system

$$\mathcal{L}_{c_0(1,w)}V + \mu\mathcal{G}V + (1+\mu)\mathcal{Q}\mathcal{G}\mathcal{N}(X+V, X+V) = 0, \tag{51}$$

$$\langle \mu\mathcal{G}X + w\mathcal{L}^{(1)}X + (1+\mu)\mathcal{G}\mathcal{N}(X+V,X+V), \xi_{\pm\mathbf{k}_j}\rangle = 0, \ j=1,2. \tag{52}$$

We notice that (51) may be solved by the implicit function theorem in $G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, for any $p \geq 3$, with respect to V. Indeed, equation (51) is of the form

$$\mathcal{L}_{c_0(1,w)}V + \mathcal{F}(X,V,\mu) = 0$$

in G_{p-3}, with $\mathcal{F}$ analytic in its arguments as a function from $E \times \big(G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}\big) \times \mathbb{R}$ into $G_{p-1} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, satisfying

$$\mathcal{F}(0,0,\mu) = 0, \quad D_V\mathcal{F}(0,0,0) = 0,$$

and thanks to Lemma 2, the operator $\mathcal{L}_{c_0(1,w)}$ has a bounded inverse from $G_{p-1} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$ to $G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, *its bound being uniform in function of* w, *provided that* w satisfies (35), $\delta < \tau_2 < \delta^{-1}$, $l_2 < \delta$, *and* s *is bounded by* *some fixed* σ. Due to the bound of $\{\widetilde{\mathcal{L}}_{c_0(1,w)}\}^{-1}$ found at Lemma 2 we need to assume

$$|\mu|\ln\sigma << 1, \quad ||X||\ln\sigma << 1, \tag{53}$$

in such a way that

$$||V||^2_{G_p}\ln\sigma \ \text{ and } |\mu|||V||_{G_p}\ln\sigma << ||V||_{G_p}$$

and then obtain

$$||V|| = O(||X||^2\ln\sigma). \tag{54}$$

It then results that for A, B close enough to 0, and for w satisfying (35), $\delta < \tau_2 < \delta^{-1}$, $l_2 < \delta$, $s \leq \sigma$, we obtain

$$V = \mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w) \in G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$$

which is analytic in $(A, \overline{A}, B, \overline{B}, \mu)$, the dependancy in w being more subtle. In fact $\mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w)$ has in $G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, $p \geq 3$, an asymptotic expansion in powers of w in the neighborhood of 0. To prove this, let us define

$$\begin{aligned}\mathcal{V}_0 &= \mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, 0),\\ \mathcal{V}_1 &= \mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w) - \mathcal{V}_0.\end{aligned}$$

Then $\mathcal{V}_1$ satisfies

$$\begin{aligned}0 =&\mathcal{L}_{c_0(1,w)}\mathcal{V}_1 + w\mathcal{L}^{(1)}\mathcal{V}_0 + \mu\mathcal{G}\mathcal{V}_1 + 2(1+\mu)\mathcal{Q}\mathcal{G}\mathcal{N}(X + \mathcal{V}_0, \mathcal{V}_1)\\ &+ (1+\mu)\mathcal{Q}\mathcal{G}\mathcal{N}(\mathcal{V}_1, \mathcal{V}_1).\end{aligned} \tag{55}$$

Since $w\mathcal{L}^{(1)}\mathcal{V}_0 \in G_{p-3} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, with a small norm, we can solve equation (55) with respect to $\mathcal{V}_1$ in $G_{p-2} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, provided that $p \geq 5$. Denoting by $\mathcal{V}_{10}$ the value of the solution $\mathcal{V}_1$ when one replaces $\mathcal{L}_{c_0(1,w)}$ by $\mathcal{L}_{\mathbf{c}_0}$, we can set $\mathcal{V}_1 = \mathcal{V}_{10} + \mathcal{V}_2$ and obtain $\mathcal{V}_2$ by the implicit function theorem in $G_{p-4} \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, and so on. Now we have estimates of the form

$$\begin{aligned}||\mathcal{V}_0||_{G_p} &\leq \gamma c(\sigma)||X||^2,\\ ||\mathcal{V}_1||_{G_{p-2}} &\leq \gamma c(\sigma)|w|||X||^2,\\ ||\mathcal{V}_2||_{G_{p-4}} &\leq \gamma c(\sigma)|w|^2||X||^2,\end{aligned}$$

and so on. This proves the assertion on the asymptotic expansion in powers of w (not converging in general) for $\mathcal{V}(A, \overline{A}, B, \overline{B}, \mu, w)$ in any space $G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, $p \geq 3$ (notice that the choice of p is arbitrary, but we need to stop the expansion at some order to insure the existence of the solution in some space G_p).

Now, we have the symmetry properties (10) of the basic equation (50) and (31), and we also have easily

$$\mathcal{T}_{\mathbf{y}}\mathcal{Q} = \mathcal{Q}\mathcal{T}_{\mathbf{y}}, \ \mathcal{S}_0\mathcal{Q} = \mathcal{Q}\mathcal{S}_0.$$

Then, the uniqueness of $\mathcal{V}$, leads to the following properties:

$$\begin{aligned} \mathcal{T}_{\mathbf{y}}\mathcal{V}(A,\overline{A},B,\overline{B},\mu,w) =& \mathcal{V}(Ae^{i\mathbf{k}_1\cdot\mathbf{y}},\overline{A}e^{-i\mathbf{k}_1\cdot\mathbf{y}},Be^{i\mathbf{k}_2\cdot\mathbf{y}},\overline{B}e^{-i\mathbf{k}_2\cdot\mathbf{y}},\mu,w), \\ \mathcal{S}_0\mathcal{V}(A,\overline{A},B,\overline{B},\mu,w) =& \mathcal{V}(\overline{A},A,\overline{B},B,\mu,w). \end{aligned} \tag{56}$$

More precisely, we have in any $G_p \cap \{\ker \mathcal{L}_{\mathbf{c}_0}\}^{\perp}_{H^0}$, $p \geq 3$

$$\begin{aligned} \mathcal{V}(A,\overline{A},B,\overline{B},\mu,w) =& -\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{Q}\mathcal{G}\mathcal{N}(X,X) \\ & + O((|\mu|+|w|)||X||^2+||X||^3). \end{aligned} \tag{57}$$

Now replacing V by $\mathcal{V}(A,\overline{A},B,\overline{B},\mu,w)$ in the system of 4 equations (52), we obtain in fact 2 complex equations, with their complex conjugates, of the form

$$\begin{aligned} h_1(A,\overline{A},B,\overline{B},\mu,w) &= 0, \\ h_2(A,\overline{A},B,\overline{B},\mu,w) &= 0, \end{aligned}$$

where h_1 is obtained with $\mathbf{k}_1$ in (52) and h_2 with $\mathbf{k}_2$, and h_j, $j = 1,2$, is analytic in $(A,\overline{A},B,\overline{B},\mu)$ and C^l at the origin with respect to w (l is arbitrary). The symmetry properties (10), (31) and (56) lead, for any $\mathbf{y} \in \mathbb{R}^2$, to the following relationships

$$\begin{aligned} h_1(Ae^{i\mathbf{k}_1\cdot\mathbf{y}},\overline{A}e^{-i\mathbf{k}_1\cdot\mathbf{y}},Be^{i\mathbf{k}_2\cdot\mathbf{y}},\overline{B}e^{-i\mathbf{k}_2\cdot\mathbf{y}},\mu,w) =& e^{i\mathbf{k}_1\cdot\mathbf{y}}h_1(A,\overline{A},B,\overline{B},\mu,w), \\ h_2(Ae^{i\mathbf{k}_1\cdot\mathbf{y}},\overline{A}e^{-i\mathbf{k}_1\cdot\mathbf{y}},Be^{i\mathbf{k}_2\cdot\mathbf{y}},\overline{B}e^{-i\mathbf{k}_2\cdot\mathbf{y}},\mu,w) =& e^{i\mathbf{k}_2\cdot\mathbf{y}}h_2(A,\overline{A},B,\overline{B},\mu,w), \\ h_1(\overline{A},A,\overline{B},B,\mu,w) =& -\overline{h_1}(A,\overline{A},B,\overline{B},\mu,w). \end{aligned}$$

It results classically that

$$\begin{aligned} h_1(A,\overline{A},B,\overline{B},\mu,w) &= iAg_1(|A|^2,|B|^2,\mu,w), \\ h_2(A,\overline{A},B,\overline{B},\mu,w) &= iBg_2(|A|^2,|B|^2,\mu,w), \end{aligned}$$

where g_1 and g_2 are *real valued* smooth functions of their arguments. When $B = 0$ (or $A = 0$) one obtains plane waves, with basic wave vector $\mathbf{k}_1$ (or $\mathbf{k}_2$), the direction of propagation being somewhat arbitrary (provided it is not orthogonal to $\mathbf{k}_1$ (or $\mathbf{k}_2$). When $AB \neq 0$, one obtains the bi-periodic travelling waves, which are the main object of our study. For concluding on their existence, we need to solve the real system of two equations:

$$\begin{aligned} g_1(|A|^2,|B|^2,\mu,w) =& 0, \\ g_2(|A|^2,|B|^2,\mu,w) =& 0. \end{aligned} \tag{58}$$

In the case when the lattice Γ has a diamond structure, and we choose the x_1- axis such that $\mathbf{k}_1$ and $\mathbf{k}_2$ are symmetric with respect to this axis, we have the additional symmetry properties (11), and (32) which, thanks to the uniqueness of $\mathcal{V}$, and for $w = 0$ (i.e. when $\mathbf{c}$ is in the x_1- direction), leads to

$$\mathcal{S}_1\mathcal{V}(A,\overline{A},B,\overline{B},\mu,0) = \mathcal{V}(B,\overline{B},A,\overline{A},\mu,0).$$

This implies
$$h_1(B,\overline{B},A,\overline{A},\mu,0)=h_2(A,\overline{A},B,\overline{B},\mu,0),$$
hence finally
$$g_1(|B|^2,|A|^2,\mu,0)=g_2(|A|^2,|B|^2,\mu,0). \tag{59}$$
The next section is devoted to the computation of the principal part of the system (58), leading to the existence of non-symmetric travelling waves for (1).

5 Bifurcating solutions

We first notice that (57), with the symmetry properties (56), leads to

$$\begin{aligned} V =&\zeta_{2,0}(A^2e^{2i\mathbf{k}_1\cdot\mathbf{x}}+\overline{A}^2e^{-2i\mathbf{k}_1\cdot\mathbf{x}})+\zeta_{0,2}(B^2e^{2i\mathbf{k}_2\cdot\mathbf{x}}+\overline{B}^2e^{-2i\mathbf{k}_2\cdot\mathbf{x}})+\\ &+\zeta_{1,1}(ABe^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}+\overline{AB}e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}})+\\ &+\zeta_{1,-1}(A\overline{B}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}+\overline{A}Be^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}})+h.o.t. \end{aligned} \tag{60}$$

where

$$\begin{aligned} \zeta_{2,0}e^{2i\mathbf{k}_1\cdot\mathbf{x}} &= -\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{GN}(\xi_{\mathbf{k}_1},\xi_{\mathbf{k}_1}),\\ \zeta_{0,2}e^{2i\mathbf{k}_2\cdot\mathbf{x}} &= -\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{GN}(\xi_{\mathbf{k}_2},\xi_{\mathbf{k}_2}),\\ \zeta_{1,1}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} &= -2\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{GN}(\xi_{\mathbf{k}_1},\xi_{\mathbf{k}_2}),\\ \zeta_{1,-1}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}} &= -2\widetilde{\mathcal{L}}_{\mathbf{c}_0}^{-1}\mathcal{GN}(\xi_{\mathbf{k}_1},\xi_{-\mathbf{k}_2}), \end{aligned}$$

the suppression of the projection $\mathcal{Q}$ coming from the non resonance of $2\mathbf{k}_1, 2\mathbf{k}_2, \mathbf{k}_1\pm\mathbf{k}_2$ with $\pm\mathbf{k}_j$, and where we notice that

$$\mathcal{GN}(\xi_{\mathbf{k}_j},\xi_{-\mathbf{k}_j})=0,\quad j=1,2.$$

We obtain

$$\begin{aligned} \mathcal{GN}(\xi_{\mathbf{k}_1},\xi_{\mathbf{k}_1}) &= \begin{pmatrix} 2il_1(1+\tau_1^2)^{3/2}\\ il_1(1+\tau_1^2)\\ i\tau_1l_1(1+\tau_1^2)\end{pmatrix}e^{2i\mathbf{k}_1\cdot\mathbf{x}},\\ \mathcal{GN}(\xi_{\mathbf{k}_2},\xi_{\mathbf{k}_2}) &= \begin{pmatrix} 2il_2(1+\tau_2^2)^{3/2}\\ il_2(1+\tau_2^2)\\ -i\tau_2l_2(1+\tau_2^2)\end{pmatrix}e^{2i\mathbf{k}_2\cdot\mathbf{x}}, \end{aligned}$$

$$\begin{aligned} 2\mathcal{GN}(\xi_{\mathbf{k}_1},\xi_{\mathbf{k}_2}) = &\; i(1-\tau_1\tau_2)\begin{pmatrix} l_1\sqrt{1+\tau_1^2}+l_2\sqrt{1+\tau_2^2}\\ l_1+l_2\\ (\tau_1l_1-l_2\tau_2)\end{pmatrix}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}+\\ &+i\sqrt{1+\tau_1^2}\sqrt{1+\tau_2^2}\begin{pmatrix} l_1\sqrt{1+\tau_1^2}+l_2\sqrt{1+\tau_2^2}\\ 0\\ 0\end{pmatrix}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}, \end{aligned}$$

$$2\mathcal{GN}(\xi_{\mathbf{k}_1},\xi_{-\mathbf{k}_2}) = i(1-\tau_1\tau_2)\begin{pmatrix} l_1\sqrt{1+\tau_1^2} - l_2\sqrt{1+\tau_2^2} \\ l_1 - l_2 \\ (\tau_1 l_1 + l_2\tau_2) \end{pmatrix} e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}} +$$

$$+\, i\sqrt{1+\tau_1^2}\sqrt{1+\tau_2^2}\begin{pmatrix} l_1\sqrt{1+\tau_1^2} - l_2\sqrt{1+\tau_2^2} \\ 0 \\ 0 \end{pmatrix} e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}},$$

and therefore, we find by using (17), (18)

$$\zeta_{2,0} = \frac{2l_1^2(1+\tau_1^2)}{D_{2,0}}\begin{pmatrix} 2c_0\sqrt{1+\tau_1^2}D_{1,0} + 1 + \tau_1^2 \\ c_0 D_{1,0} + 2\sqrt{1+\tau_1^2} \\ \tau_1(c_0 D_{1,0} + 2\sqrt{1+\tau_1^2}) \end{pmatrix}, \tag{61}$$

$$\zeta_{0,2} = \frac{2l_2^2(1+\tau_2^2)}{D_{0,2}}\begin{pmatrix} 2c_0\sqrt{1+\tau_2^2}D_{0,1} + 1 + \tau_2^2 \\ c_0 D_{0,1} + 2\sqrt{1+\tau_2^2} \\ -\tau_2(c_0 D_{0,1} + 2\sqrt{1+\tau_2^2}) \end{pmatrix}, \tag{62}$$

$$\zeta_{1,1} = \frac{L_+}{D_{1,1}}\begin{pmatrix} D_+ c_0(l_1+l_2) \\ l_1 + l_2 \\ \tau_1 l_1 - \tau_2 l_2 \end{pmatrix} + \frac{1-\tau_1\tau_2}{D_{1,1}}\begin{pmatrix} 6D_+ - 1 \\ D_+ c_0 (l_1+l_2)^2 \\ D_+ c_0 (l_1+l_2)(\tau_1 l_1 - \tau_2 l_2) \end{pmatrix}, \tag{63}$$

$$\begin{aligned} \zeta_{1,-1} = \frac{L_-}{D_{1,-1}}&\begin{pmatrix} D_- c_0(l_1-l_2) \\ l_1 - l_2 \\ \tau_1 l_1 + \tau_2 l_2 \end{pmatrix} + \\ &+ \frac{1-\tau_1\tau_2}{D_{1,-1}}\begin{pmatrix} 6D_- - 1 \\ D_- c_0 (l_1-l_2)^2 \\ D_- c_0 (l_1-l_2)(\tau_1 l_1 + \tau_2 l_2) \end{pmatrix}, \end{aligned} \tag{64}$$

where

$$L_+ = \left(1-\tau_1\tau_2 + \sqrt{1+\tau_1^2}\sqrt{1+\tau_2^2}\right)\left(l_1\sqrt{1+\tau_1^2} + l_2\sqrt{1+\tau_2^2}\right),$$

$$L_- = \left(1-\tau_1\tau_2 + \sqrt{1+\tau_1^2}\sqrt{1+\tau_2^2}\right)\left(l_1\sqrt{1+\tau_1^2} - l_2\sqrt{1+\tau_2^2}\right),$$

$$\begin{aligned} D_{1,0} &= 1 + \frac{2l_1^2}{3}(1+\tau_1^2), \quad D_{0,1} = 1 + \frac{2l_2^2}{3}(1+\tau_2^2), \\ D_{2,0} &= 4l_1^2[(D_{1,0})^2 c_0^2 - (1+\tau_1^2)], \\ D_{0,2} &= 4l_2^2[(D_{0,1})^2 c_0^2 - (1+\tau_2^2)], \end{aligned}$$

$$\begin{aligned} D_+ &= 1 + \frac{1}{6}[(l_1+l_2)^2 + (l_1\tau_1 - l_2\tau_2)^2], \\ D_- &= 1 + \frac{1}{6}[(l_1-l_2)^2 + (l_1\tau_1 + l_2\tau_2)^2], \\ D_{1,1} &= c_0^2(l_1+l_2)^2 D_+^2 - (l_1+l_2)^2 - (l_1\tau_1 - l_2\tau_2)^2, \\ D_{1,-1} &= c_0^2(l_1-l_2)^2 D_-^2 - (l_1-l_2)^2 - (l_1\tau_1 + l_2\tau_2)^2. \end{aligned}$$

Let us now calculate the leading terms in (58). Let us notice that

$$\langle \xi_{\mathbf{k}_j}, \xi_{\mathbf{k}_j} \rangle = 2(1+\tau_j^2)\Omega,$$

where Ω denotes the area of the paralellogram formed with $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$ (see the definition of the lattice of periods Γ' in (6)). In fact, we have

$$\Omega = \frac{4\pi^2}{l_1 l_2(\tau_1+\tau_2)}.$$

Now, from (48) and (49) we have the following identities

$$\begin{aligned}
\mu\langle \mathcal{G}\xi_{\mathbf{k}_j}, \xi_{\mathbf{k}_j} \rangle &= 2i\mu l_j(1+\tau_j^2)^{3/2}\Omega, \\
\langle w\mathcal{L}^{(1)}\xi_{\mathbf{k}_j}, \xi_{\mathbf{k}_j} \rangle &= 2i(-1)^j w\tau_j l_j(1+\tau_j^2)^{3/2}\Omega,
\end{aligned}$$

and it is clear that with our non resonance assumption

$$\langle \mathcal{G}\mathcal{N}(X,X), \xi_{\mathbf{k}_j} \rangle = 0.$$

For deriving the principal parts of g_1 and g_2 in (58), we write as follows

$$g_j = 2l_j(1+\tau_j^2)^{3/2}\Omega\left\{\mu + (-1)^j w\tau_j + a_j|A|^2 + b_j|B|^2 + h.o.t.\right\}, \tag{65}$$

with

$$\begin{aligned}
2il_1(1+\tau_1^2)^{3/2}\Omega a_1 &= \langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_1}, \zeta_{2,0}e^{2i\mathbf{k}_1\cdot\mathbf{x}}), \xi_{\mathbf{k}_1} \rangle, \\
2il_2(1+\tau_2^2)^{3/2}\Omega b_2 &= \langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_2}, \zeta_{0,2}e^{2i\mathbf{k}_2\cdot\mathbf{x}}), \xi_{\mathbf{k}_2} \rangle,
\end{aligned}$$

$$\begin{aligned}
2il_1(1+\tau_1^2)^{3/2}\Omega b_1 =& \langle 2\mathcal{G}\left\{\mathcal{N}(\xi_{\mathbf{k}_2}, \zeta_{1,-1}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}) + \mathcal{N}(\xi_{-\mathbf{k}_2}, \zeta_{1,1}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}})\right\}, \xi_{\mathbf{k}_1} \rangle, \\
2il_2(1+\tau_2^2)^{3/2}\Omega a_2 =& \langle 2\mathcal{G}\left\{\mathcal{N}(\xi_{\mathbf{k}_1}, \zeta_{1,-1}e^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}) + \mathcal{N}(\xi_{-\mathbf{k}_1}, \zeta_{1,1}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}})\right\}, \xi_{\mathbf{k}_2} \rangle.
\end{aligned}$$

Solving (58) with respect to μ and w and denoting $|A| = \varepsilon_1$, $|B| = \varepsilon_2$, leads to

$$\begin{aligned}
\mu =& -\frac{a_1+a_2}{2}\varepsilon_1^2 - \frac{b_1+b_2}{2}\varepsilon_2^2 + O(\varepsilon_1^2+\varepsilon_2^2)^2, \\
w(\tau_1+\tau_2) =& (a_1-a_2)\varepsilon_1^2 + (b_1-b_2)\varepsilon_2^2 + O(\varepsilon_1^2+\varepsilon_2^2)^2,
\end{aligned} \tag{66}$$

where we need not to forget the restriction

$$w = \frac{\frac{r}{s} - \frac{l_1}{l_2}}{\tau_1\frac{l_1}{l_2} + \tau_2\frac{r}{s}}, \quad \frac{r}{s} \in \mathbb{Q}^+, \ s \leq \sigma.$$

It results from the bounds (53) and (54) that

$$\varepsilon_1 + \varepsilon_2 = O(|\mu|^{1/2}(\ln\sigma)^{-1/2}). \tag{67}$$

Notice that the value $w = 0$ leads to asymmetrical waves provided that

$$(a_1 - a_2)(b_1 - b_2) < 0.$$

Then this particular case gives (the propagation direction is the x_1- axis)

$$\begin{aligned}
\varepsilon_2^2 =& \frac{a_1 - a_2}{b_2 - b_1}\varepsilon_1^2 + O(\varepsilon_1^4), \\
\mu =& -\frac{\varepsilon_1^2}{2(b_2 - b_1)}\{(a_1 + a_2)(b_2 - b_1) + (a_1 - a_2)(b_1 + b_2)\} + O(\varepsilon_1^4).
\end{aligned} \tag{68}$$

In the *case when the lattice* Γ *has a diamond structure*, and we choose the x_1- axis such that $\mathbf{k}_1$ and $\mathbf{k}_2$ are symmetric with respect to this axis, we have the additional symmetry (59) which implies

$$a_1 = b_2, \ a_2 = b_1,$$

and

$$\begin{aligned}
\mu =& -\frac{a_1 + a_2}{2}(\varepsilon_1^2 + \varepsilon_2^2) + O(\varepsilon_1^2 + \varepsilon_2^2)^2, \\
w\tau =& (\varepsilon_1^2 - \varepsilon_2^2)\left\{\frac{(a_1 - a_2)}{2} + O(\varepsilon_1^2 + \varepsilon_2^2)\right\},
\end{aligned} \tag{69}$$

where only rational values of the small parameter $w\tau = (r - s)/(r + s), \ \ s \leq \sigma$ are allowed, which leads to a restricted choice for the amplitudes ε_1 and ε_2. Here the case $\varepsilon_1 = \varepsilon_2$ gives the symmetrical waves propagating in the x_1- direction as described in [6].

It remains to compute the coefficients a_j and b_j. We have

$$\begin{aligned}
\langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_1}, \zeta_{2,0}e^{2i\mathbf{k}_1\cdot\mathbf{x}}), \xi_{\mathbf{k}_1}\rangle =& \frac{2il_1^3(1 + \tau_1^2)^3\Omega}{D_{2,0}}(4c_0 D_{1,0} + 5\sqrt{1 + \tau_1^2}), \\
\langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_2}, \zeta_{0,2}e^{2i\mathbf{k}_2\cdot\mathbf{x}}), \xi_{\mathbf{k}_2}\rangle =& \frac{2il_2^3(1 + \tau_2^2)^3\Omega}{D_{0,2}}(4c_0 D_{0,1} + 5\sqrt{1 + \tau_2^2}),
\end{aligned}$$

$$\begin{aligned}
\langle 2\mathcal{G}\mathcal{N}(\xi_{\mathbf{k}_2}, \zeta_{1,-1}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}), \xi_{\mathbf{k}_1}\rangle =& \frac{il_1(1 + \tau_1^2)^{1/2}\Omega}{D_{1.-1}}\{L_-^2 + \\
&+ 2L_-(1 - \tau_1\tau_2)D_-c_0(l_1 - l_2) + 6(1 - \tau_1\tau_2)^2(D_- - 1)\}, \\
\langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_2}, \zeta_{1,1}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}), \xi_{\mathbf{k}_1}\rangle =& \frac{il_1(1 + \tau_1^2)^{1/2}\Omega}{D_{1.1}}\{L_+^2 + \\
&+ 2L_+(1 - \tau_1\tau_2)D_+c_0(l_1 + l_2) + 6(1 - \tau_1\tau_2)^2(D_+ - 1)\},
\end{aligned}$$

$$\begin{aligned}
\langle 2\mathcal{G}\mathcal{N}(\xi_{\mathbf{k}_1}, \zeta_{1,-1}e^{i(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{x}}), \xi_{\mathbf{k}_2}\rangle =& \frac{il_2(1 + \tau_2^2)^{1/2}\Omega}{D_{1.-1}}\{L_-^2 + \\
&+ 2L_-(1 - \tau_1\tau_2)D_-c_0(l_1 - l_2) + 6(1 - \tau_1\tau_2)^2(D_- - 1)\}, \\
\langle 2\mathcal{G}\mathcal{N}(\xi_{-\mathbf{k}_1}, \zeta_{1,1}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}), \xi_{\mathbf{k}_2}\rangle =& \frac{il_2(1 + \tau_2^2)^{1/2}\Omega}{D_{1.1}}\{L_+^2 + \\
&+ 2L_+(1 - \tau_1\tau_2)D_+c_0(l_1 + l_2) + 6(1 - \tau_1\tau_2)^2(D_+ - 1)\},
\end{aligned}$$

Hence

$$
\begin{aligned}
a_1 &= \frac{l_1^2(1+\tau_1^2)^{3/2}}{D_{2,0}}(4c_0 D_{1,0} + 5\sqrt{1+\tau_1^2}), \\
b_2 &= \frac{l_2^2(1+\tau_2^2)^{3/2}}{D_{0,2}}(4c_0 D_{0,1} + 5\sqrt{1+\tau_2^2}),
\end{aligned}
$$

$$
\begin{aligned}
a_2 =& \frac{1}{2(1+\tau_2^2)}\left\{\frac{L_+^2}{D_{1,1}} + \frac{L_-^2}{D_{1,-1}} + 2(1-\tau_1\tau_2)c_0(\frac{L_+D_+}{D_{1,1}}(l_1+l_2) + \right. \\
&\left. + \frac{L_-D_-}{D_{1,-1}}(l_1-l_2)) + 6(1-\tau_1\tau_2)^2(\frac{D_+-1}{D_{1,1}} + \frac{D_--1}{D_{1,-1}})\right\}, \\
b_1 =& \frac{1}{2(1+\tau_1^2)}\left\{\frac{L_+^2}{D_{1,1}} + \frac{L_-^2}{D_{1,-1}} + 2(1-\tau_1\tau_2)c_0(\frac{L_+D_+}{D_{1,1}}(l_1+l_2) + \right. \\
&\left. + \frac{L_-D_-}{D_{1,-1}}(l_1-l_2)) + 6(1-\tau_1\tau_2)^2(\frac{D_+-1}{D_{1,1}} + \frac{D_--1}{D_{1,-1}})\right\},
\end{aligned}
$$

where we need not to forget the 2 relations (23) between $c_0, l_j, \tau_j, j=1,2$.

In the *case when the lattice* Γ *has a diamond structure*, and we choose the x_1- axis such that $\mathbf{k}_1$ and $\mathbf{k}_2$ are symmetric with respect to this axis, these formulas become

$$
\begin{aligned}
a_1 &= b_2 = \frac{l^2(1+\tau^2)^{3/2}}{D_{2,0}}(4c_0 D_{1,0} + 5\sqrt{1+\tau^2}), \\
a_2 &= b_1 = \frac{1}{2(c_0^2 D_+^2 - 1)}\left\{4c_0 D_+ \frac{(1-\tau^2)}{\sqrt{1+\tau^2}} + \frac{5+2\tau^2+\tau^4}{1+\tau^2}\right\} - \frac{(1-\tau^2)^2}{2(1+\tau^2)},
\end{aligned}
$$

where

$$
\begin{aligned}
D_{1,0} &= D_{0,1} = 1 + \frac{2l^2}{3}(1+\tau^2), \\
D_{2,0} &= D_{0,2} = 4l^2(c_0^2 D_{1,0}^2 - (1+\tau^2)), \\
D_+ &= 1 + \frac{2l^2}{3}.
\end{aligned}
$$

Theorem 3. *Assume that* $\mathbf{c}_0 = c_0(1,0)$, $\mathbf{k}_1 = l_1(1,\tau_1)$ *and* $\mathbf{k}_2 = l_2(1,-\tau_2)$ *are such that the dispersion relation* $\Delta(\mathbf{k},\mathbf{c}_0) = 0$ *is satisfied only with* $\mathbf{k} = \mathbf{k}_j, j = 1,2$, *and assume that*

$$
w = \frac{\frac{r}{s} - \frac{l_1}{l_2}}{\tau_1\frac{l_1}{l_2} + \tau_2\frac{r}{s}}, \quad r,s \in \mathbb{N}.
$$

Let the propagation velocity be $\mathbf{c} = \frac{c_0}{1+\mu}(1,w)$ *and fix* $\delta \in (0,1)$ *and* $\sigma \in \mathbb{N}$ *large enough. Choose* $l_2 < \delta$ *and* $|w| < \delta/5$ *with* $1 \le s \le \sigma$ *and choose values of* $\tau_2 \in (\delta, \delta^{-1})$, *except in a small neighborhood of a finite set* $\tau_2^{(p)}(\tau_1, l_1, l_2)$ *of* *cardinal at most $O(\ln s)$. Then, for any $p \geq 3$, there is a family of bifurcating bi-periodic traveling waves, $U = (\eta, \mathbf{v})$ which are solutions of (2) in G_p, in general non symmetric with respect to the propagation direction $\mathbf{c}$, and of the form*

$$\begin{aligned} U =& A\xi_{\mathbf{k}_1} + \overline{A}\xi_{-\mathbf{k}_1} + B\xi_{\mathbf{k}_2} + \overline{B}\xi_{-\mathbf{k}_2} + \zeta_{2,0}(A^2 e^{2i\mathbf{k}_1\cdot\mathbf{x}} + \overline{A}^2 e^{-2i\mathbf{k}_1\cdot\mathbf{x}}) + \\ &+ \zeta_{0,2}(B^2 e^{2i\mathbf{k}_2\cdot\mathbf{x}} + \overline{B}^2 e^{-2i\mathbf{k}_2\cdot\mathbf{x}}) + \zeta_{1,1}(ABe^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} + \overline{AB}e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}) + \\ &+ \zeta_{1,-1}(A\overline{B}e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}} + \overline{A}Be^{-i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}}) + h.o.t. \end{aligned}$$

where $A, B \in \mathbb{C}$, and $\xi_{\pm\mathbf{k}_j}$, $\zeta_{l,n}$ are defined in (29), (61), (62), (63), (64). The amplitudes $|A| = \varepsilon_1$, $|B| = \varepsilon_2$ are uniformly bounded by $O\{(|\mu|/\ln\sigma)^{1/2}\}$ where $|\mu| << (\ln\sigma)^{-1}$, and μ and w satisfy (66). Any solution of the family, corresponding to

$$A = \varepsilon_1 e^{i\mathbf{k}_1\cdot\mathbf{y}}, \quad B = \varepsilon_2 e^{i\mathbf{k}_2\cdot\mathbf{y}}$$

is deduced from the one with $A = \varepsilon_1 > 0$, $B = \varepsilon_2 > 0$ in applying the translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{y}$.

We can now plot the traveling surfaces in the (z_1, z_2) plane, where z_2 is the traveling direction and points downward, i.e.

$$x_1 = \frac{wz_1 + z_2}{\sqrt{1+w^2}}, \quad x_2 = \frac{-z_1 + wz_2}{\sqrt{1+w^2}}.$$

By choosing the waves of the bifurcating family with

$$A = \varepsilon_1, \quad B = \varepsilon_2,$$

the elevation η of the waves indicated in the pictures is computed with terms up to degree 2 in $(\varepsilon_1, \varepsilon_2)$:

$$\begin{aligned} \eta \approx\ & 2\varepsilon_1\sqrt{1+\tau_1^2}\cos(\mathbf{k}_1\cdot\mathbf{x}) + 2\varepsilon_2\sqrt{1+\tau_2^2}\cos(\mathbf{k}_2\cdot\mathbf{x}) \\ &+ 2\varepsilon_1^2(\zeta_{2,0})_1\cos(2\mathbf{k}_1\cdot\mathbf{x}) + 2\varepsilon_2^2(\zeta_{0,2})_1\cos(2\mathbf{k}_2\cdot\mathbf{x}) \\ &+ 2\varepsilon_1\varepsilon_2(\zeta_{1,1})_1\cos((\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}) + 2\varepsilon_1\varepsilon_2(\zeta_{1,-1})_1\cos((\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}). \end{aligned}$$

For fixed values of l_1, τ_1, τ_2, we compute l_2 with formula (23), and once ε_1 and ε_2 are fixed, we compute w with (66). When $\tau_1 = \tau_2 = \tau$ and $l_1 = l_2$, the lattice Γ is symmetric. Figure 1 shows the influence of the ratio $\varepsilon_1/\varepsilon_2$ when the lattice Γ is symmetric. When $\varepsilon_2/\varepsilon_1 = 1$, the wave pattern is symmetric with respect to the propagation direction (here the vertical direction). Figures 2, 3, 4 show also cases with a symmetric lattice Γ for different values of τ, and compare the *asymmetrical pattern* for $\varepsilon_2/\varepsilon_1 = 0.5$ with the symmetric one for $\varepsilon_2/\varepsilon_1 = 1$. Figures 5 and 6 show cases with a non symmetric lattice Γ. Figure 7 provides two examples of waves where $w = 0$, i.e. once ε_1 is fixed, we compute ε_2 with (66) in such a way that $w = 0$ (at main order). Notice that in view of Theorem 3, these solutions exist for l_1/l_2 being rational. In our computed examples this ratio is indeed not rational, so we need to take for r/s a rational approximation of l_1/l_2 in such a way that w is very close to 0.

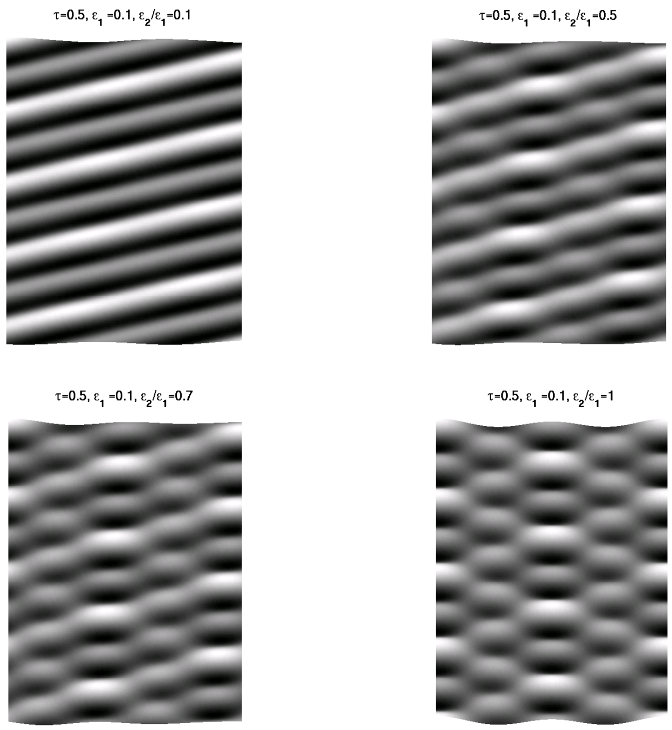

Figure 1: Γ symmetric, $\tau = 0.5, l_1 = l_2 = 0.25, \varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.1$, ii) $\varepsilon_2/\varepsilon_1 = 0.5$, iii) $\varepsilon_2/\varepsilon_1 = 0.7$ (asymmetrical waves), iv $\varepsilon_2/\varepsilon_1 = 1$ (symmetric waves). The direction of propagation of the waves is the vertical axis. Crests are white and troughs are dark.

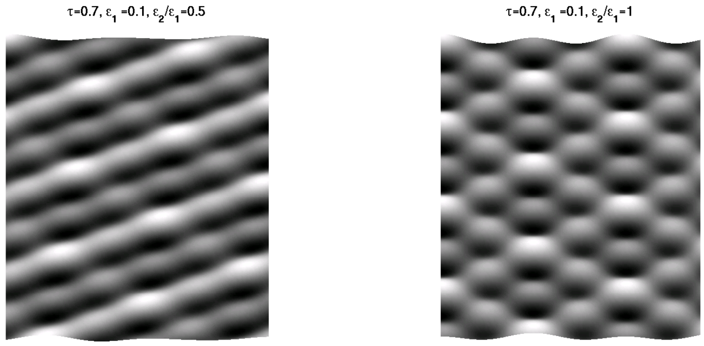

Figure 2: Γ symmetric, $\tau = 0.7, l_1 = l_2 = 0.25, \varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.5$ (asymmetrical waves), ii) $\varepsilon_2/\varepsilon_1 = 1$ (symmetric waves).

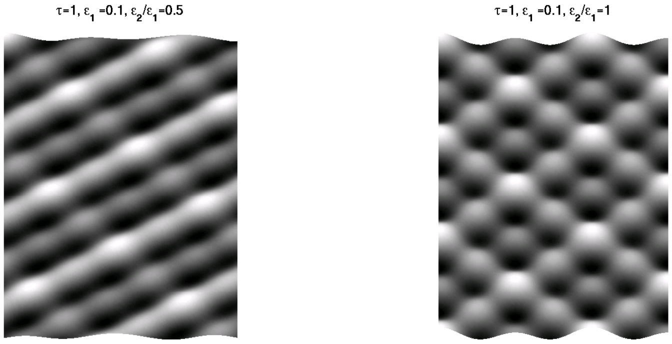

Figure 3: Γ symmetric, $\tau = 1, l_1 = l_2 = 0.25, \varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.5$ (asymmetrical waves), ii) $\varepsilon_2/\varepsilon_1 = 1$ (symmetric waves). The direction of propagation of the waves is the vertical axis

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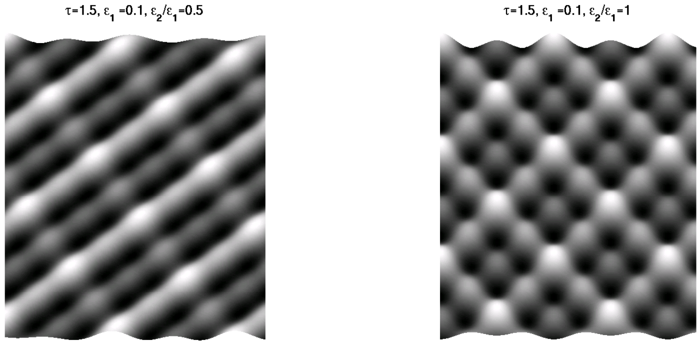

Figure 4: Γ symmetric, $\tau = 1.5, l_1 = l_2 = 0.25, \varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.5$ (asymmetrical waves), ii) $\varepsilon_2/\varepsilon_1 = 1$ (symmetric waves).

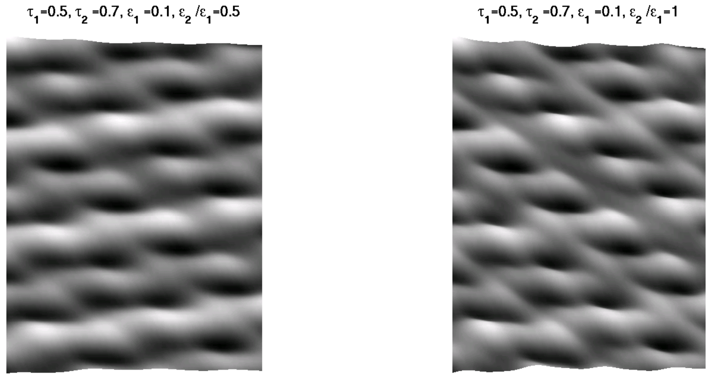

Figure 5: Γ asymmetrical, $\tau_1 = 0.5, \tau_2 = 0.7, l_1 = 0.25, \varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.5$, ii) $\varepsilon_2/\varepsilon_1 = 1$. The direction of propagation of the waves is the vertical axis

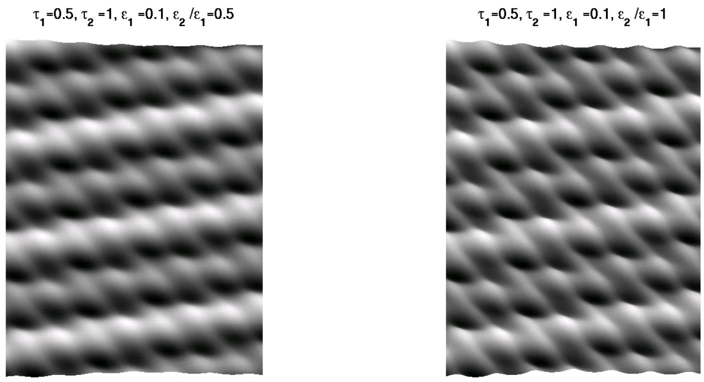

Figure 6: Γ asymmetrical, $\tau_1 = 0.5, \tau_2 = 1, l_1 = 0.25, \varepsilon_1 = 0.1$, i) $\varepsilon_2/\varepsilon_1 = 0.5$, ii) $\varepsilon_2/\varepsilon_1 = 1$. The direction of propagation of the waves is the vertical axis

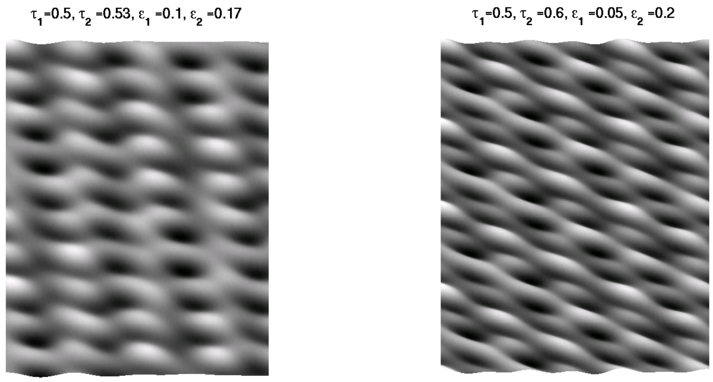

Figure 7: Γ asymmetrical, here $w = 0, l_1 = 0.25, \varepsilon_1 = 0.1$, i) $\tau_1 = 0.5, \tau_2 = 0.53, \varepsilon_2 = 0.15$, ii) $\tau_1 = 0.5, \tau_2 = 0.6, \varepsilon_1 = 0.05, \varepsilon_2 = 0.2$.