INCREMENTAL UNKNOWNS IN FINITE DIFFERENCES IN THREE SPACE DIMENSIONS

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ABSTRACT. The utilization of incremental unknowns was proposed in [21] as a tool for the approximation of inertial manifolds ([6], [7]) when finite differences discretizations are used. In a first step the emphasis has been put on their utilization for the solution of linear elliptic problems (more specifically in space dimension two), where they appear as a preconditioner for the corresponding linear systems. Thanks to their flexibility, they are related sometimes to the hierarchical bases in finite elements (see [22] and [23]) or to wavelets (see [9] and [15]), and other classes of Incremental unknowns occur as well.

In this article, we describe the application of incremental unknowns for solving the Laplace problem in space dimension three. We introduce and study here the second-order incremental unknowns, and prove by deriving suitable a priori estimates that the incremental unknowns are small as expected. We then analyze the condition number of the matrix corresponding to the five-points discretization of the Laplace operator. We show that this number is $0(h^{-1}(lnh)^4)$ instead of $0(h^{-2})$ when the usual nodal unknowns are used, h being the fine grid mesh size.

Introduction

The computation of turbulent flows necessitates a large number of unknowns. For instance, in space dimension three, it is not anymore unthinkable to consider billions of unknowns (i.e. 1024^3 grid points). In this context, it does not seem reasonable to handle all unknowns in the same way and to spend as much computing time with the small wavelengths as with the large wavelengths carrying most of the energy.

In the case of spectral methods, the distinction between small and large wavelengths is straightforward when considering Fourier series expansions. However, this is no longer the case when the discretization is made by finite differences, since all nodal values play

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the same role. The concept of incremental unknowns that stems from the dynamical system theory has been introduced in [21] in order to overcome this difficulty. Incremental unknowns appear in this context as the natural tool to construct inertial manifolds and approximate inertial manifolds (see for example [6] and [7]) when finite differences are used. Incremental unknowns can be defined when multilevel discretizations are used. For example, if two levels of discretizations are used, the incremental unknowns consist of usual nodal values at the coarse grid points, and at the fine grid points that do not belong to the coarse grid the incremental unknown is an increment to the values of suitable neighboring points.

In a first step the emphasis has been put on their utilization for the solution of linear elliptic problems, more specifically in space dimension two, where they appear as a preconditioner for the corresponding linear systems. Thanks to their flexibility, they are related sometimes to the hierarchical bases in finite elements (see [22] and [23]) or to wavelets (see [9] and [15]). Several types of incremental unknowns have been defined and studied (see for instance [4]) in order to suit certain specific requirements from the original physical problems or the design of the numerical schemes. For instance, the wavelet-like incremental unknowns considered in [4] (see also [18]) have the L_2-orthogonality property between the different levels of mesh.

The incremental unknowns have been applied to many problems and situations in space dimension two and have led to subsequent improvements when compared to methods using the usual nodal unknowns (see [2], [3], [4] and the references therein). Firstly it is proved in [3] that the condition number of the matrix associated to the discretization of linear problems by the second-order incremental unknowns is of order d^{-2}, when d levels of discretization are used. Hence the condition number is $0((lnh)^2)$ instead of $0(h^{-2})$ with the usual unknowns, where h is the size of the finest grid. This result is also confirmed by numerical computations. Secondly a number of numerical simulations have been made using incremental unknowns for linear problems. These results show that the incremental unknowns based algorithms are comparable to multigrid methods, which is already a satisfactory result since they are intended for nonlinear evolution problems and their utilization for stationary problems is just a preliminary step to this study. For the use of incremental unknowns for evolution equations, see [10], [11], [12], [16] and [18]. However, no such studies have been made yet for three-dimensional flows, which is the case of interest for actual flows.

Our aim in this article is to address the study of incremental unknowns in space dimension three in the context of stationary problems. For the sake of simplicity, we restrict ourselves to the Laplace equation on the cube $(0,1)^3$.

In section 1, we define the second-order incremental unknowns. As for the one and two-dimensional cases, they consist of the nodal values at the coarse grid points and of the increment of the nodal value to the average of the values at the neighboring points for the fine grid points that do not belong to the coarse grid. Then, in section 2 we obtain a priori estimates based on energy methods. These estimates enable us to prove in section 3 that the incremental unknowns are indeed small as expected.

We then study the condition number of the matrix associated to the discretization of the Laplace operator. Let A be the matrix associated to the five-points discretization of the Laplace operator using the nodal values and let $\bar{A}$ be the matrix associated to the discretization by the incremental unknowns. Our aim is to prove that the condition number of $\bar{A}$ is at most $0(h^{-1}(lnh)^4)$, where h is the size of the finest grid mesh, instead of $0(h^{-2})$ with the usual finite differences matrix A. As for the two-dimensional case, this result is obtained by deriving appropriate bounds on the smallest and largest eigenvalues of $\bar{A}$ through the associated bilinear form. We note however that some inequalities used in [3] for the two-dimensional case could not be used in an optimal way here.

In section 4, we describe the mathematical setting and obtain the bounds on the eigenvalues, which give the upper bound on the condition number of $\bar{A}$. Then in section 5, we present some numerical results. These results are in agreement with our theoretical results, and show that the condition number of $\bar{A}$ is of order $h^{-1}|lnh|$. We also define another type of incremental unknowns, namely the third-order incremental unknowns, and conjecture the condition number of the matrix associated to the discretization by these incremental unknowns through numerical simulations. These results make us believe that this number is again $0(h^{-1}|lnh|)$.

1. Incremental Unknowns.

We restrict ourselves here to the Dirichlet problem in the cube $\Omega = (0,1)^3$:

$$-\Delta u = f \text{ in } \Omega, \tag{1}$$

$$u = 0 \text{ on } \partial\Omega. \tag{2}$$

We set $h = \frac{1}{2N}$, $N \in \mathbb{N}^*$ and we consider the usual discretization of the Dirichlet problem on the grid of mesh h :

$$\begin{aligned}(2u_{\alpha,\beta,\gamma} - u_{\alpha-1,\beta,\gamma} - u_{\alpha+1,\beta,\gamma}) + (2u_{\alpha,\beta,\gamma} - u_{\alpha,\beta-1,\gamma} - u_{\alpha,\beta+1,\gamma})\\ + (2u_{\alpha,\beta,\gamma} - u_{\alpha,\beta,\gamma-1} - u_{\alpha,\beta,\gamma+1}) = h^2 f_{\alpha,\beta,\gamma},\end{aligned} \tag{3}$$

for $\alpha, \beta, \gamma = 1, \dots, 2N-1$,

$$u_{\alpha,\beta,\gamma} = 0, \quad \text{for } \alpha, \beta \text{ or } \gamma = 0 \text{ or } 2N, \tag{4}$$

where $f_{\alpha,\beta,\gamma} = f(\alpha h, \beta h, \gamma h)$ and $u_{\alpha,\beta,\gamma}$ is the approximate value of u at $(\alpha h, \beta h, \gamma h)$.

We then consider a coarse grid of mesh $2h = 1/N$ and we introduce the incremental unknowns which consist of the nodal values $y_{2i,2j,2k} = u_{2i,2j,2k}$ at the coarse grid points $(2ih, 2jh, 2kh), i, j, k = 0, \dots, N$, and of appropriate incremental quantities $z_{\alpha,\beta,\gamma}$ at the other points. As for the one and two-dimensional cases (see [2]), $z_{\alpha,\beta,\gamma}$ is the increment of u to the average of the values at the neighboring points (see figure 1). We then obtain three sorts of incremental unknowns:

(i) mid edge:

$$z_{2i,2j,2k+1} = u_{2i,2j,2k+1} - \frac{1}{2}(u_{2i,2j,2k} + u_{2i,2j,2k+2}), \tag{5}$$

$$z_{2i,2j+1,2k} = u_{2i,2j+1,2k} - \frac{1}{2}(u_{2i,2j,2k} + u_{2i,2j+2,2k}), \tag{6}$$

$$z_{2i+1,2j,2k} = u_{2i+1,2j,2k} - \frac{1}{2}(u_{2i,2j,2k} + u_{2i+2,2j,2k}), \tag{7}$$

(ii) center of a face:

$$\begin{aligned} z_{2i,2j+1,2k+1} =& u_{2i,2j+1,2k+1} - \frac{1}{4}(u_{2i,2j,2k} + u_{2i,2j,2k+2} \\ &+ u_{2i,2j+2,2k} + u_{2i,2j+2,2k+2}), \end{aligned} \tag{8}$$

$$\begin{aligned} z_{2i+1,2j,2k+1} =& u_{2i+1,2j,2k+1} - \frac{1}{4}(u_{2i,2j,2k} + u_{2i,2j,2k+2} \\ &+ u_{2i+2,2j,2k} + u_{2i+2,2j,2k+2}), \end{aligned} \tag{9}$$

$$\begin{aligned} z_{2i+1,2j+1,2k} =& u_{2i+1,2j+1,2k} - \frac{1}{4}(u_{2i,2j,2k} + u_{2i,2j+2,2k} \\ &+ u_{2i+2,2j,2k} + u_{2i+2,2j+2,2k}), \end{aligned} \tag{10}$$

(iii) center of a cube:

$$z_{2i+1,2j+1,2k+1} = u_{2i+1,2j+1,2k+1} - \frac{1}{8}(u_{2i,2j,2k} + u_{2i,2j,2k+2} + u_{2i,2j+2,2k} \tag{11}$$

$$+u_{2i+2,2j,2k} + u_{2i,2j+2,2k+2} + u_{2i+2,2j,2k+2} + u_{2i+2,2j+2,2k} + u_{2i+2,2j+2,2k+2}).$$

Let $\tilde{U}, \tilde{b} \in \mathbb{R}^{(2N-1)^3}$ denote the $u_{\alpha,\beta,\gamma}, f_{\alpha,\beta,\gamma}, \alpha, \beta, \gamma = 1, \dots, 2N-1$ in lexicographic order. We first reorder the $u_{\alpha,\beta,\gamma}$ and $f_{\alpha,\beta,\gamma}$ by considering the coarse grid points first and then the other points of each type, with lexicographic order in each family. Let U and b be the reordered vectors of $\mathbb{R}^{(2N-1)^3}$, we have

$$U = \begin{pmatrix} U_c \\ U_f \end{pmatrix}, b = \begin{pmatrix} b_c \\ b_f \end{pmatrix}.$$

The matrix form of (3)-(4) is

$$\tilde{A}\tilde{U} = \tilde{b}, \tag{12}$$

which is equivalent to

$$AU = b, \tag{13}$$

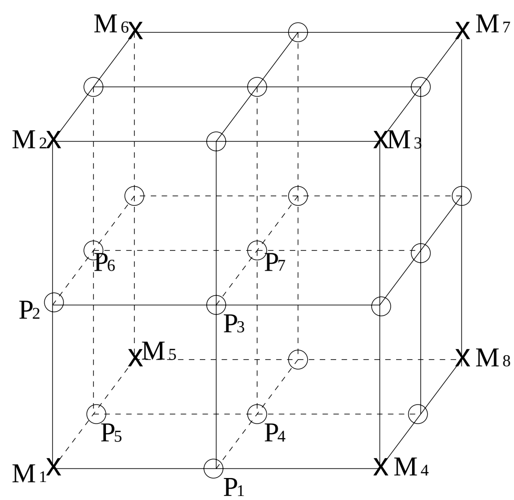


FIGURE 1. Coarse grid points (x) and fine grid points (x,o).

where

$$A = \begin{pmatrix} A_{cc} & A_{cf} \\ A_{fc} & A_{ff} \end{pmatrix}.$$

We then introduce the incremental unknowns studied above. We set $\bar{U} = \binom{Y}{Z}$ with $Y = U_c$ and Z the vector of incremental unknowns with the order described above. We denote by S the transformation matrix

$$U = S\bar{U}. \tag{14}$$

We can rewrite (13) as

$$AS\bar{U} = b, \tag{15}$$

or

$$\bar{A}\bar{U} = \bar{b}, \tag{16}$$

where $\bar{A} =^t SAS$ and $\bar{b} =^t Sb$. The matrix $\bar{A}$ is symmetric, positive definite, and the general form of S and S^{-1} is

$$S = \begin{pmatrix} I & 0 \\ S_{fc} & I \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} I & 0 \\ -S_{fc} & I \end{pmatrix}.$$

Our aim is to study properties of $\bar{A}$ such as the smallest eigenvalue of $\bar{A}$ (coercivity) and the condition number of $\bar{A}$. Throughout this article the letter c denotes constants which are independent of h and $|\Omega|$ and may change from line to line.

2. A priori estimates.

We multiply (3) by $u_{\alpha,\beta,\gamma}$ and sum for $\alpha,\beta,\gamma=1,\dots,2N-1$. A reordering which is equivalent to a discrete integration by parts then yields, taking into account (4):

$$\begin{aligned}\sum_{\alpha=0}^{2N-1}\sum_{\beta=1}^{2N-1}\sum_{\gamma=1}^{2N-1}\left(u_{\alpha+1,\beta,\gamma}-u_{\alpha,\beta,\gamma}\right)^2+\sum_{\alpha=1}^{2N-1}\sum_{\beta=0}^{2N-1}\sum_{\gamma=1}^{2N-1}\left(u_{\alpha,\beta+1,\gamma}-u_{\alpha,\beta,\gamma}\right)^2\\ +\sum_{\alpha=1}^{2N-1}\sum_{\beta=1}^{2N-1}\sum_{\gamma=0}^{2N-1}\left(u_{\alpha,\beta,\gamma+1}-u_{\alpha,\beta,\gamma}\right)^2=h^2\sum_{\alpha,\beta,\gamma=1}^{2N-1}f_{\alpha,\beta,\gamma}u_{\alpha,\beta,\gamma}.\end{aligned}\tag{17}$$

We have three discrete Poincaré inequalities:

$$\begin{aligned}h\sum_{\alpha,\beta,\gamma=1}^{2N-1}u^2_{\alpha,\beta,\gamma}&\le\sum_{\alpha=0}^{2N-1}\sum_{\beta=1}^{2N-1}\sum_{\gamma=1}^{2N-1}\frac{1}{h}\left(u_{\alpha+1,\beta,\gamma}-u_{\alpha,\beta,\gamma}\right)^2,\\ h\sum_{\alpha,\beta,\gamma=1}^{2N-1}u^2_{\alpha,\beta,\gamma}&\le\sum_{\alpha=1}^{2N-1}\sum_{\beta=0}^{2N-1}\sum_{\gamma=1}^{2N-1}\frac{1}{h}\left(u_{\alpha,\beta+1,\gamma}-u_{\alpha,\beta,\gamma}\right)^2,\\ h\sum_{\alpha,\beta,\gamma=1}^{2N-1}u^2_{\alpha,\beta,\gamma}&\le\sum_{\alpha=1}^{2N-1}\sum_{\beta=1}^{2N-1}\sum_{\gamma=0}^{2N-1}\frac{1}{h}\left(u_{\alpha,\beta,\gamma+1}-u_{\alpha,\beta,\gamma}\right)^2.\end{aligned}$$

Therefore

$$\begin{aligned}h\sum_{\alpha,\beta,\gamma=1}^{2N-1}u^2_{\alpha,\beta,\gamma}\le\frac{1}{3h}\Bigg(\sum_{\alpha=0}^{2N-1}\sum_{\beta=1}^{2N-1}\sum_{\gamma=1}^{2N-1}\left(u_{\alpha+1,\beta,\gamma}-u_{\alpha,\beta,\gamma}\right)^2\\ +\sum_{\alpha=1}^{2N-1}\sum_{\beta=0}^{2N-1}\sum_{\gamma=1}^{2N-1}\left(u_{\alpha,\beta+1,\gamma}-u_{\alpha,\beta,\gamma}\right)^2+\sum_{\alpha=1}^{2N-1}\sum_{\beta=1}^{2N-1}\sum_{\gamma=0}^{2N-1}\left(u_{\alpha,\beta,\gamma+1}-u_{\alpha,\beta,\gamma}\right)^2\Bigg).\end{aligned}\tag{18}$$

Thanks to Cauchy-Schwarz inequality, we can bound the right-hand-side of (17) by

$$\begin{aligned}&\left(\sum_{\alpha,\beta,\gamma=1}^{2N-1}h^2u^2_{\alpha,\beta,\gamma}\right)^{\frac12}\left(\sum_{\alpha,\beta,\gamma=1}^{2N-1}h^2f^2_{\alpha,\beta,\gamma}\right)^{\frac12}\\ &\le\frac{1}{\sqrt3}\Bigg(\sum_{\alpha=0}^{2N-1}\sum_{\beta=1}^{2N-1}\sum_{\gamma=1}^{2N-1}\left(u_{\alpha+1,\beta,\gamma}-u_{\alpha,\beta,\gamma}\right)^2+\sum_{\alpha=1}^{2N-1}\sum_{\beta=0}^{2N-1}\sum_{\gamma=1}^{2N-1}\left(u_{\alpha,\beta+1,\gamma}-u_{\alpha,\beta,\gamma}\right)^2\\ &\qquad+\sum_{\alpha=1}^{2N-1}\sum_{\beta=1}^{2N-1}\sum_{\gamma=0}^{2N-1}\left(u_{\alpha,\beta,\gamma+1}-u_{\alpha,\beta,\gamma}\right)^2\Bigg)^{\frac12}\left(\sum_{\alpha,\beta,\gamma=1}^{2N-1}h^2f^2_{\alpha,\beta,\gamma}\right)^{\frac12},\end{aligned}$$

and thus

$$\begin{aligned}\sum_{\alpha=0}^{2N-1}\sum_{\beta=1}^{2N-1}\sum_{\gamma=1}^{2N-1}(u_{\alpha+1,\beta,\gamma}-u_{\alpha,\beta,\gamma})^2+\sum_{\alpha=1}^{2N-1}\sum_{\beta=0}^{2N-1}\sum_{\gamma=1}^{2N-1}(u_{\alpha,\beta+1,\gamma}-u_{\alpha,\beta,\gamma})^2\\ +\sum_{\alpha=1}^{2N-1}\sum_{\beta=1}^{2N-1}\sum_{\gamma=0}^{2N-1}(u_{\alpha,\beta,\gamma+1}-u_{\alpha,\beta,\gamma})^2\leq\frac{1}{3}\sum_{\alpha,\beta,\gamma=1}^{2N-1}h^2f^2_{\alpha,\beta,\gamma}.\end{aligned}\tag{19}$$

Denoting the right-hand-side of (19) by $\mathcal{L}(y,z)$, we have

$$\mathcal{L}(y,z)\leq\frac{1}{3}\sum_{\alpha,\beta,\gamma=1}^{2N-1}h^2f^2_{\alpha,\beta,\gamma}.$$

We then introduce the incremental unknowns defined by (5)-(11) and we split the sum in the left-hand-side of (19) into eight sums depending on whether α,β,γ are odd or even. We write

$$\begin{aligned}&\mathcal{L}=\mathcal{L}_a+\mathcal{L}_b+\cdots+\mathcal{L}_h,\ \text{with}\\ &\mathcal{L}_a\ \text{for}\ \alpha=2i,\beta=2j,\gamma=2k,\\ &\mathcal{L}_b\ \text{for}\ \alpha=2i,\beta=2j,\gamma=2k+1,\\ &\mathcal{L}_c\ \text{for}\ \alpha=2i,\beta=2j+1,\gamma=2k,\\ &\mathcal{L}_d\ \text{for}\ \alpha=2i+1,\beta=2j,\gamma=2k,\\ &\mathcal{L}_e\ \text{for}\ \alpha=2i,\beta=2j+1,\gamma=2k+1,\\ &\mathcal{L}_f\ \text{for}\ \alpha=2i+1,\beta=2j,\gamma=2k+1,\\ &\mathcal{L}_g\ \text{for}\ \alpha=2i+1,\beta=2j+1,\gamma=2k,\\ &\mathcal{L}_h\ \text{for}\ \alpha=2i+1,\beta=2j+1,\gamma=2k+1,\end{aligned}$$

$i,j,k=0,\ldots,N-1$. This yields

$$\begin{aligned}\mathcal{L}_a=&\sum_{i,j,k=0}^{N-1}\left(z_{2i+1,2j,2k}+\frac{1}{2}\left(y_{2i+2,2j,2k}-y_{2i,2j,2k}\right)\right)^2\\ &+\sum_{i,j,k=0}^{N-1}\left(z_{2i,2j+1,2k}+\frac{1}{2}\left(y_{2i,2j+2,2k}-y_{2i,2j,2k}\right)\right)^2\\ &+\sum_{i,j,k=0}^{N-1}\left(z_{2i,2j,2k+1}+\frac{1}{2}\left(y_{2i,2j,2k+2}-y_{2i,2j,2k}\right)\right)^2,\end{aligned}$$

$$\mathcal{L}_b = \sum_{i,j,k=0}^{N-1} \left(z_{2i+1,2j,2k+1} - z_{2i,2j,2k+1} + \frac{1}{4} g_{i,j,k} \right)^2 + \sum_{i,j,k=0}^{N-1} \left(z_{2i,2j+1,2k+1} - z_{2i,2j,2k+1} + \frac{1}{4} h_{i,j,k} \right)^2 + \sum_{i,j,k=0}^{N-1} \left(-z_{2i,2j,2k+1} + \frac{1}{2} \left(y_{2i,2j,2k+2} - y_{2i,2j,2k} \right) \right)^2,$$

where

$$g_{i,j,k} = y_{2i+2,2j,2k+2} + y_{2i+2,2j,2k} - y_{2i,2j,2k+2} - y_{2i,2j,2k},$$
$$h_{i,j,k} = y_{2i,2j+2,2k+2} + y_{2i,2j+2,2k} - y_{2i,2j,2k+2} - y_{2i,2j,2k},$$

$$\mathcal{L}_c = \sum_{i,j,k=0}^{N-1} \left(z_{2i+1,2j+1,2k} - z_{2i,2j+1,2k} + \frac{1}{4} g'_{i,j,k} \right)^2 + \sum_{i,j,k=0}^{N-1} \left(-z_{2i,2j+1,2k} + \frac{1}{2} \left(y_{2i,2j+2,2k} - y_{2i,2j,2k} \right) \right)^2 + \sum_{i,j,k=0}^{N-1} \left(z_{2i,2j+1,2k+1} - z_{2i,2j+1,2k} + \frac{1}{4} h'_{i,j,k} \right)^2,$$

where

$$g'_{i,j,k} = y_{2i+2,2j+2,2k} + y_{2i+2,2j,2k} - y_{2i,2j+2,2k} - y_{2i,2j,2k},$$
$$h'_{i,j,k} = y_{2i,2j+2,2k+2} + y_{2i,2j,2k+2} - y_{2i,2j+2,2k} - y_{2i,2j,2k},$$

$$\mathcal{L}_d = \sum_{i,j,k=0}^{N-1} \left(-z_{2i+1,2j,2k} + \frac{1}{2} \left(y_{2i+2,2j,2k} - y_{2i,2j,2k} \right) \right)^2 + \sum_{i,j,k=0}^{N-1} \left(z_{2i+1,2j+1,2k} - z_{2i+1,2j,2k} + \frac{1}{4} g''_{i,j,k} \right)^2 + \sum_{i,j,k=0}^{N-1} \left(z_{2i+1,2j,2k+1} - z_{2i+1,2j,2k} + \frac{1}{4} h''_{i,j,k} \right)^2,$$

where

$$g''_{i,j,k} = y_{2i+2,2j+2,2k} + y_{2i,2j+2,2k} - y_{2i+2,2j,2k} - y_{2i,2j,2k},$$
$$h''_{i,j,k} = y_{2i+2,2j,2k+2} + y_{2i,2j,2k+2} - y_{2i+2,2j,2k} - y_{2i,2j,2k},$$

$$\begin{aligned}\mathcal{L}e = &\sum_{i,j,k=0}^{N-1}\left(z_{2i+1,2j+1,2k+1} - z_{2i,2j+1,2k+1} + \frac{1}{8}q_{i,j,k}\right)^2\\ &+ \sum_{i,j,k=0}^{N-1}\left(z_{2i,2j+2,2k+1} - z_{2i,2j+1,2k+1} + \frac{1}{4}h_{i,j,k}\right)^2\\ &+ \sum_{i,j,k=0}^{N-1}\left(z_{2i,2j+1,2k+2} - z_{2i,2j+1,2k+1} + \frac{1}{4}h'_{i,j,k}\right)^2,\end{aligned}$$

where

$$\begin{aligned}q_{i,j,k} =& y_{2i+2,2j,2k} + y_{2i+2,2j+2,2k} + y_{2i+2,2j,2k+2} + y_{2i+2,2j+2,2k+2}\\ &- y_{2i,2j+2,2k+2} - y_{2i,2j+2,2k} - y_{2i,2j,2k+2} - y_{2i,2j,2k},\end{aligned}$$

$$\begin{aligned}\mathcal{L}_f = &\sum_{i,j,k=0}^{N-1}\left(z_{2i+2,2j,2k+1} - z_{2i+1,2j,2k+1} + \frac{1}{4}g_{i,j,k}\right)^2\\ &+ \sum_{i,j,k=0}^{N-1}\left(z_{2i+1,2j+1,2k+1} - z_{2i+1,2j,2k+1} + \frac{1}{8}q'_{i,j,k}\right)^2\\ &+ \sum_{i,j,k=0}^{N-1}\left(z_{2i+1,2j,2k+2} - z_{2i+1,2j,2k+1} + \frac{1}{4}h''_{i,j,k}\right)^2,\end{aligned}$$

where

$$\begin{aligned}q'_{i,j,k} =& y_{2i+2,2j+2,2k+2} + y_{2i,2j+2,2k+2} + y_{2i+2,2j+2,2k} + y_{2i,2j+2,2k}\\ &- y_{2i+2,2j,2k+2} - y_{2i+2,2j,2k} - y_{2i,2j,2k+2} - y_{2i,2j,2k},\end{aligned}$$

$$\begin{aligned}\mathcal{L}_g = &\sum_{i,j,k=0}^{N-1}\left(z_{2i+2,2j+1,2k} - z_{2i+1,2j+1,2k} + \frac{1}{4}g'_{i,j,k}\right)^2\\ &+ \sum_{i,j,k=0}^{N-1}\left(z_{2i+1,2j+2,2k} - z_{2i+1,2j+1,2k} + \frac{1}{4}g''_{i,j,k}\right)^2\\ &+ \sum_{i,j,k=0}^{N-1}\left(z_{2i+1,2j+1,2k+1} - z_{2i+1,2j+1,2k} + \frac{1}{8}q''_{i,j,k}\right)^2,\end{aligned}$$

where

$$\begin{aligned}q''_{i,j,k} =& y_{2i+2,2j+2,2k+2} + y_{2i+2,2j,2k+2} + y_{2i,2j+2,2k+2} + y_{2i,2j,2k+2}\\ &- y_{2i+2,2j+2,2k} - y_{2i+2,2j,2k} - y_{2i,2j+2,2k} - y_{2i,2j,2k},\end{aligned}$$

$$\mathcal{L}_h = \sum_{i,j,k=0}^{N-1} \left(z_{2i+2,2j+1,2k+1} - z_{2i+1,2j+1,2k+1} + \frac{1}{8} q_{i,j,k} \right)^2 + \sum_{i,j,k=0}^{N-1} \left(z_{2i+1,2j+2,2k+1} - z_{2i+1,2j+1,2k+1} + \frac{1}{8} q'_{i,j,k} \right)^2 + \sum_{i,j,k=0}^{N-1} \left(z_{2i+1,2j+1,2k+2} - z_{2i+1,2j+1,2k+1} + \frac{1}{8} q''_{i,j,k} \right)^2 .$$

We thus obtain

$$\mathcal{L}(y,z) = \mathcal{L}_a + \cdots + \mathcal{L}_h = \sum_{i,j,k=0}^{N-1} \mathcal{L}_{i,j,k}(y,z),$$

where

$$\begin{aligned}
\mathcal{L}_{i,j,k}(y,z) &= 2z^2_{2i+1,2j,2k} + 2z^2_{2i,2j+1,2k} + 2z^2_{2i,2j,2k+1} \qquad (20)\\
&+ \frac{1}{2}(y_{2i+2,2j,2k} - y_{2i,2j,2k})^2 + \frac{1}{2}(y_{2i,2j+2,2k} - y_{2i,2j,2k})^2 + \frac{1}{2}(y_{2i,2j,2k+2} - y_{2i,2j,2k})^2\\
&+ (z_{2i+1,2j,2k+1} - z_{2i,2j,2k+1} + \frac{1}{4} g_{i,j,k})^2 + (z_{2i+2,2j,2k+1} - z_{2i+1,2j,2k+1} + \frac{1}{4} g_{i,j,k})^2\\
&+ (z_{2i+1,2j+1,2k} - z_{2i,2j+1,2k} + \frac{1}{4} g'_{i,j,k})^2 + (z_{2i+2,2j+1,2k} - z_{2i+1,2j+1,2k} + \frac{1}{4} g'_{i,j,k})^2\\
&+ (z_{2i+1,2j+1,2k} - z_{2i+1,2j,2k} + \frac{1}{4} g''_{i,j,k})^2 + (z_{2i+1,2j+2,2k} - z_{2i+1,2j+1,2k} + \frac{1}{4} g''_{i,j,k})^2\\
&+ (z_{2i,2j+1,2k+1} - z_{2i,2j,2k+1} + \frac{1}{4} h_{i,j,k})^2 + (z_{2i,2j+2,2k+1} - z_{2i,2j+1,2k+1} + \frac{1}{4} h_{i,j,k})^2\\
&+ (z_{2i,2j+1,2k+1} - z_{2i,2j+1,2k} + \frac{1}{4} h'_{i,j,k})^2 + (z_{2i,2j+1,2k+2} - z_{2i,2j+1,2k+1} + \frac{1}{4} h'_{i,j,k})^2\\
&+ (z_{2i+1,2j,2k+1} - z_{2i+1,2j,2k} + \frac{1}{4} h''_{i,j,k})^2 + (z_{2i+1,2j,2k+2} - z_{2i+1,2j,2k+1} + \frac{1}{4} h''_{i,j,k})^2\\
&+ (z_{2i+1,2j+1,2k+1} - z_{2i,2j+1,2k+1} + \frac{1}{8} q_{i,j,k})^2 + (z_{2i+2,2j+1,2k+1} - z_{2i+1,2j+1,2k+1} + \frac{1}{8} q_{i,j,k})^2\\
&+ (z_{2i+1,2j+1,2k+1} - z_{2i+1,2j,2k+1} + \frac{1}{8} q'_{i,j,k})^2 + (z_{2i+1,2j+2,2k+1} - z_{2i+1,2j+1,2k+1} + \frac{1}{8} q'_{i,j,k})^2\\
&+ (z_{2i+1,2j+1,2k+1} - z_{2i+1,2j+1,2k} + \frac{1}{8} q''_{i,j,k})^2 + (z_{2i+1,2j+1,2k+2} - z_{2i+1,2j+1,2k+1} + \frac{1}{8} q''_{i,j,k})^2.
\end{aligned}$$

We have the following result (see also [2]):

Lemma 1: For every $(a, b, c, g) \in \mathbb{R}^4$, and for every $0 \le \gamma \le 1$

$$(a - b + g)^2 + (c - a + g)^2 \ge \gamma(a^2 - b^2 - c^2).$$

Proof: Since the left-hand side of this inequality is positive, it suffices to prove the inequality for $\gamma = 1$ which can be done by observing that

$$(a-b+g)^2+(c-a+g)^2 = (a-b-c)^2+(-b+c+g)^2+a^2-b^2-c^2+g^2 \geq a^2-b^2-c^2. \quad \square$$

We use Lemma 1 with $\gamma = \frac{1}{4}$ for the terms involving $\frac{1}{4}\{g, g', g'', h, h', h''\}$ and with $\gamma = \frac{1}{8}$ for the terms involving $\frac{1}{8}\{q, q', q''\}$. This yields:

$$\begin{aligned}\sum_{i,j,k=0}^{N-1} \mathcal{L}_{i,j,k} \geq \sum_{i,j,k=0}^{N-1} \Big\{ & 2z^2_{2i+1,2j,2k} + 2z^2_{2i,2j+1,2k} + 2z^2_{2i,2j,2k+1} \\
& + \frac{1}{2}(y_{2i+2,2j,2k} - y_{2i,2j,2k})^2 + \frac{1}{2}(y_{2i,2j+2,2k} - y_{2i,2j,2k})^2 \\
& + \frac{1}{2}(y_{2i,2j,2k+2} - y_{2i,2j,2k})^2 + \frac{1}{4}z^2_{2i+1,2j,2k+1} - \frac{1}{4}z^2_{2i,2j,2k+1} - \frac{1}{4}z^2_{2i+2,2j,2k+1} \\
& + \frac{1}{4}z^2_{2i+1,2j+1,2k} - \frac{1}{4}z^2_{2i,2j+1,2k} - \frac{1}{4}z^2_{2i+2,2j+1,2k} + \frac{1}{4}z^2_{2i+1,2j+1,2k} \\
& - \frac{1}{4}z^2_{2i+1,2j,2k} - \frac{1}{4}z^2_{2i+1,2j+2,2k} + \frac{1}{4}z^2_{2i,2j+1,2k+1} - \frac{1}{4}z^2_{2i,2j,2k+1} \\
& - \frac{1}{4}z^2_{2i,2j+2,2k+1} + \frac{1}{4}z^2_{2i,2j+1,2k+1} - \frac{1}{4}z^2_{2i,2j+1,2k} - \frac{1}{4}z^2_{2i,2j+1,2k+2} \\
& + \frac{1}{4}z^2_{2i+1,2j,2k+1} - \frac{1}{4}z^2_{2i+1,2j,2k} - \frac{1}{4}z^2_{2i+1,2j,2k+2} + \frac{1}{8}z^2_{2i+1,2j+1,2k+1} \\
& - \frac{1}{8}z^2_{2i,2j+1,2k+1} - \frac{1}{8}z^2_{2i+2,2j+1,2k+1} + \frac{1}{8}z^2_{2i+1,2j+1,2k+1} - \frac{1}{8}z^2_{2i+1,2j,2k+1} \\
& - \frac{1}{8}z^2_{2i+1,2j+2,2k+1} + \frac{1}{8}z^2_{2i+1,2j+1,2k+1} - \frac{1}{8}z^2_{2i+1,2j+1,2k} - \frac{1}{8}z^2_{2i+1,2j+1,2k+2} \Big\}.\end{aligned}$$

Using (4), we can replace, for example, $z^2_{2i+1,2j+2,2k}$ by $z^2_{2i+1,2j,2k}$ in the sum, which gives

$$\sum_{i,j,k=0}^{N-1} \mathcal{L}_{i,j,k} \geq \sum_{i,j,k=0}^{N-1} \mathcal{L}'_{i,j,k},$$

where

$$\begin{aligned}\mathcal{L}'_{i,j,k}(y,z) = {} & z^2_{2i+1,2j,2k} + z^2_{2i,2j+1,2k} + z^2_{2i,2j,2k+1} + \frac{1}{4}z^2_{2i+1,2j+1,2k} + \frac{1}{4}z^2_{2i+1,2j,2k+1} \\
& + \frac{1}{4}z^2_{2i,2j+1,2k+1} + \frac{3}{8}z^2_{2i+1,2j+1,2k+1} + \frac{1}{2}(y_{2i+2,2j,2k} - y_{2i,2j,2k})^2 \\
& + \frac{1}{2}(y_{2i,2j+2,2k} - y_{2i,2j,2k})^2 + \frac{1}{2}(y_{2i,2j,2k+2} - y_{2i,2j,2k})^2.\end{aligned}$$

If we replace $f_{\alpha,\beta,\gamma} = f(\alpha h, \beta h, \gamma h)$ by $f^*_{\alpha,\beta,\gamma} = \frac{1}{h^3}\int_{\alpha h}^{(\alpha+1)h}\int_{\beta h}^{(\beta+1)h}\int_{\gamma h}^{(\gamma+1)h} f(x)dx$, then the right-hand-side of (19) is bounded by $\frac{1}{3}\frac{1}{h} \mid f \mid^2 = \frac{1}{3h}\int_{(0,1)^3} \mid f(x_1,x_2,x_3) \mid^2 dx$, and thus

$$\begin{aligned}\sum_{i,j,k=0}^{N-1} \{ & z^2_{2i+1,2j,2k} + z^2_{2i,2j+1,2k} + z^2_{2i,2j,2k+1} + z^2_{2i+1,2j+1,2k} + z^2_{2i+1,2j,2k+1} \\ & + z^2_{2i,2j+1,2k+1} + z^2_{2i+1,2j+1,2k+1} + (y_{2i+2,2j,2k} - y_{2i,2j,2k})^2 \\ & + (y_{2i,2j+2,2k} - y_{2i,2j,2k})^2 + (y_{2i,2j,2k+2} - y_{2i,2j,2k})^2\} \\ & \le \frac{4}{3h} \mid f \mid^2 .\end{aligned} \tag{21}$$

In conclusion, we have for $h \to 0$

$$\begin{aligned} h^3 \sum_{i,j,k=0}^{N-1} & \big(z^2_{2i+1,2j,2k} + z^2_{2i,2j+1,2k} + z^2_{2i,2j,2k+1} + z^2_{2i+1,2j+1,2k} \\ & + z^2_{2i+1,2j,2k+1} + z^2_{2i,2j+1,2k+1} + z^2_{2i+1,2j+1,2k+1}\big) \le \text{ const. } h^2, \end{aligned} \tag{22}$$

$$\sum_{i=0}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} (y_{2i+2,2j,2k} - y_{2i,2j,2k})^2 \le \frac{\text{const.}}{h}, \tag{23}$$

$$\sum_{i=1}^{N-1}\sum_{j=0}^{N-1}\sum_{k=1}^{N-1} (y_{2i,2j+2,2k} - y_{2i,2j,2k})^2 \le \frac{\text{const.}}{h}, \tag{24}$$

$$\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=0}^{N-1} (y_{2i,2j,2k+2} - y_{2i,2j,2k})^2 \le \frac{\text{const.}}{h}. \tag{25}$$

3. Variational formulation.

We consider the spaces $H = L^2(\Omega), V = H^1_0(\Omega)$ which are endowed with their usual scalar products which we denote $(\cdot,\cdot)$ for H and $((\cdot,\cdot))$ for V ($|\cdot|$ and $||\cdot||$ denote the corresponding norms). The variational formulation of (1)-(2) is classical. We look for $v \in V$ such that

$$((u, v)) = (f, v), \forall v \in V.$$

Let now V_h be the finite dimensional space which consists of functions u_h which are constant on each cube $[\alpha h, (\alpha+1)h) \times [\beta h, (\beta+1)h) \times [\gamma h, (\gamma+1)h)$. This space is spanned by the functions $w_{h,M}$, $M = (\alpha h, \beta h, \gamma h), \alpha, \beta, \gamma = 1, \ldots, 2N-1$, which are equal to 1 on the cube $[\alpha h, (\alpha+1)h) \times [\beta h, (\beta+1)h) \times [\gamma h, (\gamma+1)h)$ and 0 outside this cube. Thus

$$u_h(x) = \sum_{M \in \Omega_h} u_h(M) w_{h,M}(x), x \in \Omega,$$

where $\Omega_h = \{(\alpha h, \beta h, \gamma h), \alpha, \beta, \gamma = 1, \dots, 2N-1\}$. We introduce the finite differences operators

$$\nabla_{i,h}\phi(x) = \frac{1}{h}(\phi(x + he_i) - \phi(x)), i = 1, 2, 3,$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. We then endow V_h with the discrete scalar product

$$((u_h, v_h))_h = \sum_{i=1}^{3} (\nabla_{i,h} u_h, \nabla_{i,h} v_h).$$

We then consider the following variational problem:

Find $u_h \in V_h$ such that

$$((u_h, v_h))_h = (f, v_h), \forall v_h \in V_h.$$

We now consider the space decomposition

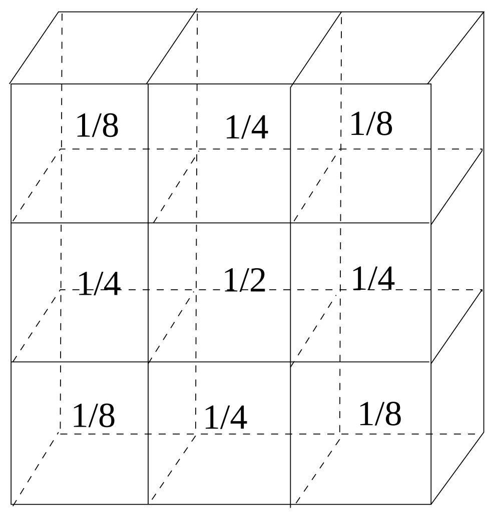
$$V_h = Y_h \oplus Z_h,$$

where Y_h is the space spanned by the functions $\phi_{2h,M}, M \in \Omega_{2h}$, where $\phi_{2h,M}$ is given by (see figure 2):

$$\begin{aligned}
\phi_{2h,M} =& w_{h,M} + \frac{1}{2}\{w_{h,M+he_1} + w_{h,M+he_2} + w_{h,M+he_3} + w_{h,M-he_1} \\
&+ w_{h,M-he_2} + w_{h,M-he_3}\} + \frac{1}{4}\{w_{h,M+he_1+he_2} + w_{h,M+he_1-he_2} \\
&+ w_{h,M-he_1+he_2} + w_{h,M-he_1-he_2} + w_{h,M+he_1+he_3} + w_{h,M+he_1-he_3} \\
&+ w_{h,M-he_1+he_3} + w_{h,M-he_1-he_3} + w_{h,M+he_2+he_3} + w_{h,M+he_2-he_3} \\
&+ w_{h,M-he_2+he_3} + w_{h,M-he_2-he_3}\} + \frac{1}{8}\{w_{h,M+he_1+he_2+he_3} \\
&+ w_{h,M+he_1+he_2-he_3} + w_{h,M+he_1-he_2-he_3} + w_{h,M+he_1-he_2+he_3} \\
&+ w_{h,M-he_1+he_2+he_3} + w_{h,M-he_1-he_2+he_3} + w_{h,M-he_1+he_2-he_3} \\
&+ w_{h,M-he_1-he_2-he_3}\},
\end{aligned}$$

and Z_h is the space spanned by the functions $w_{h,M}, M \in \Omega_h \backslash \Omega_{2h}$. Writing $u_h = y_h + z_h, y_h \in Y_h, z_h \in Z_h$, we obtain the incremental unknowns introduced above by using the same method as in [4] for the two-dimensional case.

south and north faces



middle face

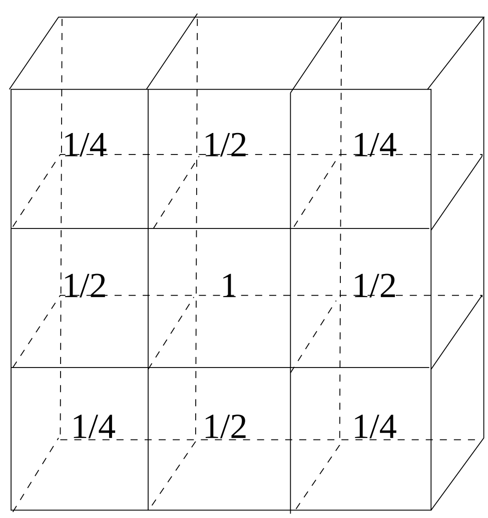


FIGURE 2. The values of the basis function $\phi_{2h,M}$.

Now we have

$$\| y_h + z_h \|_h^2 = \| u_h \|_h^2 = h\,[\sum_{\alpha=0}^{2N-1}\sum_{\beta=0}^{2N-1}\sum_{\gamma=0}^{2N-1}(u_{\alpha+1,\beta,\gamma}-u_{\alpha,\beta,\gamma})^2$$

$$+\sum_{\alpha=0}^{2N-1}\sum_{\beta=0}^{2N-1}\sum_{\gamma=0}^{2N-1}(u_{\alpha,\beta+1,\gamma}-u_{\alpha,\beta,\gamma})^2+\sum_{\alpha=0}^{2N-1}\sum_{\beta=0}^{2N-1}\sum_{\gamma=0}^{2N-1}(u_{\alpha,\beta,\gamma+1}-u_{\alpha,\beta,\gamma})^2\,]$$

$$= h\sum_{i,j,k=0}^{N-1}\mathcal{L}_{i,j,k} \geq h\sum_{i,j,k=0}^{N-1}\mathcal{L}'_{i,j,k},$$

that is to say

$$\| y_h + z_h \|_h^2 \geq \frac{h}{4}\sum_{i,j,k=0}^{N-1}\{z^2_{2i+1,2j,2k}+z^2_{2i,2j+1,2k}+z^2_{2i,2j,2k+1}+z^2_{2i+1,2j+1,2k}$$

$$+z^2_{2i+1,2j,2k+1}+z^2_{2i,2j+1,2k+1}+z^2_{2i+1,2j+1,2k+1}+(y_{2i+2,2j,2k}-y_{2i,2j,2k})^2$$

$$+(y_{2i,2j+2,2k}-y_{2i,2j,2k})^2+(y_{2i,2j,2k+2}-y_{2i,2j,2k})^2\}$$

$$=\frac{1}{4h^2}|z_h|_h^2+\frac{1}{4}||y_h||_{2h}^2.$$

Now

$$\| y_h \|_h^2 = (\text{ we take } z=0 \text{ in } \mathcal{L}_{i,j,k}(y,z))$$

$$= h\sum_{i,j,k=0}^{2N-1}\{\frac{1}{2}(y_{2i+2,2j,2k}-y_{2i,2j,2k})^2$$

$$+\frac{1}{2}(y_{2i,2j+2,2k}-y_{2i,2j,2k})^2+\frac{1}{2}(y_{2i,2j,2k+2}-y_{2i,2j,2k})^2$$

$$+\frac{1}{8}(y_{2i+2,2j,2k+2}-y_{2i,2j,2k+2}+y_{2i+2,2j,2k}-y_{2i,2j,2k})^2$$

$$+\frac{1}{8}(y_{2i,2j+2,2k+2}-y_{2i,2j,2k+2}+y_{2i,2j+2,2k}-y_{2i,2j,2k})^2$$

$$+\frac{1}{8}(y_{2i+2,2j+2,2k}-y_{2i,2j+2,2k}+y_{2i+2,2j,2k}-y_{2i,2j,2k})^2$$

$$+\frac{1}{8}(y_{2i,2j+2,2k+2}-y_{2i,2j+2,2k}+y_{2i,2j,2k+2}-y_{2i,2j,2k})^2$$

$$+\frac{1}{8}(y_{2i+2,2j+2,2k}-y_{2i+2,2j,2k}+y_{2i,2j+2,2k}-y_{2i,2j,2k})^2$$

$$+\frac{1}{8}(y_{2i+2,2j,2k+2}-y_{2i+2,2j,2k}+y_{2i,2j,2k+2}-y_{2i,2j,2k})^2$$

$$+\frac{1}{32}(y_{2i+2,2j,2k}-y_{2i,2j,2k}+y_{2i+2,2j+2,2k}-y_{2i,2j+2,2k}$$

$$+y_{2i+2,2j,2k+2}-y_{2i,2j,2k+2}+y_{2i+2,2j+2,2k+2}-y_{2i,2j+2,2k+2})^2$$

$$+\frac{1}{32}(y_{2i,2j+2,2k}-y_{2i,2j,2k}+y_{2i,2j+2,2k+2}-y_{2i,2j,2k+2}$$
$$+y_{2i+2,2j+2,2k}-y_{2i+2,2j,2k}+y_{2i+2,2j+2,2k+2}-y_{2i+2,2j,2k+2})^2$$
$$+\frac{1}{32}(y_{2i,2j,2k+2}-y_{2i,2j,2k}+y_{2i,2j+2,2k+2}-y_{2i,2j+2,2k}$$
$$+y_{2i+2,2j,2k+2}-y_{2i+2,2j,2k}+y_{2i+2,2j+2,2k+2}-y_{2i+2,2j+2,2k})^2\}.$$

We have, for instance

$$\begin{aligned}
&\frac{1}{8}\sum_{i,j,k=0}^{N-1}(y_{2i+2,2j,2k+2}-y_{2i,2j,2k+2}+y_{2i+2,2j,2k}-y_{2i,2j,2k})^2\\
&\leq\frac{1}{4}\sum_{i,j,k=0}^{N-1}\left[(y_{2i+2,2j,2k+2}-y_{2i,2j,2k+2})^2+(y_{2i+2,2j,2k}-y_{2i,2j,2k})^2\right]\\
&\leq\frac{1}{2}\sum_{i,j,k=0}^{N-1}(y_{2i+2,2j,2k}-y_{2i,2j,2k})^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\frac{1}{32}\sum_{i,j,k=0}^{N-1}(y_{2i+2,2j,2k}-y_{2i,2j,2k}+y_{2i+2,2j+2,2k}-y_{2i,2j+2,2k}\\
&\qquad+y_{2i+2,2j,2k+2}-y_{2i,2j,2k+2}+y_{2i+2,2j+2,2k+2}-y_{2i,2j+2,2k+2})^2\\
&\leq\frac{1}{8}\sum_{i,j,k=0}^{N-1}\left[(y_{2i+2,2j,2k}-y_{2i,2j,2k})^2+(y_{2i+2,2j+2,2k}-y_{2i,2j+2,2k})^2\right.\\
&\qquad\left.+(y_{2i+2,2j,2k+2}-y_{2i,2j,2k+2})^2+(y_{2i+2,2j+2,2k+2}-y_{2i,2j+2,2k+2})^2\right]\\
&\leq\frac{1}{2}\sum_{i,j,k=0}^{N-1}(y_{2i+2,2j,2k}-y_{2i,2j,2k})^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\| y_h \|_h^2\leq 2h\sum_{i,j,k=0}^{N-1}&\left[(y_{2i+2,2j,2k}-y_{2i,2j,2k})^2\right.\\
&\left.+\,(y_{2i,2j+2,2k}-y_{2i,2j,2k})^2+(y_{2i,2j,2k+2}-y_{2i,2j,2k})^2\right],
\end{aligned}$$

that is to say

$$\| y_h \|_h^2\leq 2 \| y_h \|_{2h}^2 . \tag{26}$$

Moreover

$$\begin{aligned}
&\| z_h \|_h^2 = (\text{ we take } y=0 \text{ in } \mathcal{L}_{i,j,k}(y,z))\\
&= h \sum_{i,j,k=0}^{N-1} \big[2z^2_{2i+1,2j,2k} + 2z^2_{2i,2j+1,2k} + 2z^2_{2i,2j,2k+1}\\
&+ \left(z_{2i+1,2j,2k+1} - z_{2i,2j,2k+1}\right)^2 + \left(z_{2i+2,2j,2k+1} - z_{2i+1,2j,2k+1}\right)^2\\
&+ \left(z_{2i+1,2j+1,2k} - z_{2i,2j+1,2k}\right)^2 + \left(z_{2i+2,2j+1,2k} - z_{2i+1,2j+1,2k}\right)^2\\
&+ \left(z_{2i+1,2j+1,2k} - z_{2i+1,2j,2k}\right)^2 + \left(z_{2i+1,2j+2,2k} - z_{2i+1,2j+1,2k}\right)^2\\
&+ \left(z_{2i,2j+1,2k+1} - z_{2i,2j,2k+1}\right)^2 + \left(z_{2i,2j+2,2k+1} - z_{2i,2j+1,2k+1}\right)^2\\
&+ \left(z_{2i,2j+1,2k+1} - z_{2i,2j+1,2k}\right)^2 + \left(z_{2i,2j+1,2k+2} - z_{2i,2j+1,2k+1}\right)^2\\
&+ \left(z_{2i+1,2j,2k+1} - z_{2i+1,2j,2k}\right)^2 + \left(z_{2i+1,2j,2k+2} - z_{2i+1,2j,2k+1}\right)^2\\
&+ \left(z_{2i+1,2j+1,2k+1} - z_{2i,2j+1,2k+1}\right)^2 + \left(z_{2i+2,2j+1,2k+1} - z_{2i+1,2j+1,2k+1}\right)^2\\
&+ \left(z_{2i+1,2j+1,2k+1} - z_{2i+1,2j,2k+1}\right)^2 + \left(z_{2i+1,2j+2,2k+1} - z_{2i+1,2j+1,2k+1}\right)^2\\
&+ \left(z_{2i+1,2j+1,2k+1} - z_{2i+1,2j+1,2k}\right)^2 + \left(z_{2i+1,2j+1,2k+2} - z_{2i+1,2j+1,2k+1}\right)^2 \big]\\
&\\
&\le (\text{ with the same techniques as above })\\
&\le h \sum_{i,j,k=0}^{N-1} \big(10z^2_{2i+1,2j,2k} + 10z^2_{2i,2j+1,2k} + 10z^2_{2i,2j,2k+1}\\
&+ 10z^2_{2i+1,2j+1,2k} + 10z^2_{2i+1,2j,2k+1} + 10z^2_{2i,2j+1,2k+1} + 12z^2_{2i+1,2j+1,2k+1}\ \big),
\end{aligned}$$

which gives

$$\begin{aligned}
\| z_h \|_h^2 \le 12h \sum_{i,j,k=0}^{N-1} (&z^2_{2i+1,2j,2k} + z^2_{2i,2j+1,2k} + z^2_{2i,2j,2k+1} + z^2_{2i+1,2j+1,2k}\\
&+ z^2_{2i+1,2j,2k+1} + z^2_{2i,2j+1,2k+1} + z^2_{2i+1,2j+1,2k+1}),
\end{aligned}$$

that is to say

$$\| z_h \|_h^2 \le \frac{12}{h^2} \mid z_h \mid^2 . \tag{27}$$

We then obtain

$$\| y_h + z_h \|_h^2 \ge \frac{1}{48} \| z_h \|_h^2 + \frac{1}{8} \| y_h \|_h^2 .$$

Therefore, we have the inverse triangle inequality:

$$\| y_h + z_h \|_h \ge \frac{1}{4\sqrt{3}} (\| z_h \|_h^2 + \| y_h \|_h^2)^{\frac{1}{2}}, \tag{28}$$

$\forall y_h \in Y_h, z_h \in Z_h$. If we take $\| y_h \|_h = \| z_h \|_h = 1$ in (28) we find

$$\| y_h + z_h \|_h^2 \geq \frac{1}{24},$$

which gives

$$-((y_h, z_h))_h \leq 1 - \frac{1}{48}.$$

Therefore, changing z_h into $-z_h$ we find

$$((y_h, z_h))_h \leq \frac{47}{48}.$$

Now if z_h and y_h are $\neq 0$, then

$$\left| \left(\left(\frac{y_h}{\| y_h \|_h}, \frac{z_h}{\| z_h \|_h}\right)\right)_h \right| \leq \frac{47}{48},$$

and we obtain the enhanced Cauchy-Schwarz inequality:

$$|((y_h, z_h))_h| \leq \frac{47}{48} \| y_h \|_h \| z_h \|_h, \tag{29}$$

$\forall y_h \in Y_h, z_h \in Z_h$. Finally

$$\begin{aligned} | z_h |^2 = h^3 \sum_{i,j,k=0}^{N-1} (& z_{2i+1,2j,2k}^2 + z_{2i,2j+1,2k}^2 + z_{2i,2j,2k+1}^2 + z_{2i+1,2j,2k+1}^2 \\ & + z_{2i+1,2j+1,2k}^2 + z_{2i,2j+1,2k+1}^2 + z_{2i+1,2j+1,2k+1}^2), \end{aligned}$$

and since

$$\begin{aligned} \| z_h \|_h^2 \geq \frac{h}{4} \sum_{i,j,k=0}^{N-1} (& z_{2i+1,2j,2k}^2 + z_{2i,2j+1,2k}^2 + z_{2i,2j,2k+1}^2 + z_{2i+1,2j+1,2k}^2 \\ & + z_{2i+1,2j,2k+1}^2 + z_{2i,2j+1,2k+1}^2 + z_{2i+1,2j+1,2k+1}^2), \end{aligned}$$

we find

$$\frac{1}{4h^2} | z_h |^2 \leq \| z_h \|_h^2 .$$

We then obtain the enhanced Poincaré inequality

$$\frac{1}{2h} | z_h | \leq \| z_h \|_h, \forall z_h \in Z_h. \tag{30}$$

A first consequence of (28), (30) is that the L^2- norm of z_h is small. We have

$$((u_h, v_h))_h = (f, v_h), \forall v_h \in V_h,$$

which gives for $v_h = u_h$

$$\| u_h \|_h^2 = (f, u_h) \leq | f || u_h | .$$

We have the discrete Poincaré inequality (see [21])

$$| u_h | \leq c \| u_h \|_h, \forall u_h \in V_h, \tag{31}$$

where c is independent of h. Therefore

$$\| u_h \|_h^2 \leq c | f | \| u_h \|_h,$$

and

$$\| u_h \|_h^2 \leq c^2 | f |^2,$$

that is to say

$$\| y_h + z_h \|_h^2 \leq c^2 | f |^2 .$$

Thanks to (28) we have

$$\| y_h \|_h^2 + \| z_h \|_h^2 \leq c | f |^2,$$

and thus, thanks to (30)

$$| z_h |^2 \leq ch^2 | f |^2,$$

where c is independent of h, hence the result.

4. Condition number of the matrix.

4.1. Mathematical setting.

We consider here two discretization meshes h_0 and $h_d, h_d = \frac{h_0}{2^d}, d > 0$. We associate to these meshes the grids

$$\mathcal{R}_0 = \mathcal{R}_{h_0} \text{ made of points } (j_1 h_0, j_2 h_0, j_3 h_0),$$
$$\mathcal{R}_d = \mathcal{R}_{h_d} \text{ made of points } (j_1 h_d, j_2 h_d, j_3 h_d),$$

where $j_1, j_2, j_3 \in \mathbb{Z}$. Here $\mathcal{R}_0$ is the coarse grid and $\mathcal{R}_d$ the fine grid. We set, for $j = (j_1, j_2, j_3) \in \mathbb{Z}^3$,

$$K_{j,d} = (j_1 h_d, (j_1 + 1)h_d) \times (j_2 h_d, (j_2 + 1)h_d) \times (j_3 h_d, (j_3 + 1)h_d).$$

We denote by V_d the space of continuous real functions on $\bar{\Omega}$ that are Q_1 (i.e. affine with respect to x_1, x_2 and x_3 separately) on each cube $K_{j,d} \subset \Omega$ and by U_d the set of nodal points

$$U_d = \mathcal{R}_d \cap \bar{\Omega}.$$

Similarly we define the spaces $V_\ell, 0 \leq \ell \leq d$, and we observe that $V_0 \subset V_1 \subset \ldots \subset V_d$. Moreover, these spaces are endowed with the scalar products and norms induced by L^2 and H^1. In this section, $((\cdot,\cdot))$ denotes the scalar product in H^1 and $||\cdot||$ is the corresponding norm:

$$(u,v) = \int_\Omega u \cdot v dx, \mid u \mid = (u,u)^{\frac{1}{2}},$$
$$((u,v)) = (u,v) + \int_\Omega \nabla u \cdot \nabla v dx, \| u \| = ((u,u))^{\frac{1}{2}}.$$

Since $V_{d-1} \subset V_d$ we can define a supplement W_{d-1} of V_{d-1} in V_d. We write

$$V_d = V_{d-1} \oplus W_{d-1},$$

where W_{d-1} is the subspace of V_d consisting of the functions $z \in V_d$ that vanish at the coarse grid points, i.e.

$$z(M) = 0, \forall M \in U_{d-1}.$$

Since the functions of V_d are uniquely defined by their values at the nodal points on U_d, every function $u \in V_d$ can be uniquely written as the sum

$$u = y + z, y \in V_{d-1}, z \in W_{d-1}, \tag{32}$$

so that

$$y(M) = u(M), \forall M \in U_{d-1}, \tag{33}$$

$$z(P) = u(P) - y(P), \forall P \in U_d \backslash U_{d-1}. \tag{34}$$

Let $\mathcal{F}_d$ be the set of all the cubes $K_{j,d}$. The cubes of $\mathcal{F}_d$ are obtained by dividing the cubes of $\mathcal{F}_{d-1}$ into eight equal cubes (see figure 1). Since y is affine along the edges of $K \in \mathcal{F}_{d-1}$, we have with (33) and (34) (see figure 1):

$$\begin{aligned}
z(P_1) &= u(P_1) - \frac{1}{2}(u(M_1) + u(M_4)),\\
z(P_2) &= u(P_2) - \frac{1}{2}(u(M_1) + u(M_2)),\\
z(P_5) &= u(P_5) - \frac{1}{2}(u(M_1) + u(M_5)),\\
z(P_3) &= u(P_3) - \frac{1}{4}(u(M_1) + u(M_2) + u(M_3) + u(M_4)),\\
z(P_4) &= u(P_4) - \frac{1}{4}(u(M_1) + u(M_4) + u(M_5) + u(M_8)),\\
z(P_6) &= u(P_6) - \frac{1}{4}(u(M_1) + u(M_2) + u(M_5) + u(M_6)),\\
z(P_7) &= u(P_7) - \frac{1}{8}(u(M_1) + u(M_2) + u(M_3) + u(M_4) + u(M_5) + u(M_6)\\
&\qquad + u(M_7) + u(M_8)).
\end{aligned} \tag{35}$$

Hence the nodal values of z at the points $P_i \in U_d \backslash U_{d-1}$ are the incremental values of u as defined in Section 1.

We can reiterate the decomposition and write

$$V_{d-1} = V_{d-2} \oplus W_{d-2}, \dots ,$$

and finally

$$V_d = V_0 \oplus W_1 \oplus \cdots \oplus W_{d-1}. \tag{36}$$

We define the linear interpolation operator $r_\ell, 0 \leq \ell \leq d$, which associates to any continuous function u in $\bar{\Omega}$ the function $r_\ell u \in V_\ell$ defined by its nodal values

$$r_\ell u(M) = u(M), \forall M \in U_\ell.$$

Of course, if $u \in V_d, u = r_d u$, and we have if $u \in V_d$

$$u = r_d u = r_0 u + \sum_{\ell=1}^{d} (r_\ell u - r_{\ell-1} u), \tag{37}$$

which gives the decomposition of u corresponding to (36) since $r_0 u \in V_0$ and $r_\ell u - r_{\ell-1} u \in W_{\ell-1}$. Furthermore the $(r_\ell u - r_{\ell-1} u)(x), x \in U_\ell \backslash U_{\ell-1}, \ell = 1, \cdots d$, are the incremental values of u at the different levels of discretization. If $u \in V_d$, we set

$$[u]_d^2 = \sum_{\ell=1}^{d} \sum_{x \in U_\ell \backslash \ U_{\ell-1}} (r_\ell u - r_{\ell-1} u)(x) \ |^2, \tag{38}$$

$$[[u]]_d^2 = \sum_{x \in U_0} \mid r_0 u(x) \mid^2 + [u]_d^2. \tag{39}$$

Here $[\cdot]_d$ is a semi-norm on V_d and $[[\cdot]]_d$ is a norm. To a function $u \in V_d$, we associate the step function $\tilde{u}$ such that

$$\tilde{u}(x) = u(j_1 h_d, j_2 h_d, j_3 h_d), \forall x \in K_{j,d}.$$

We then define the discrete H^1-scalar product and norm:

$$((u, v))_d = \int_\Omega \tilde{u}(x) \cdot \tilde{v}(x) dx + \sum_{i=1}^{3} \int_\Omega \nabla_{i,h_d} \tilde{u}(x) \cdot \nabla_{i,h_d} \tilde{v}(x) dx,$$
$$\| u \|_d = [((u, u))_d]^{\frac{1}{2}}, \forall u, v \in V_d.$$

We now state the following result which will be proved in the next subsection:

Theorem 1: There exist two constants c_1 and c_2 that depend only on the shape of Ω such that for every $u \in V_d$

$$\frac{c_1 h_d}{d^3}(\mid \nabla r_0 u \mid^2 + [u]_d^2) \leq \mid \nabla u \mid^2 \leq c_2 d(\mid \nabla r_0 u \mid^2 + |\Omega|^{\frac{1}{3}} [u]_d^2), \tag{40}$$

$$\begin{aligned} \frac{c_1 h_d}{d^3}(\mid \nabla r_0 u \mid^2 + \mid \Omega \mid^{-\frac{2}{3}} \mid r_0 u \mid^2 + [u]_d^2) &\leq (\mid \nabla u \mid^2 + \mid \Omega \mid^{-\frac{2}{3}} \mid u \mid^2) \\ &\leq c_2 d(\mid \nabla r_0 u \mid^2 + \mid \Omega \mid^{\frac{1}{3}} [u]_d^2 + \mid \Omega \mid^{-\frac{2}{3}} \mid r_0 u \mid^2). \end{aligned} \tag{41}$$

4.2. Proof of Theorem 1.

We first establish the following result:

Lemma 2: There exist three constants c_1, c_2, c_3 that depend only on the shape of Ω such that

$$\mid u \mid_{L^\infty(\Omega)} \leq \frac{c_1}{h_d^{\frac{1}{2}}} \ell n \left(\frac{\mid \Omega \mid}{h_d^3} \right) \mid \nabla u \mid, \forall u \in V_d \cap H_0^1(\Omega), \tag{42}$$

$$\mid u - \bar{u} \mid_{L^\infty(\Omega)} \leq \frac{c_2}{h_d^{\frac{1}{2}}} \ell n \left(\frac{\mid \Omega \mid}{h_d^3} \right) \mid \nabla u \mid, \forall u \in V_d, \tag{43}$$

$$\mid u \mid_{L^\infty(\Omega)} \leq \frac{c_3}{h_d^{\frac{1}{2}}} \ell n \left(\frac{\mid \Omega \mid}{h_d^3} \right) (\mid \nabla u \mid^2 + \mid \Omega \mid^{-1} \mid u \mid^2)^{\frac{1}{2}}, \forall u \in V_d, \tag{44}$$

where $\bar{u}$ is the average of u on Ω and $\mid \Omega \mid$ is the volume of Ω.

Proof: If $u \in H_0^1(\Omega)$, then $r_d u \in V_d \cap H_0^1(\Omega)$. Moreover we have

$$\begin{aligned} r_d u(x) = u(A_0) + \frac{u(A_1)-u(A_0)}{h_d}x_1 + \frac{u(A_7)-u(A_0)}{h_d}x_2 + \frac{u(A_3)-u(A_0)}{h_d}x_3 \\ + \frac{u(A_4)+u(A_0)-u(A_1)-u(A_7)}{h_d^2}x_1x_2 + \frac{u(A_2)+u(A_0)-u(A_1)-u(A_3)}{h_d^2}x_1x_3 \\ + \frac{u(A_6)+u(A_0)-u(A_3)-u(A_7)}{h_d^2}x_2x_3 \\ + \frac{u(A_1)+u(A_3)+u(A_5)+u(A_7)-u(A_0)-u(A_2)-u(A_4)-u(A_6)}{h_d^3}x_1x_2x_3, \end{aligned} \tag{45}$$

on the cube K (see figure 3).

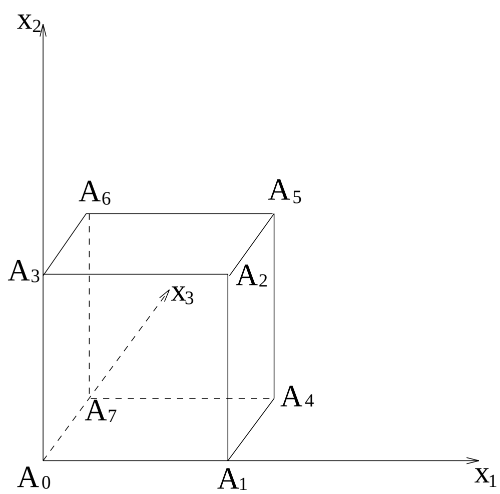


FIGURE 3. The cube K.

Therefore

$$\begin{aligned} \frac{\partial r_d u}{\partial x_1} =& (1-\frac{x_2}{h_d}-\frac{x_3}{h_d}+\frac{x_2x_3}{h_d^2})\frac{u(A_1)-u(A_0)}{h_d} - (1-\frac{x_3}{h_d})\frac{x_2}{h_d}\frac{u(A_7)-u(A_4)}{h_d} \\ & - (1-\frac{x_2}{h_d})\frac{x_3}{h_d}\frac{u(A_3)-u(A_2)}{h_d} + \frac{x_2x_3}{h_d^2}\frac{u(A_5)-u(A_6)}{h_d}, \end{aligned} \tag{46}$$

$$\begin{aligned}\frac{\partial r_d u}{\partial x_2} =&(1-\frac{x_1}{h_d}-\frac{x_3}{h_d}+\frac{x_1x_3}{h_d^2})\frac{u(A_7)-u(A_0)}{h_d}-(1-\frac{x_3}{h_d})\frac{x_1}{h_d}\frac{u(A_1)-u(A_4)}{h_d}\\&-(1-\frac{x_1}{h_d})\frac{x_3}{h_d}\frac{u(A_3)-u(A_6)}{h_d}+\frac{x_1x_3}{h_d^2}\frac{u(A_5)-u(A_2)}{h_d},\end{aligned}\tag{47}$$

$$\begin{aligned}\frac{\partial r_d u}{\partial x_3} =&(1-\frac{x_1}{h_d}-\frac{x_2}{h_d}+\frac{x_1x_2}{h_d^2})\frac{u(A_3)-u(A_0)}{h_d}-(1-\frac{x_2}{h_d})\frac{x_1}{h_d}\frac{u(A_1)-u(A_2)}{h_d}\\&-(1-\frac{x_1}{h_d})\frac{x_2}{h_d}\frac{u(A_7)-u(A_6)}{h_d}+\frac{x_1x_2}{h_d^2}\frac{u(A_5)-u(A_4)}{h_d},\end{aligned}\tag{48}$$

on the cube $K \in \mathcal{F}_d$. Thus, if $u \in V_d$

$$\int_K \mid \nabla u \mid^3 dx \leq \frac{c}{h_d} \mid u \mid_{L^\infty(\Omega)} \int_K \mid \nabla u \mid^2 dx, \tag{49}$$

where c is a numerical constant. Furthermore, thanks to [8], Lemma 7.12 we have

$$\mid u \mid_{L^q(\Omega)} \leq c q^{\frac{2}{3}} \mid \Omega \mid^{\frac{1}{q}} \mid u \mid_{W_0^{1,3}(\Omega)}, \forall q \geq 3, \tag{50}$$

where c is an absolute constant. Now, it is proved in [1] that

$$\mid u \mid_{L^\infty(\Omega)} \leq c h_d^{\frac{-3}{q}} \mid u \mid_{L^q(\Omega)}, \tag{51}$$

$\forall u \in V_d$, where c is an absolute constant. We sum (49) for all $K \in \mathcal{F}_d$ and we obtain thanks to (50) and (51)

$$\mid u \mid^2_{L^\infty(\Omega)} \leq \frac{c}{h_d} q^2 \mid \Omega \mid^{\frac{3}{q}} h_d^{-\frac{9}{q}} \mid \nabla u \mid^2 .$$

Taking $q = \ell n(\frac{|\Omega|}{h_d^3})$, we find (42).

Now, thanks to [8], Lemma 7.16 we obtain

$$\mid u - \bar{u} \mid_{L^q(\Omega)} \leq c q^{\frac{2}{3}} \mid \Omega \mid^{\frac{1}{q}} \mid \nabla u \mid_{L^3(\Omega)}, \tag{52}$$

$\forall u \in V_d, \forall q \geq 3$, where c depends only on the shape of Ω, and we obtain (43) and (44) as above. $\square$

Inequalities (43) and (44) are also valid if we replace Ω by a cube $K \in \mathcal{F}_\ell$, $\mid \Omega \mid$ being replaced by h_ℓ^3 and $\frac{|\Omega|}{h_d^3}$ by $8^{d-\ell}$. In that case, (43) and (44) become

$$\mid u - \bar{u}_K \mid_{L^\infty(K)} \leq \frac{c}{h_\ell^{\frac{1}{2}}} (d-\ell) \mid \nabla u \mid_{L^2(K)}, \tag{53}$$

$$| u |_{L^\infty(K)} \leq \frac{c}{h_\ell^{\frac{1}{2}}}(d-\ell)(| \nabla u |^2_{L^2(K)} + h_\ell^{-3} | u |^2_{L^2(K)})^{\frac{1}{2}}. \tag{54}$$

Now, if $u \in V_d$, we have

$$| \nabla r_\ell u |_{L^3(K)} \leq c' | r_\ell u - \overline{r_\ell u}_K |_{L^\infty(K)}, \tag{55}$$

where c' depends only on the shape of K (i.e. of Ω), (55) being invariant by homothety. Here $\bar{u}_K$ and $\overline{r_\ell u}_K$ are the average on K. More precisely, we have

$$| \nabla(r_\ell u) |_{L^3(K)} \leq c' Inf_{\alpha \in \mathbb{R}} | r_\ell u - \alpha |_{L^\infty(K)}, \tag{56}$$

and thus for $\alpha = \bar{u}_K$

$$| \nabla(r_\ell u) |_{L^3(K)} \leq c' | r_\ell u - \bar{u}_K |_{L^\infty(K)} \leq c' | u - \bar{u}_K |_{L^\infty(K)},$$

since $r_\ell u$ agrees with u at the vertices of K. Therefore, thanks to (53) we obtain

$$| \nabla(r_\ell u) |_{L^3(K)} \leq \frac{c}{h_\ell^{\frac{1}{2}}}(d-\ell) | \nabla u |_{L^2(K)} .$$

Since

$$| \nabla(r_\ell u) |_{L^2(K)} \leq | K |^{\frac{1}{6}} | \nabla(r_\ell u) |_{L^3(K)} \leq h_\ell^{\frac{1}{2}} | \nabla(r_\ell u) |_{L^3(K)},$$

we find

$$| \nabla(r_\ell u) |^2_{L^2(K)} \leq c(d-\ell)^2 | \nabla u |^2_{L^2(K)} . \tag{57}$$

Moreover, if $u \in V_d$, we have

$$\begin{aligned} | r_\ell u |^2_{L^2(K)} &\leq h_\ell^3 | u |^2_{L^\infty(K)} \leq (\text{ thanks to (54)}) \\ &\leq c h_\ell^2 (d-\ell)^2 (| \nabla u |^2_{L^2(K)} + h_\ell^{-3} | u |^2_{L^2(K)}), \end{aligned}$$

and thus

$$| r_\ell u |^2_{L^2(K)} \leq c(d-\ell)^2 (h_\ell^2 | \nabla u |^2_{L^2(K)} + h_\ell^{-1} | u |^2_{L^2(K)}). \tag{58}$$

Summing (57) and (58) for $K \in \mathcal{F}_\ell$ we then obtain:

Lemma 3: There exists a constant c that depends only on the shape of Ω such that for every $u \in V_d, 0 \leq \ell \leq d$

$$| \nabla(r_\ell u) | \leq c(d-\ell) | \nabla u |, \tag{59}$$

$$| r_\ell u | \leq c(d-\ell)(h_\ell^2 \mid \nabla u \mid^2 + h_\ell^{-1} \mid u \mid^2)^{\frac{1}{2}}. \tag{60}$$

We can now prove the following results:

Lemma 4: There exist two constants c and c' that depend only on the shape of Ω such that for every $u \in V_d$

$$c h_d [u]_d^2 \leq \sum_{k=1}^{d} \mid \nabla(r_k u - r_{k-1} u) \mid^2 \leq c' h_0 [u]_d^2. \tag{61}$$

Proof: We fix $k, 1 \leq k \leq d$. Let K be a cube of $\mathcal{F}_{d-1}$ (see figure 1). Then, for every continuous function φ that is Q_1 on each of the eight cubes of $\mathcal{F}_d$ that are in K and that vanish at the points M_i, we have

$$c \sum_i \mid \varphi(P_i) \mid^2 \leq \mid K \mid^{-\frac{1}{3}} \mid \nabla \varphi \mid^2_{L^2(K)} \leq c' \sum_i \mid \varphi(P_i) \mid^2 . \tag{62}$$

Indeed, $\mid K \mid^{-\frac{1}{6}} \mid \nabla \varphi \mid_{L^2(K)}$ and $(\sum_i \mid \varphi(P_i) \mid^2)^{\frac{1}{2}}$ are two norms on the space of such functions. Moreover, (62) is invariant by homothety. Thus c and c' depend only on the shape of K (i.e. of Ω) and are independent of h_{k-1}. Therefore, for every $v \in W_{k-1}$, we have

$$c \sum_{x \in U_k \setminus U_{k-1}} \mid v(x) \mid^2 \leq h_{k-1}^{-1} \mid \nabla v \mid^2_{L^2(K)} \leq c' \sum_{x \in U_k \setminus U_{k-1}} \mid v(x) \mid^2 .$$

We then obtain

$$c \sum_{x \in U_k \setminus U_{k-1}} \mid v(x) \mid^2 \leq h_d^{-1} \mid \nabla v \mid^2_{L^2(K)} .$$

We now add these relations for $K \in \mathcal{F}_{k-1}$, and then for $k = 1, \cdots, d$, with $v = r_k u - r_{k-1} u$, and find

$$c[u]_d^2 \leq h_d^{-1} \sum_{k=1}^{d} \mid \nabla(r_k u - r_{k-1} u) \mid^2 .$$

Similarly

$$h_0^{-1} \mid \nabla v \mid^2_{L^2(K)} \leq c' \sum_{x \in U_k \setminus U_{k-1}} \mid v(x) \mid^2,$$

hence

$$c'[u]_d^2 \geq h_0^{-1} \sum_{k=1}^{d} \mid \nabla(r_k u - r_{k-1} u) \mid^2 . \quad \square$$

Lemma 5: There exists a constant c that depends only on the shape of Ω such that for every $u \in V_d$

$$| u |\leq c(| r_0 u |^2 + | \Omega | [u]_d^2)^{\frac{1}{2}}. \tag{63}$$

Proof: Let $u \in V_d$. Then

$$u = r_d u = r_0 u + \sum_{l=1}^{d}(r_l u - r_{l-1} u),$$

where $r_l u - r_{l-1} u \in W_{l-1} \subset V_l$. Thus

$$| u |\leq| r_0 u | + \sum_{l=1}^{d} | r_l u - r_{l-1} u | . \tag{64}$$

If $v \in V_l$, we have

$$| v |^2 = \int_\Omega | v(x) |^2 \, dx = \sum_{K \in \mathcal{F}_l} \int_K v^2 dx.$$

Now, if $K \in \mathcal{F}_l, | v |_{L^2(K)}$ and $(h_l^3 \sum_i | v(A_i) |^2)^{\frac{1}{2}}$ (see figure 3) are two norms that are equivalent on the finite dimensional space consisting of the functions that are Q_1 on K. Therefore, there exist two constants c and c' that depend only on the shape of Ω such that

$$c(h_l^3 \sum_i | v(A_i) |^2)^{\frac{1}{2}} \leq | v |_{L^2(K)} \leq c'(h_l^3 \sum_i | v(A_i) |^2)^{\frac{1}{2}}, \tag{65}$$

since (65) is invariant by homothety. Thus

$$| v |_{L^2(K)}^2 \leq c h_l^3 \sum_i | v(A_i) |^2 \leq \frac{c}{8^l} h_0^3 \sum_i | v(A_i) |^2 \leq \frac{c}{8^l} | \Omega | \sum_i | v(A_i) |^2 . \tag{66}$$

We now sum the inequalities (66) for $K \in \mathcal{F}_l$ and obtain, since each vertex A_i belongs to at most eight cubes

$$| v |^2 \leq \frac{8c}{8^l} | \Omega | \sum_{x \in U_l} | v(x) |^2, \forall v \in V_l. \tag{67}$$

If $v \in W_{l-1}$, then $v(x) = 0, \forall x \in U_{l-1}$, and therefore thanks to (67)

$$| v |^2 \leq \frac{c}{8^l} | \Omega | \sum_{x \in U_l \setminus U_{l-1}} | v(x) |^2, \forall v \in W_{l-1}. \tag{68}$$

We set $v = r_l u - r_{l-1} u$ in (68) and obtain thanks to (64)

$$\begin{aligned}
\mid u \mid &\leq \mid r_0 u \mid + c \mid \Omega \mid^{\frac{1}{2}} \left(\sum_{l=1}^{d} \frac{1}{(2\sqrt{2})^l} \left(\sum_{x \in U_l \setminus U_{l-1}} \mid (r_l u - r_{l-1} u)(x) \mid^2 \right)^{\frac{1}{2}} \right) \\
&\leq \mid r_0 u \mid + c \mid \Omega \mid^{\frac{1}{2}} \left(\sum_{l=1}^{d} \frac{1}{8^l} \right)^{\frac{1}{2}} \left(\sum_{l=1}^{d} \sum_{x \in U_l \setminus U_{l-1}} \mid (r_l u - r_{l-1} u)(x) \mid^2 \right)^{\frac{1}{2}} \\
&\leq \mid r_0 u \mid + c \mid \Omega \mid^{\frac{1}{2}} [u]_d. \quad \square
\end{aligned}$$

We can now prove Theorem 1. Thanks to Lemma 4, we have

$$\begin{aligned}
\mid \nabla (r_0 u) \mid^2 + [u]_d^2 &\leq \frac{c}{h_d} \left(\mid \nabla (r_0 u) \mid^2 + \sum_{k=1}^{d} \mid \nabla (r_k u - r_{k-1} u) \mid^2 \right) \\
&\leq \frac{c}{h_d} \left(\mid \nabla (r_0 u) \mid^2 + 2 \sum_{k=1}^{d} \left(\mid \nabla r_k u \mid^2 + \mid \nabla r_{k-1} u \mid^2 \right) \right) \\
&\leq \frac{4c}{h_d} \sum_{k=0}^{d} \mid \nabla r_k u \mid^2 .
\end{aligned}$$

Therefore, thanks to (59)

$$\mid \nabla r_0 u \mid^2 + [u]_d^2 \leq \frac{c}{h_d} \sum_{l=1}^{d} (d-l)^2 \mid \nabla u \mid^2 \leq \frac{cd^3}{h_d} \mid \nabla u \mid^2 .$$

We now take $\ell = 0$ in (60) and obtain

$$\mid r_0 u \mid^2 \leq cd^2 (h_0^2 \mid \nabla u \mid^2 + h_0^{-1} \mid u \mid^2). \tag{69}$$

Since $h_0^3 \leq \mid \Omega \mid$,

$$\mid \Omega \mid^{\frac{-2}{3}} \mid r_0 u \mid^2 \leq cd^2 (\mid \nabla u \mid^2 + h_0^{-1} \mid \Omega \mid^{\frac{-2}{3}} \mid u \mid^2),$$

and thus

$$\begin{aligned}
\mid \nabla r_0 u \mid^2 + \mid \Omega \mid^{\frac{-2}{3}} \mid r_0 u \mid^2 + [u]_d^2 &\leq \frac{cd^3}{h_d} \mid \nabla u \mid^2 + cd^2 (\mid \nabla u \mid^2 + h_0^{-1} \mid \Omega \mid^{\frac{-2}{3}} \mid u \mid^2) \\
&\leq \frac{cd^3}{h_d} (\mid \nabla u \mid^2 + \mid \Omega \mid^{\frac{-2}{3}} \mid u \mid^2).
\end{aligned}$$

We have then proved the first inequalities in (40) and (41). Now if $u \in V_d$, we write

$$u = \sum_{l=0}^{d} v_l,$$

where $v_0 = r_0 u, v_l = r_l u - r_{l-1} u, 1 \leq l \leq d$. We obtain

$$| \nabla u |^2 = | \sum_{l=0}^{d} \nabla v_l |^2 \leq \sum_{j,k=0}^{d} (\nabla v_j, \nabla v_k) \leq (d+1) \sum_{l=0}^{d} | \nabla v_l |^2 \leq 2d \sum_{l=0}^{d} | \nabla v_l |^2 .$$

Therefore, thanks to Lemma 4

$$| \nabla u |^2 \leq cd(h_0 [u]_d^2 + | \nabla r_0 u |^2). \tag{70}$$

The second inequality in (40) is then satisfied. Finally, if $u \in V_d$ we have

$$\begin{aligned} | \nabla u |^2 + | \Omega |^{\frac{-2}{3}} | u |^2 &\leq cd(| \nabla r_0 u |^2 + h_0 [u]_d^2) + | \Omega |^{\frac{-2}{3}} | u |^2 \\ &\leq (\text{ Thanks to Lemma 5}) \\ &\leq cd(| \nabla r_0 u |^2 + h_0 [u]_d^2) + c(| \Omega |^{\frac{-2}{3}} | r_0 u |^2 + | \Omega |^{\frac{1}{3}} [u]_d^2) \\ &\leq cd(| \nabla r_0 u |^2 + | \Omega |^{\frac{1}{3}} [u]_d^2 + | \Omega |^{-\frac{2}{3}} | r_0 u |^2), \end{aligned}$$

which proves the second inequality in (41).

4.3. Application to incremental unknowns.

4.3.1 Preliminary results.

Lemma 6: For every $u \in V_d$, we have

$$\frac{\sqrt{2}}{6} | \nabla_d \tilde{u} | \leq | \nabla u |, \tag{71}$$

where $\nabla_d u = (\nabla_{1,h_d} u, \nabla_{2,h_d} u, \nabla_{3,h_d} u)$.

Proof: In order to prove (71), it suffices to prove

$$\frac{1}{18} \int_K | \nabla_d \tilde{u} |^2 \, dx \leq \int_K | \nabla u |^2 \, dx,$$

where K is a cube of $\mathcal{F}_d$. We assume for simplicity that $K = (0, h_d)^3$. We then have (see figure 3):

$$\nabla_d \tilde{u} = \begin{pmatrix} \frac{1}{h_d}(u(A_1) - u(A_0)) \\ \frac{1}{h_d}(u(A_7) - u(A_0)) \\ \frac{1}{h_d}(u(A_3) - u(A_0)) \end{pmatrix},$$

and

$$\int_K |\nabla_d \tilde{u}|^2 \, dx = h_d[(u(A_1) - u(A_0))^2 + (u(A_7) - u(A_0))^2 + (u(A_3) - u(A_0))^2].$$

Furthermore, thanks to (46), (47) and (48) we have

$$\nabla u = \begin{pmatrix} \left(1 - \frac{x_2}{h_d} - \frac{x_3}{h_d} + \frac{x_2 x_3}{h_d^2}\right)\frac{\alpha_1}{h_d} - \left(1 - \frac{x_3}{h_d}\right)\frac{x_2}{h_d}\frac{\beta_1}{h_d} - \left(1 - \frac{x_2}{h_d}\right)\frac{x_3}{h_d}\frac{\gamma_1}{h_d} + \frac{x_2 x_3}{h_d^2}\frac{\delta_1}{h_d} \\ \left(1 - \frac{x_1}{h_d} - \frac{x_3}{h_d} + \frac{x_1 x_3}{h_d^2}\right)\frac{\alpha_2}{h_d} - \left(1 - \frac{x_3}{h_d}\right)\frac{x_1}{h_d}\frac{\beta_2}{h_d} - \left(1 - \frac{x_2}{h_d}\right)\frac{x_3}{h_d}\frac{\gamma_2}{h_d} + \frac{x_1 x_3}{h_d^2}\frac{\delta_2}{h_d} \\ \left(1 - \frac{x_1}{h_d} - \frac{x_2}{h_d} + \frac{x_1 x_2}{h_d^2}\right)\frac{\alpha_3}{h_d} - \left(1 - \frac{x_2}{h_d}\right)\frac{x_1}{h_d}\frac{\beta_3}{h_d} - \left(1 - \frac{x_1}{h_d}\right)\frac{x_2}{h_d}\frac{\gamma_3}{h_d} + \frac{x_1 x_2}{h_d^2}\frac{\delta_3}{h_d} \end{pmatrix},$$

where

$$\begin{aligned} &\alpha_1 = u(A_1) - u(A_0), \alpha_2 = u(A_7) - u(A_0), \alpha_3 = u(A_3) - u(A_0), \\ &\beta_1 = u(A_7) - u(A_4), \beta_2 = u(A_1) - u(A_4), \beta_3 = u(A_1) - u(A_2), \\ &\gamma_1 = u(A_3) - u(A_2), \gamma_2 = u(A_3) - u(A_6), \gamma_3 = u(A_7) - u(A_6), \\ &\delta_1 = u(A_5) - u(A_6), \delta_2 = u(A_5) - u(A_2), \delta_3 = u(A_5) - u(A_4). \end{aligned}$$

Therefore

$$\begin{aligned}
\int_K |\nabla_d \tilde{u}|^2\,dx &= h_d(\alpha_1^2+\alpha_2^2+\alpha_3^2),\\
\int_K |\nabla u|^2\,dx &= \frac{h_d}{9}\sum_{i=1}^{3}\Big(\alpha_i^2+\beta_i^2+\gamma_i^2+\delta_i^2-\alpha_i\beta_i-\alpha_i\gamma_i+\frac{1}{2}\alpha_i\delta_i\\
&\quad+\frac{1}{2}\beta_i\gamma_i-\beta_i\delta_i-\gamma_i\delta_i\Big)\\
&= \frac{h_d}{9}\sum_{i=1}^{3}\left(\frac{1}{2}\left(\alpha_i-\beta_i-\gamma_i+\frac{\delta_i}{2}\right)^2+\frac{1}{2}\alpha_i^2+\frac{1}{2}\beta_i^2+\frac{1}{2}\gamma_i^2\right.\\
&\quad\left.+\frac{3}{4}\delta_i^2-\frac{1}{2}\beta_i\delta_i-\frac{1}{2}\gamma_i\delta_i-\frac{1}{2}\beta_i\gamma_i\right)\\
&\geq \text{(thanks to Cauchy-Schwarz inequality)}\\
&\geq \frac{h_d}{18}\left(\left(\alpha_i-\beta_i-\gamma_i+\frac{\delta_i}{2}\right)^2+\alpha_i^2+\frac{\delta_i^2}{2}\right).
\end{aligned} \tag{72}$$

We then deduce (71). □

In order to establish the inverse inequality of (71), we introduce the extended norms

$$\begin{aligned}
|\nabla_{i,d}^e \tilde{u}| &= \left(\int_{\Omega\cup(\Omega+h_d e_i)} |\nabla_{i,h_d}\tilde{u}|^2\,dx\right)^{\frac{1}{2}},\\
|\nabla_d^e \tilde{u}| &= \left(|\nabla_{1,d}^e\tilde{u}|^2+|\nabla_{2,d}^e\tilde{u}|^2\right)^{1/2},\\
\Omega_d^* &= (0,1+h_d)^3.
\end{aligned}$$

As in [3] (Lemma 4.2 and Lemma 4.3) we can prove the following results:

Lemma 7: There exist two numerical constants c_1 and c_2 such that for every $u \in V_d$

$$c_1 |\nabla_d^e\tilde{u}| \leq |\nabla u| \leq c_2 |\nabla_d^e \tilde{u}|. \tag{73}$$

Lemma 8: There exist two constants c_1 and c_2 that depend only on the shape of Ω such that for every $u \in V_d$

$$c_1\left(\int_{\Omega_d^*} |\tilde{u}|^2\,dx\right)^{\frac{1}{2}} \leq |u| \leq c_2\left(\int_{\Omega_d^*}|\tilde{u}|^2\,dx\right)^{\frac{1}{2}}. \tag{74}$$

We note that for functions vanishing on $\partial\Omega$, the extended domains are not needed and $|\nabla_d^e\tilde{u}| = |\nabla_d\tilde{u}|$. Thus, the left inequality in (73) is replaced by (71), and for the second inequality in (73), we can replace $\nabla_d^e \tilde{u}$ by $\nabla_d \tilde{u}$. We then have the inverse inequality of (71).

4.3.2. Incremental unknowns for boundary value problems.

The discretization of (1)-(2) by the usual finite differences scheme leads to the classical linear problem

$$A_d U_d = b_d. \tag{75}$$

We actually have several systems

$$A_l U_l = b_l, \quad l = 0, \cdots, d, \tag{76}$$

corresponding to the different levels of discretization, but we are here interested in the resolution of (75). At each level l, we consider the set $\bar{U}_l$ of incremental unknowns which consists of the following:

• the set properly ordered (see Section 1) $Y^l = Y_0$ of the approximate nodal values of u at the coarse grid,

• the set properly ordered (see Section 1) Z_j of the incremental unknowns at level j.

Thus

$$\bar{U}_l = \begin{pmatrix} Y^l \\ Z^l \end{pmatrix}, \qquad Z^l = \begin{pmatrix} Z_1 \\ \vdots \\ Z_l \end{pmatrix}.$$

We can pass from U_l to $\bar{U}_l$ by using a transformation matrix S_l and we obtain the systems

$$\bar{A}_l \bar{U}_l = \bar{b}_l, \quad 0 \le l \le d. \tag{77}$$

If $\kappa(\bar{A}_d)$ is the condition number of $\bar{A}_d$, then we have

$$\kappa(\bar{A}_d) = \bar{\lambda}(\bar{A}_d)/\underline{\lambda}(\bar{A}_d),$$

where $\bar{\lambda}(\bar{A}_d)$ is the largest eigenvalue of $\bar{A}_d$ and $\underline{\lambda}(\bar{A}_d)$ the smallest eigenvalue. As in [3] we have

$$\begin{aligned} < \bar{A}_d \bar{U}_d, \bar{U}_d > &= \int_\Omega \mid \nabla_d \tilde{u}_d \mid^2 dx \ge c \int_\Omega \mid \nabla u_d \mid^2 dx \ge \text{(thanks to (40))} \\ &\ge \frac{c h_d}{d^3} (\mid \nabla r_0 u_d \mid^2 + [u_d]_d^2) \ge \text{(thanks to (71))} \\ &\ge \frac{c h_d}{d^3} (\mid \nabla r_0 \tilde{u}_d \mid^2 + [u_d]_d^2) \ge \frac{c h_d}{d^3} \min\, (\underline{\lambda}(A_0), 1) < \bar{U}_d, \bar{U}_d > . \end{aligned}$$

Therefore

$$\underline{\lambda}(\bar{A}_d) \geq \frac{ch_d}{d^3} \min\ (\underline{\lambda}(A_0), 1).$$

Similarly we obtain

$$\bar{\lambda}(\bar{A}_d) \leq cd \max\ (\bar{\lambda}(A_0), 1).$$

Thus

$$\kappa(\bar{A}_d) \leq c\frac{d^4}{h_d}\frac{\max\ (\bar{\lambda}(A_0), 1)}{\min\ (\underline{\lambda}(A_0), 1)}. \tag{78}$$

We set $h = h_d$. We then deduce that $\kappa(\bar{A}_d)$ is at most of order $\frac{1}{h}(lnh)^4$, whereas $\kappa(A_d)$ is of order $\frac{1}{h^2}$, hence an improvement in the case of incremental unknowns.

Remark 1: As in [3], we can consider more general domains or operators and obtain similar results. Moreover, thanks to (41), Lemma 7 and 8, we can obtain similar results in the case of Neumann boundary conditions.

5. Numerical results and other types of Incremental Unknowns.

In this section, we shall present the numerical results from the computation of $\kappa(\bar{A}_d)$ introduced in earlier sections. We then introduce another type of incremental unknowns and conjecture the behavior of the condition number $\kappa(\bar{A}_d)$ through numerical computations.

The incremental unknowns introduced in Section 1 were referred to as the second-order incremental unknowns in [4] since the Taylor's expansion of $z_{\alpha,\beta,\gamma}$ at $(\alpha h, \beta h, \gamma h)$ is of order h^2. In Figure 4 and 5, $\kappa(\bar{A}_d)$ and $\kappa(\bar{A}_d)/(h^{-1}|\log_2(h)|)$ are plotted against d respectively with $h = h_d = 1/2^d$ being the mesh size and $\Omega = (0,1)^3$. The graph in Figure 4(a) shows that $\kappa(\bar{A}_d)$ is growing with d and the graph in Figure 4(b) shows that $\kappa(\bar{A}_d)/(h^{-1}|\log_2(h)|)$ is almost a constant, which means that $\kappa(\bar{A}_d)$ is of order $h^{-1}|\log_2(h)|$ which confirms our theoretical results obtained in the previous sections.

Inspired by the idea of the second-order incremental unknowns and the first-order incremental unknowns introduced in [4] for the two-dimensional case, we can introduce third-order or even higher order incremental unknowns. For $h = 1/(2N)$, we let as for the second-order incremental unknowns:

$$y_{2i,2j,2k} = u_{2i,2j,2k},\ \text{for } i = 0, 1, \dots, N.$$

For a mesh point $A_0 = (\alpha, \beta, \gamma)$ which is not a coarse grid point, there exist three distinct coarse grid points A_1, A_2, A_3 on a line and $dist(A_1, A_2) = dist(A_2, A_3) = 2\lambda h$, A_0 being the midpoint of $\overline{A_1A_2}$, and 3 $dist(A_0, A_1) = 3\ dist(A_0, A_2) = dist(A_0, A_3) = 3\lambda h$, where $\lambda = 1$ or $\sqrt{2}$ correspond to the cases where $\overline{A_1A_2A_3}$ is a horizontal\ vertical line segment or a diagonal line segment respectively. Let

$$z_{\alpha,\beta,\gamma} = u_{\alpha,\beta,\gamma} - (\frac{3}{8}u_{A_1} + \frac{3}{4}u_{A_2} - \frac{1}{8}u_{A_3}),$$

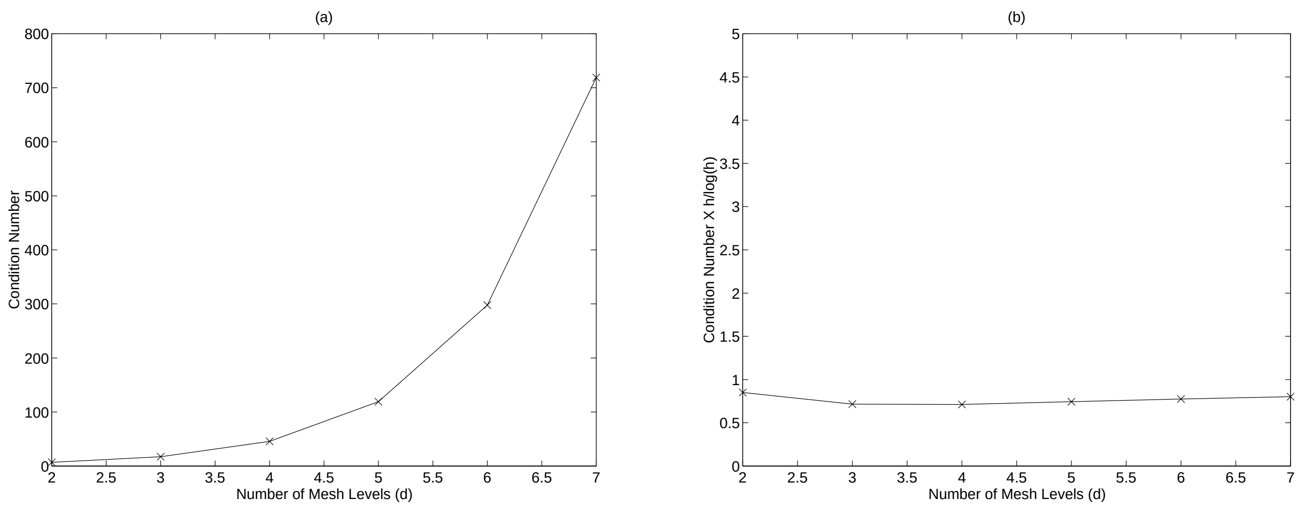


FIGURE 4. Condition numbers of the second-order IU in the three dimensional case.

for α or β or γ odd, A_1, A_2, and A_3 as described. The Taylor's expansion of $z_{\alpha,\beta,\gamma}$ at $(\alpha h, \beta h, \gamma h)$ is of order h^3. Numerical results presented in Figure 5 make us believe that $\kappa(\bar{A}_d)$ in this case is also of order $h^{-1}|\log_2(h)|$.

Many other types of incremental unknowns can be introduced to suit certain specific requirements from the original physical problem or the design of the numerical schemes. In [5], we have used the wavelet-like incremental unknowns which have the L_2 orthogonality property between different levels of mesh to design numerical schemes which have significantly improved stability when compared to the existing schemes.

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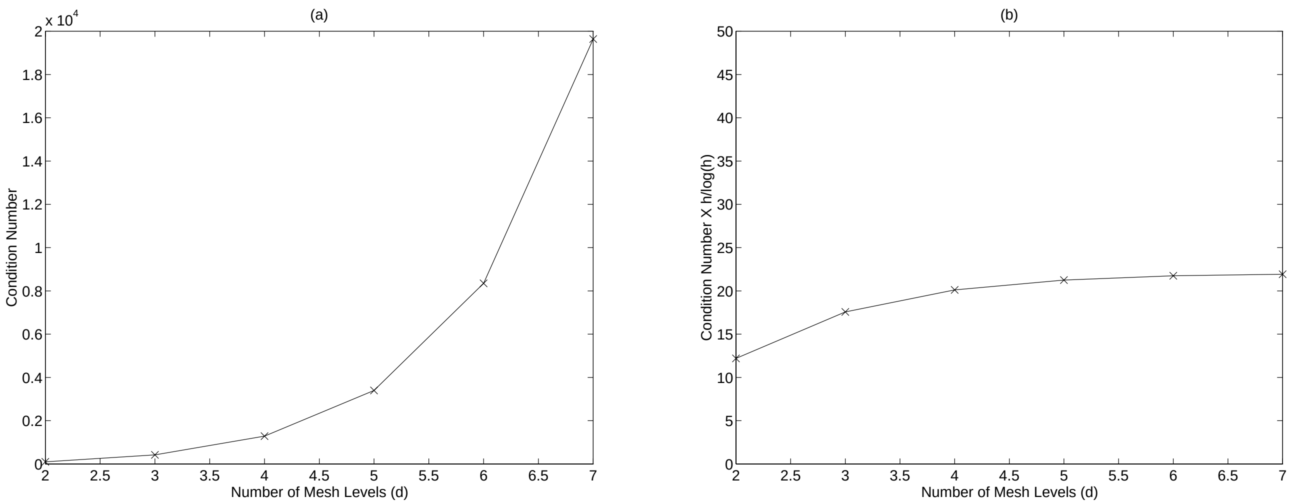

FIGURE 5. Condition numbers of the third-order IU in the three dimensional case.

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