Equations for Bi-directional Waves Over an Uneven Bottom

M. Chen
Department of Mathematics, University of Central Florida,
Orlando, FL 32816, USA

Abstract

This paper is centered at deriving and studying systems modeling bi-directional surface waves over an uneven or a moving bottom. Compared with the huge amount of work on model equations describing one-way propagation of water waves, much less attention has been given to Boussinesq systems describing two-way propagation of water waves, especially to the systems which are externally forced.

Key words: Water waves, Boussinesq systems, bottom topography.

1 Introduction

Boussinesq systems over a flat bottom have been studied in [2-4] and it was demonstrated that the systems have the capability of predicting physical phenomena and have the advantages of being simpler than Euler equations and applicable to more general situations than the KdV equation. Since in practical applications, the wave reflection due to the bottom topography and the wave interaction have to be considered, we will derive, without ad hoc assumptions, a class of model equations which have the same formal accuracy as KdV-type equations, but model bi-directional surface waves over an uneven or moving bottom.

2 Equations for Bi-directional Waves Over an Uneven Bottom

Consider nonlinear dispersive waves in a water channel of length L with an uneven and possibly moving bottom. The bottom may actually be moving, as in the experiments of [10], or it may result from an imposed flow at infinity which is brought to rest via a change to traveling coordinates in which an uneven bottom will appear to move. Let x be the coordinate along the channel and y the vertical coordinate pointing upward with the undisturbed free surface located at $y=0$. Let $\eta(x,t)$ be the free surface which is a fundamental unknown of the problem and $-h(x,t)$ be the possibly moving bottom topography, the flow domain is then $\Omega_t = (0,L)\times(-h(x,t),\eta(x,t))$. Let a_0 denote a typical wave height, λ_0 denote a typical wave length, h_0 denote the average still water depth and g be the gravity constant. By assuming the initial flow is irrotational, the system describing two-dimensional gravity waves on the free surface may be written in the form

$$
\begin{array}{lll}
\text{momentum conservation in } x: & u_t + uu_x + vu_y + \dfrac{1}{\rho}P_x = 0, & \text{for } (x,y)\in\Omega_t, \\
\text{momentum conservation in } y: & v_t + uv_x + vv_y + \dfrac{1}{\rho}P_y = -g, & \text{for } (x,y)\in\Omega_t, \\
\text{mass conservation:} & u_x + v_y = 0, & \text{for } (x,y)\in\Omega_t, \\
\text{irrotational flow:} & u_y - v_x = 0, & \text{for } (x,y)\in\Omega_t,
\end{array}
$$

where (u,v) denotes the velocity in the x and y directions, respectively, P denotes the pressure field and ρ denotes the density. The boundary conditions on the surfaces are

$$
\begin{array}{ll}
\eta_t + u\eta_x - v = 0; \quad P(x,t) = P_0(x,t); & \text{on } y = \eta(x,t), \\
h_t + uh_x + v = 0; & \text{on } y = -h(x,t).
\end{array}
$$

Since the object of current study is on the small amplitude and long waves, one can scale the variables so that the magnitude of each term be more explicit. It is not always clear, initially, how all the variables should be scaled and sometime there may be a genuine ambiguity corresponding to different physical situation. A natural scaling of x, y and η is chosen, so

$$\widetilde{x} = \frac{x}{\lambda_0}; \quad \widetilde{y} = \frac{y}{h_0}; \quad \widetilde{\eta} = \frac{\eta}{a_0}.$$

In the context of waves we are considering, such as the waves on a beach, the waves progress with a velocity (i.e phase velocity) which is of order 1, so the time should be scaled as x and we therefore set the non-dimensional time as

$$\widetilde{t} = \frac{c_0 t}{\lambda_0}$$

where $c_0 = \sqrt{gh_0}$. To maintain the approximation that would obtain in the absence of bottom variation or bottom motion, $h - h_0$ is assumed to behave similarly to η, so we let

$$\widetilde{h} = \frac{h - h_0}{a_0}.$$

By assuming the horizontal velocity u has the same order of magnitude as the surface variation η,

$$\widetilde{u}=\frac{uh_0}{a_0c_0}$$

is used as corresponding non-dimensional variable. From mass conservation and the kinematic condition on the surface, one finds

$$\begin{aligned}&\frac{a_0c_0}{\lambda_0}\widetilde{u}_{\widetilde{x}}+v_{\widetilde{y}}=0,\\&\frac{a_0c_0}{\lambda_0}\widetilde{\eta}_{\widetilde{t}}+\frac{a_0^2c_0}{\lambda_0h_0}\widetilde{u}\widetilde{\eta}_{\widetilde{x}}-v=0,\end{aligned}$$

so v is of order $\dfrac{a_0}{\lambda_0}$ and the nondimensionalized vertical velocity is chosen to be

$$\widetilde{v}=\frac{v\lambda_0}{a_0c_0}.$$

Substitute the non-dimensional, scaled variables into the governing equations of the flow in $\widetilde{\Omega}_{\widetilde{t}}=\left(0,\dfrac{L}{\lambda_0}\right)\times\left(-\left(1+\alpha\widetilde{h}\left(\widetilde{x},\widetilde{t}\right)\right),\alpha\widetilde{\eta}\left(\widetilde{x},\widetilde{t}\right)\right)$, and denote

$$\alpha=\frac{a_0}{h_0}\quad\text{and}\quad\beta=\frac{h_0^2}{\lambda_0^2},$$

which are small parameters, one obtains

$$\alpha\widetilde{u}_{\widetilde{t}}+\alpha^2\widetilde{u}\widetilde{u}_{\widetilde{x}}+\alpha^2\widetilde{v}\widetilde{u}_{\widetilde{y}}+\frac{1}{\rho c_0^2}P_{\widetilde{x}}=0,\qquad\text{for }(\widetilde{x},\widetilde{y})\in\widetilde{\Omega}_{\widetilde{t}},\tag{1}$$

$$\alpha\beta\widetilde{v}_{\widetilde{t}}+\alpha^2\beta\widetilde{u}\widetilde{v}_{\widetilde{x}}+\alpha^2\beta\widetilde{v}\widetilde{v}_{\widetilde{y}}+\frac{1}{\rho c_0^2}P_{\widetilde{y}}=-1,\qquad\text{for }(\widetilde{x},\widetilde{y})\in\widetilde{\Omega}_{\widetilde{t}},\tag{2}$$

$$\widetilde{u}_{\widetilde{x}}+\widetilde{v}_{\widetilde{y}}=0,\qquad\text{for }(\widetilde{x},\widetilde{y})\in\widetilde{\Omega}_{\widetilde{t}},\tag{3}$$

$$\widetilde{u}_{\widetilde{y}}-\beta\widetilde{v}_{\widetilde{x}}=0,\qquad\text{for }(\widetilde{x},\widetilde{y})\in\widetilde{\Omega}_{\widetilde{t}},\tag{4}$$

$$\widetilde{\eta}_t+\alpha\widetilde{u}\widetilde{\eta}_{\widetilde{x}}-\widetilde{v}=0,\qquad\text{on }\widetilde{y}=\alpha\widetilde{\eta}(\widetilde{x},\widetilde{t}),\tag{5}$$

$$P\left(\widetilde{x},\widetilde{t}\right)=P_0\left(\widetilde{x},\widetilde{t}\right),\qquad\text{on }\widetilde{y}=\alpha\widetilde{\eta}(\widetilde{x},\widetilde{t}),\tag{6}$$

$$\widetilde{h}_{\widetilde{t}}+\alpha\widetilde{u}\widetilde{h}_{\widetilde{x}}+\widetilde{v}=0,\qquad\text{on }\widetilde{y}=-(1+\alpha\widetilde{h}(\widetilde{x},\widetilde{t})).\tag{7}$$

Since we are considering the wave motion for which the classical Stokes number $S=\alpha/\beta$ is of order one, the two small parameters α and β may be treated on an equal footing and we seek to write approximate equations corresponding to the orders of accuracy characterized by α^n or β^n for $n=1,2,\cdots$.

Let $\widetilde{U}(x,t)$ denote the horizontal velocity at the bottom of the channel $\widetilde{y}=-(1+\alpha\widetilde{h})$. Integrate the relation (4) along $\widetilde{y}$ to determine that

$\widetilde{u} - \widetilde{U} = O(\beta)$. Observing from equation (7) that $\widetilde{h}_{\widetilde{t}} + \widetilde{v}(\widetilde{x}, -(1+\alpha\widetilde{h}), \widetilde{t}) = O(\alpha)$, the leading order approximation of $\widetilde{v}$ is then obtained by integrating (3) in $\widetilde{y}$,

$$\widetilde{v} + \widetilde{h}_{\widetilde{t}} = -\int_{-(1+\alpha\widetilde{h})}^{\widetilde{y}} \widetilde{u}_{\widetilde{x}} d\widetilde{y} + O(\alpha,\beta) = -(\widetilde{y}+1)\widetilde{U}_{\widetilde{x}} + O(\alpha,\beta). \qquad (8)$$

The next order of approximation keeps all the terms which are of order one and of order α and β. Using (4) and (8), one obtains

$$\widetilde{u}_{\widetilde{y}} = \beta\widetilde{v}_{\widetilde{x}} = -\beta(\widetilde{y}+1)\widetilde{U}_{\widetilde{x}\widetilde{x}} - \beta\widetilde{h}_{\widetilde{x}\widetilde{t}} + O(\alpha^2, \alpha\beta, \beta^2)$$

which yields by integrating in $\widetilde{y}$

$$\widetilde{u} - \widetilde{U} = -\frac{1}{2}\beta(\widetilde{y}+1)^2\widetilde{U}_{\widetilde{x}\widetilde{x}} - \beta(\widetilde{y}+1)\widetilde{h}_{\widetilde{x}\widetilde{t}} + O(\alpha^2, \alpha\beta, \beta^2). \qquad (9)$$

Integrating the momentum equation (2) with respect to $\widetilde{y}$ from $\widetilde{y}$ to $\alpha\widetilde{\eta}$ and using (8), one derives by keeping all the terms of order $\alpha^2, \alpha\beta$ and β^2, that

$$\alpha\beta\widetilde{y}\widetilde{h}_{\widetilde{t}\widetilde{t}} + \frac{1}{2}\alpha\beta(\widetilde{y}^2 + 2\widetilde{y})\widetilde{U}_{\widetilde{x}\widetilde{t}} + \frac{1}{\rho c_0^2}(P_0 - P) + \alpha\widetilde{\eta} - \widetilde{y} = O(\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3),$$

which means $P - P_0 = \rho c_0^2(\alpha\widetilde{\eta} - \widetilde{y}) + O(\alpha\beta)$, the pressure P consists the hydrostatic pressure and a higher order correction.

Differentiating with respect to $\widetilde{x}$ and combining with equation (1), one obtains by using approximations (9) and (8) that

$$\alpha\widetilde{U}_{\widetilde{t}} + \alpha\widetilde{\eta}_{\widetilde{x}} + \alpha^2\widetilde{U}\widetilde{U}_{\widetilde{x}} - \frac{1}{2}\alpha\beta\widetilde{U}_{\widetilde{x}\widetilde{x}\widetilde{t}} + \frac{1}{\rho c_0^2}(P_0)_{\widetilde{x}} - \alpha\beta\widetilde{h}_{\widetilde{x}\widetilde{t}\widetilde{t}} = O(\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3).$$

It is worth to note that to maintain the order of approximation, $(P_0)_{\widetilde{x}}$ should be of the order α or α^2, which means that the variation of outside pressure along the channel should be of the order $\dfrac{\alpha}{\lambda_0}$ or $\dfrac{\alpha^2}{\lambda_0}$, depending on the physical situation! In the field situation, $(P_0)_x$ is lower order. But when P_0 is used to model the effect of ships as in the experiment of [10], it could be in the higher order. As an example, we assume that $(P_0)_{\widetilde{x}}$ is of order α^2 (i.e. $(P_0)_x$ is of order α^2/λ_0). In this case, the non-dimensional variables

$$\widetilde{P}_0 = \frac{P_0 h_0}{\rho g a_0^2}, \quad \widetilde{P} = \frac{P h_0}{\rho g a_0^2},$$

can be used. We therefore obtain

$$\widetilde{U}_{\widetilde{t}} + \widetilde{\eta}_{\widetilde{x}} + \alpha\widetilde{U}\widetilde{U}_{\widetilde{x}} - \frac{1}{2}\beta\widetilde{U}_{\widetilde{x}\widetilde{x}\widetilde{t}} + \alpha(\widetilde{P}_0)_{\widetilde{x}} - \beta\widetilde{h}_{\widetilde{x}\widetilde{t}\widetilde{t}} = O(\alpha^2, \alpha\beta, \beta^2).$$

Following the work of [8], one can derive, by using mass conservation (3) and the boundary conditions (5) and (7), that

$$\widetilde{\eta}_{\widetilde{t}} + \frac{\partial}{\partial \widetilde{x}} \int_{-(1+\alpha\widetilde{h})}^{\alpha\widetilde{\eta}} \widetilde{u}\, d\widetilde{y} = \widetilde{\eta}_{\widetilde{t}} + \int_{-(1+\alpha\widetilde{h})}^{\alpha\widetilde{\eta}} \frac{\partial \widetilde{u}}{\partial \widetilde{x}} d\widetilde{y} + \alpha\widetilde{u}(\widetilde{x}, \alpha\widetilde{\eta}, \widetilde{t}) \frac{\partial \widetilde{\eta}}{\partial \widetilde{x}}$$
$$+ \alpha\widetilde{u}(\widetilde{x}, -(1+\alpha\widetilde{h}), \widetilde{t}) \frac{\partial \widetilde{h}}{\partial \widetilde{x}} = -\widetilde{h}_{\widetilde{t}}.$$

Denoting $\bar{u} = \dfrac{1}{(1+\alpha\widetilde{\eta}+\alpha\widetilde{h})} \displaystyle\int_{-(1+\alpha\widetilde{h})}^{\alpha\widetilde{\eta}} \widetilde{u}\, d\widetilde{y}$ which is the depth-averaged velocity, one finds

$$\widetilde{\eta}_{\widetilde{t}} + \Big((1+\alpha\widetilde{\eta}+\alpha\widetilde{h})\bar{u}\Big)_{\widetilde{x}} = -\widetilde{h}_{\widetilde{t}}. \tag{10}$$

Therefore, by using

$$\widetilde{U} = \bar{u} + O(\alpha, \beta), \quad \widetilde{U} = \bar{u} + \frac{\beta}{6}\bar{u}_{\widetilde{x}\widetilde{x}} + \frac{\beta}{2}\widetilde{h}_{\widetilde{x}\widetilde{t}} + O(\alpha^2, \alpha\beta, \beta^2), \tag{11}$$

which are consequences of (9) and by associating with (10), one obtains the following system of equations with respect to unknown functions $\bar{u}$ and $\widetilde{\eta}$

$$\begin{aligned} &\widetilde{\eta}_{\widetilde{t}} + \Big((1+\alpha\widetilde{\eta}+\alpha\widetilde{h})\bar{u}\Big)_{\widetilde{x}} = -\widetilde{h}_{\widetilde{t}}, \\ &\bar{u}_{\widetilde{t}} + \widetilde{\eta}_{\widetilde{x}} + \alpha\bar{u}\bar{u}_{\widetilde{x}} - \frac{1}{3}\beta\bar{u}_{\widetilde{x}\widetilde{x}\widetilde{t}} = \frac{1}{2}\beta\widetilde{h}_{\widetilde{x}\widetilde{t}\widetilde{t}} - \alpha(\widetilde{P}_0)_{\widetilde{x}}, \end{aligned} \tag{12}$$

where the higher-order terms are neglected in the second equation. System (12) appeared in [10] in a different form and was studied in [7, 6, 5] under an additional condition that the external forces $\widetilde{h}$ and $\widetilde{P}_0$ are time independent. To my knowledge, the well-posedness of this system when associated with the physically relevant, non-homogeneous initial- and Dirichlet-boundary-value conditions is still an open question, even when the external forcing is assumed to be zero. (The pure initial-value problem with zero external forcing has been analyzed, however, and found to have global solutions. See [9, 1].)

Similar to the work in [2], a regularized version of system (12) might be more suitable for imposing a physically relevant initial- and non-homogeneous boundary-value problem. Such systems, which are also correct to first order in α and β, can be obtained by considering changes in the dependent variables and by making use of lower-order relations in the higher-order terms. Letting $\widetilde{u}^\theta$ be the scaled horizontal velocity at the depth $\widetilde{y}_\theta = \theta\alpha\widetilde{\eta} - (1-\theta)(1+\alpha\widetilde{h})$, where $0 \le \theta \le 1$ and evaluating (9) at $\widetilde{y}_\theta$ and using (11), we have

$$\bar{u} = \widetilde{u}^\theta + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\beta\widetilde{u}^\theta_{\widetilde{x}\widetilde{x}} + \left(\theta - \frac{1}{2}\right)\beta\widetilde{h}_{\widetilde{x}\widetilde{t}} + O(\alpha^2, \alpha\beta, \beta^2),$$

which yields from (12) an one-parameter family of systems,

$$\begin{aligned}
&\widetilde{\eta}_{\widetilde{t}} + \widetilde{u}^{\theta}_{\widetilde{x}} + \alpha(\widetilde{\eta}\widetilde{u}^{\theta} + \widetilde{h}\widetilde{u}^{\theta})_{\widetilde{x}} + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\beta\widetilde{u}^{\theta}_{\widetilde{x}\widetilde{x}\widetilde{x}} \\
&\qquad\qquad = -\widetilde{h}_{\widetilde{t}} - \left(\theta - \frac{1}{2}\right)\beta\widetilde{h}_{\widetilde{x}\widetilde{x}\widetilde{t}} + O(\alpha^2, \alpha\beta, \beta^2), \\
&\widetilde{u}^{\theta}_{\widetilde{t}} + \widetilde{\eta}_{\widetilde{x}} + \alpha\widetilde{u}^{\theta}\widetilde{u}^{\theta}_{\widetilde{x}} + \frac{1}{2}\left(\theta^2 - 1\right)\beta\widetilde{u}^{\theta}_{\widetilde{x}\widetilde{x}\widetilde{t}} \\
&\qquad\qquad = -\alpha(\widetilde{P}_0)_{\widetilde{x}} + (1-\theta)\beta\widetilde{h}_{\widetilde{x}\widetilde{t}\widetilde{t}} + O(\alpha^2, \alpha\beta, \beta^2).
\end{aligned} \tag{13}$$

The order one relation in (13) is

$$\widetilde{\eta}_{\widetilde{t}} + \widetilde{u}^{\theta}_{\widetilde{x}} + \widetilde{h}_{\widetilde{t}} = O(\alpha, \beta), \quad \widetilde{u}^{\theta}_{\widetilde{t}} + \widetilde{\eta}_{\widetilde{x}} = O(\alpha, \beta),$$

which formally implies that

$$\widetilde{u}^{\theta}_{\widetilde{x}\widetilde{x}\widetilde{x}} = -\widetilde{\eta}_{\widetilde{x}\widetilde{x}\widetilde{t}} - \widetilde{h}_{\widetilde{x}\widetilde{x}\widetilde{t}} + O(\alpha, \beta), \quad \widetilde{u}^{\theta}_{\widetilde{x}\widetilde{x}\widetilde{t}} = -\widetilde{\eta}_{\widetilde{x}\widetilde{x}\widetilde{x}} + O(\alpha, \beta).$$

Therefore, for any $\lambda, \mu \in I\!R$, one has

$$\begin{aligned}
\widetilde{u}^{\theta}_{\widetilde{x}\widetilde{x}\widetilde{x}} &= \lambda\widetilde{u}^{\theta}_{\widetilde{x}\widetilde{x}\widetilde{x}} - (1-\lambda)(\widetilde{\eta}_{\widetilde{x}\widetilde{x}\widetilde{t}} + \widetilde{h}_{\widetilde{x}\widetilde{x}\widetilde{t}}) + O(\alpha, \beta); \\
\widetilde{u}^{\theta}_{\widetilde{x}\widetilde{x}\widetilde{t}} &= -\mu\widetilde{\eta}_{\widetilde{x}\widetilde{x}\widetilde{x}} + (1-\mu)\widetilde{u}^{\theta}_{\widetilde{x}\widetilde{x}\widetilde{t}} + O(\alpha, \beta).
\end{aligned}$$

Substituting the relations above into (13) and neglecting the higher-order terms, one derives a restricted four-parameter family of systems

$$\begin{aligned}
&\widetilde{\eta}_{\widetilde{t}} + (\widetilde{u}^{\theta})_{\widetilde{x}} + \alpha(\widetilde{\eta}\widetilde{u}^{\theta} + \widetilde{h}\widetilde{u}^{\theta})_{\widetilde{x}} + a\beta\widetilde{u}^{\theta}_{\widetilde{x}\widetilde{x}\widetilde{x}} - b\beta\widetilde{\eta}_{\widetilde{x}\widetilde{x}\widetilde{t}} \\
&\qquad = -\widetilde{h}_{\widetilde{t}} + \frac{1}{2}\beta\left((1-\lambda)\left(\theta^2 - \frac{1}{3}\right) - 2\theta + 1\right)\widetilde{h}_{\widetilde{x}\widetilde{x}\widetilde{t}}, \\
&\widetilde{u}^{\theta}_{\widetilde{t}} + \widetilde{\eta}_{\widetilde{x}} + \alpha\widetilde{u}^{\theta}\widetilde{u}^{\theta}_{\widetilde{x}} + c\beta\widetilde{\eta}_{\widetilde{x}\widetilde{x}\widetilde{x}} - d\beta\widetilde{u}^{\theta}_{\widetilde{x}\widetilde{x}\widetilde{t}} = -\alpha\left(\widetilde{P}_0\right)_{\widetilde{x}} + (1-\theta)\beta\widetilde{h}_{\widetilde{x}\widetilde{t}\widetilde{t}},
\end{aligned} \tag{14}$$

where a, b, c, d are giving by

$$\begin{aligned}
&a = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda, \quad b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\lambda), \\
&c = \frac{1}{2}(1-\theta^2)\mu, \quad d = \frac{1}{2}(1-\theta^2)(1-\mu).
\end{aligned} \tag{15}$$

The following change of variables

$$\begin{aligned}
&\widetilde{x} = \beta^{\frac{1}{2}}\widehat{x}, \quad \widetilde{t} = \beta^{\frac{1}{2}}\widehat{t}, \quad \widetilde{\eta} = \alpha^{-1}\widehat{\eta}, \\
&\widetilde{u}^{\theta} = \alpha^{-1}\widehat{u}^{\theta}, \quad \widetilde{h} = \alpha^{-1}\widehat{h}, \quad \widetilde{P}_0 = \alpha^{-2}\widehat{P}_0,
\end{aligned}$$

will simplify the equations by hiding the magnitude of each term

$$
\begin{aligned}
&\widehat{\eta}_{\widehat{t}} + (\widehat{u}^{\theta})_{\widehat{x}} + (\widehat{\eta}\widehat{u}^{\theta} + \widehat{h}\widehat{u}^{\theta})_{\widehat{x}} + a\widehat{u}^{\theta}_{\widehat{x}\widehat{x}\widehat{x}} - b\widehat{\eta}_{\widehat{x}\widehat{x}\widehat{t}} \\
&\qquad = -\widehat{h}_{\widehat{t}} + \frac{1}{2}\left((1-\lambda)\left(\theta^2 - \frac{1}{3}\right) - 2\theta + 1\right)\widehat{h}_{\widehat{x}\widehat{x}\widehat{t}}\,, \\
&\widehat{u}^{\theta}_{\widehat{t}} + \widehat{\eta}_{\widehat{x}} + \widehat{u}^{\theta}\widehat{u}^{\theta}_{\widehat{x}} + c\widehat{\eta}_{\widehat{x}\widehat{x}\widehat{x}} - d\widehat{u}^{\theta}_{\widehat{x}\widehat{x}\widehat{t}} = -\left(\widehat{P}_0\right)_{\widehat{x}} + (1-\theta)\widehat{h}_{\widehat{x}\widehat{t}\widehat{t}}\,,
\end{aligned}
\tag{16}
$$

It is worth to note that the relationship between the original physical variables $x, t, \eta, u^{\theta}, h, P_0$ and the new variables $\widehat{x}, \widehat{t}, \widehat{\eta}, \widehat{u}^{\theta}, \widehat{h}, \widehat{P}_0$ is

$$
\begin{aligned}
&x = h_0\widehat{x}, \quad t = h_0\widehat{t}/c_0, \quad \eta = h_0\widehat{\eta}, \\
&u^{\theta} = c_0\widehat{u}^{\theta}, \quad h - h_0 = h_0\widehat{h}, \quad P_0 = \rho c_0^2\widehat{P}_0.
\end{aligned}
$$

It follows that $\widehat{x}, \widehat{t}, \widehat{\eta}$ and $\widehat{u}^{\theta}$ are the standard non-dimensionalization of x, t, η and u^{θ} wherein the length scale is taken to be h_0 and the time scale to be h_0/c_0.

3 Remarks

Systems in (16)-(15) are the model equations describing two-way propagating surface waves over an uneven or a moving bottom. In general, it is more challenging to study a system than a single equation and there are fewer results available. But to describe the two-way propagation of water waves, one is obliged to study such a system.

It is clear from the derivation that (16) is valid under the assumptions that the bottom of the channel is moving slowly and has a small variation, and the variation of outside pressure along the channel is small. Therefore, the forcing terms in (16) should be of the form

$$
\widehat{h}(\widehat{x}, \widehat{t}\,) = \alpha H(\alpha^{\frac{1}{2}}\widehat{x}, \alpha^{\frac{1}{2}}\widehat{t}\,), \quad \widehat{P}_0(\widehat{x}, \widehat{t}\,) = \alpha^2 F(\alpha^{\frac{1}{2}}\widehat{x}, \alpha^{\frac{1}{2}}\widehat{t}\,)
$$

where H, F and their first few derivatives are all of order one.

If one poses the pure initial-value problem for (16), then to be physically relevant, the initial data

$$
\widehat{\eta}(\widehat{x}, 0) = \phi(\widehat{x}), \quad \widehat{u}^{\theta}(\widehat{x}, 0) = \psi(\widehat{x}),
$$

should satisfy the small-amplitude, long-wavelength assumptions inherent in the derivation of the models. That is, in principle, $\phi(\widehat{x})$ and $\psi(\widehat{x})$ should be of the form

$$
\phi(\widehat{x}) = \alpha f(\alpha^{\frac{1}{2}}\widehat{x}), \quad \psi(\widehat{x}) = \alpha g(\alpha^{\frac{1}{2}}\widehat{x}),
$$

where f, g and their first few derivatives are all of order one.

One sample system which corresponding to the regularized Boussinesq system studied in [2] reads, by taking $\lambda = \mu = 0$ and $\theta^2 = \frac{2}{3}$,

$$\widehat{\eta}_{\widehat{t}} + (\widehat{u}^\theta)_{\widehat{x}} + (\widehat{\eta}\widehat{u}^\theta + \widehat{h}\widehat{u}^\theta)_{\widehat{x}} - \frac{1}{6}\widehat{\eta}_{\widehat{x}\widehat{x}\widehat{t}} = -\widehat{h}_{\widehat{t}} + \left(\frac{1}{2} - \sqrt{\frac{2}{3}}\right)\widehat{h}_{\widehat{x}\widehat{x}\widehat{t}},$$

$$\widehat{u}^\theta_{\widehat{t}} + \widehat{\eta}_{\widehat{x}} + \widehat{u}^\theta \widehat{u}^\theta_{\widehat{x}} - \frac{1}{6}\widehat{u}^\theta_{\widehat{x}\widehat{x}\widehat{t}} = -(\widehat{P}_0)_{\widehat{x}} + \left(1 - \sqrt{\frac{2}{3}}\right)\widehat{h}_{\widehat{x}\widehat{t}\widehat{t}}.$$

The theoretical and numerical analysis of above system and the systems in (16) will be presented elsewhere.

Acknowledgement: The author wish to thank Professor Jerry Bona for his helpful comments and suggestions.

References

[1] C. J. Amick, *Regularity and uniqueness of solutions to the Boussinesq system of equations*, J. Differential Equations, 54 (1984), pp. 231–247.

[2] J. Bona and M. Chen, *A Boussinesq system for two-way propagation of nonlinear dispersive waves*, Physica D., 116 (1998), pp. 191–224.

[3] J. L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations for small-amplitude long wavelength water waves*, preprint, (2000).

[4] M. Chen, *Solitary-wave and multi-pulsed traveling-wave solutions of Boussinesq systems*, To appear in Applicable Analysis, (2000).

[5] R. S. Johnson, *On the development of a solitary wave moving over an uneven bottom*, Proc. Camb. Phil. Soc., 73 (1972), pp. 183–203.

[6] O. Madsen and C. C. Mei, *The transformation of a solitary wave over an uneven bottom*, J. Fluid Mech., 39 (1969), pp. 781–791.

[7] D. H. Peregrine, *Long waves on a beach*, J. Fluid Mech., 27 (1967), pp. 815–827.

[8] ——, *Equations for water waves and the approximation behind them*, in Waves on beaches and resulting sediment transport; proceedings of an advanced seminar conducted by the Mathematics Research Center, New York, Academic Press, 1972, pp. 95–121.

[9] M. E. Schonbek, *Existence of solutions for the Boussinesq system of equations*, J. Differential Equations, 42 (1981), pp. 325–352.

[10] T. Y. Wu, *Generation of upstream advancing solitons by moving disturbances*, Journal of fluid mechanics, 184 (1987), pp. 75–99.