DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **17**, Number **3**, March **2007**

Website: http://aimSciences.org

pp. **509–528**

LONG-TIME ASYMPTOTIC BEHAVIOR OF DISSIPATIVE BOUSSINESQ SYSTEMS

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(Communicated by Jerry Bona)

Abstract. In this paper, we study various dissipative mechanics associated with the Boussinesq systems which model two-dimensional small amplitude long wavelength water waves. We will show that the decay rate for the damped one-directional model equations, such as the KdV and BBM equations, holds for some of the damped Boussinesq systems.

1. **Introduction.** Considered here are waves on the surface of an inviscid fluid in a flat channel. When one is interested in the propagation of one-directional irrotational small amplitude long waves, it is classical to model the waves by the well-known KdV (Korteweg-de Vries) equation (see [23])

$$u_t + u_x + u_{xxx} + uu_x = 0,$$

or its regularized version, the so-called regularized long wave equation or BBM (Benjamin-Bona-Mahony) equation,

$$u_t + u_x - u_{txx} + uu_x = 0.$$

When one is dealing with two-directional waves, so the effects of wave interactions and/or wave reflections are not excluded from the study, a restricted four-parameter family of systems (see [5]),

$$\begin{aligned} &\eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} = 0, \\ &u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0, \end{aligned} \tag{1}$$

may be used. The dimensionless variables $\eta(x,t)$, $u(x,t)$, x, and t are scaled by the length scale h_0 and time scale $(h_0/g)^{\frac{1}{2}}$ where h_0 denotes the still water depth and g denotes the acceleration of gravity. The variable $\eta(x,t)$ is the non-dimensional deviation of the water surface from its undisturbed position and $u(x,t)$ is the non-dimensional horizontal velocity at a height θh_0, with $0 \le \theta \le 1$, above the bottom

2000 *Mathematics Subject Classification.* Primary: 35Q35, 35Q53,76B15; Secondary: 65M70.
Key words and phrases. waves, two-way propagation, Boussinesq systems, dissipation, long-time asymptotics.

of the channel. The constants a, b, c, d are dispersive constants which satisfy the physical relevant constraints

$$(C0) \qquad a+b+c+d = \frac{1}{3} \quad \text{and} \quad c+d = \frac{1}{2}(1-\theta^2) \geq 0.$$

This class of systems contains some of the well-known systems, such as the classical Boussinesq system ($a = b = c = 0,\ d = 1/3$) (see for example [9, 18, 22, 1, 20]) and the Bona-Smith system ($a = 0,\ b = d > 0,\ c < 0$) [8]. It is shown in [5, 6] that a physically relevant system in (1) is linearly well posed and locally nonlinearly wellposed in certain natural Sobolev spaces if the constants a, b, c, d satisfy

$$(C1) \qquad b \geq 0,\ d \geq 0,\ a \leq 0,\ c \leq 0,$$

or

$$(C2) \qquad b \geq 0,\ d \geq 0,\ a = c > 0.$$

It is also shown in [4, 7] that the above systems have the capacity to capture the main characteristics of the flow in an ideal fluid. But when the damping effect is comparable with the effects of nonlinearity and/or dispersion, as occurs in the real laboratory-scale experiments and in the fields (see [7, 16, 12, 17]), it should be considered in order for the model and its numerical results to correspond in detail with the experiments. The full system would be the Navier-Stokes equations with a free boundary, which is very difficult to handle both theoretically and numerically (cf. [21, 3]). Therefore, it is useful to construct simpler model systems which are capable of capturing the main properties of water waves under various special circumstances.

For example, many researchers have studied the dissipative one-way propagation model equations, such as the dissipative KdV and dissipative regularized long-wave equations and their generalizations. As a model to our study, we recall the results from [2] for the dissipative BBM equation,

$$\begin{aligned} u_t - u_{xxt} - \nu u_{xx} + u u_x = 0, \\ u(x,0) = u_0(x) \end{aligned}$$

where ν is a positive constant.

Theorem 1. *Assume u_0 is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then there exists a constant C such that*

$$\|u(t)\|_{L^2} \leq C(1+t)^{-1/4}. \tag{2}$$

Here $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ are the classical Banach spaces. A similar result holds for the corresponding dissipative KdV equation. See also [7], [15] and the references therein.

In this article, we aim to analyze the effect of dissipation on systems (1) and study the decay rates of solutions (η, u) toward zero. We will restrict our study to the cases where constants a, b, c, d satisfy (C0)-(C1) or (C0)-(C2). The goal of this research is to find the appropriate dissipative term (or terms) which will provide the right amount of energy dissipation for all wave numbers while keeping the mass conserved.

In this article, two kinds of dissipations will be considered:

Complete dissipation: replacing the $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in the right-hand side of (1) by the vector $\begin{pmatrix} \eta_{xx} \\ u_{xx} \end{pmatrix}$, and

Partial dissipation: replace the $\begin{pmatrix}0\\0\end{pmatrix}$ in the right-hand side of (1) by the vector $\begin{pmatrix}0\\u_{xx}\end{pmatrix}$.

The decay rates of solutions to the linearized systems supplemented with either complete or partial dissipations will be studied first. These equations read

$$\begin{aligned}&\eta_t + u_x + au_{xxx} - b\eta_{xxt} = \nu\eta_{xx},\\&u_t + \eta_x + c\eta_{xxx} - du_{xxt} = u_{xx},\end{aligned} \tag{3}$$

with $\nu = 0$ or $\nu = 1$. Systems which satisfy the *dichotomy* property in the Fourier space:

- decay as $t^{-1/4}$ for low frequencies (small ξ);
- decay as $\exp(-\beta t)$ or $\exp(-\beta\xi^2 t)$ for high frequencies (large ξ);

will be identified and studied. It is shown in Section 3 that the dichotomy property will lead to the decay rate $t^{-\frac{1}{4}}$ for $\|(\eta, u)\|_{L^2\times H^h}$ where h will be specified later.

For later use, we shall emphasize (and it is easy to check) that the dichotomy property holds true for two fundamentally different equations: the linearized BBM-Burgers equation $u_t - u_{xxt} + u_x - u_{xx} = 0$ and the linearized KdV-Burgers equation $u_t + u_{xxx} + u_x - u_{xx} = 0$. If the high frequency part of the solution to a system is damped as $\exp(-\beta t)$, we say that the system belongs to the BBM-Burgers class since solutions to linearized BBM-Burgers equation feature this property. If the high frequency part of the solution is damped as $\exp(-\beta\xi^2 t)$, which is the case for linearized KdV-Burgers equation, then we say that the system belongs to the KdV-Burgers class. The low frequency parts of the solutions to the linearized BBM-Burgers and KdV-Burgers equations behave in a similar fashion.

The main result in Section 3 is to classify the linearized systems accordingly and to prove that for systems which satisfy the dichotomy property, a decay rate comparing to (2) is valid. We shall also present some systems where the decay rates can be arbitrarily small, behaving as the solution of

$$u_t - u_{xxt} + u = 0, \quad u(x, 0) = u_0(x),$$

where by Fourier transform,

$$\widehat{u}(t, \xi) = e^{-\frac{t}{1+\xi^2}}\widehat{u}_0(\xi),$$

and therefore $\|u(., t)\|_{L^2}$ could decay arbitrarily slow.

In Section 4, we extend the linear theory to nonlinear systems and show that the decay rate as (2) is valid for weakly dispersive systems, i.e. systems with $b > 0$ and $d > 0$, and for some systems in the KdV-Burgers class with total dissipation, which include the KdV-KdV system ($a = c = \frac{1}{6}, b = d = 0$), with small initial data. In Section 5, the decay rate with respect to L^∞-norm is presented and in Section 6, spectral method is used on several systems to demonstrate that the rates obtained in Section 4 and Section 5 are sharp and the constants involved in the bounds are reasonably sized.

It is worthwhile to note that there are other methods, such as the energy methods (like the so-called Schonbek's splitting method which has been applied to the classical Boussinesq system [19] in large dimensions), can be used in proving decay rate for solutions of (1) with dissipation. We believe that those methods will be helpful especially in the cases where $b = d$ so there is a Hamiltonian(see [6]). They might also be useful in extending our local results to global ones. The authors in [2] used a kind of Cole–Hopf transformation for single equations, such as the KdV-Burgers and BBM-Burgers equations, and were able to control the extra-terms. This line of study will be carried out elsewhere.

2. **Notations and Preparations.** Throughout the paper, the standard notation on Sobolev spaces will be used. The $L^p(\mathbb{R})$ norm will be denoted as $\|\cdot\|_{L^p}$ for $1 \le p \le \infty$ and the H^s norm will be denoted as $\|\cdot\|_{H^s}$. When several variables are involved, we may also set L^p_x for $L^p(\mathbb{R})$ to specify that the norm is computed with respect to the x-variable. The product space $X \times X$ will be abbreviated by X and a function $\mathbf{f} = (f_1, f_2)$ in X carries the norm

$$\|\mathbf{f}\|_X = \left(\|f_1\|_X^2 + \|f_2\|_X^2\right)^{\frac{1}{2}}.$$

The Euclidean norm of a vector is denoted by $|\cdot|$. We will use C and β as generic positive constants whose values may change with each appearance. Fourier transform of a function f is denoted by either $\widehat{f}$ or $\mathcal{F}(f)$.

2.1. **Some notations.** Consider $\nu \in \{0, 1\}$. As stated before, we plan to first estimate the decay rates of solutions to the linear systems

$$\begin{aligned} &\eta_t + u_x + a u_{xxx} - b\eta_{xxt} = \nu \eta_{xx}, \\ &u_t + \eta_x + c\eta_{xxx} - d u_{xxt} = u_{xx}, \end{aligned} \tag{4}$$

when t goes to $+\infty$.

Following [6], we introduce the Fourier multipliers

$$\omega_1 = \frac{1 - a\xi^2}{1 + b\xi^2} \quad \text{and} \quad \omega_2 = \frac{1 - c\xi^2}{1 + d\xi^2}.$$

Since a, b, c, d satisfy (C1) or (C2), $\omega_1\omega_2$ is nonnegative and we denote

$$\widehat{H} = \left(\frac{\omega_1}{\omega_2}\right)^{1/2} \quad \text{and} \quad \sigma = (\omega_1\omega_2)^{1/2},$$

with the conventional notation $\frac{0}{0} = 1$. We also denote

$$\alpha = \frac{\xi^2}{1 + b\xi^2} \quad \text{and} \quad \varepsilon = \frac{\xi^2}{1 + d\xi^2}.$$

Remark 1. When a system satisfying (C2) assumption is the subject of the study, ω_1 and ω_2 do change signs, but $\omega_1\omega_2 \ge 0$.

Definition 1. Consider a nonnegative function $\xi \to \widehat{\kappa}(\xi)$. The *order* of $\widehat{\kappa}$ (when it exists) is defined as the number m such that

$$\widehat{\kappa}(\xi) \sim C|\xi|^m$$

when $|\xi| \to +\infty$. The (pseudo–differential) operator κ with order m is defined by setting

$$\kappa u = v \quad \text{iff} \quad \widehat{\kappa}\widehat{u} = \widehat{v}.$$

Therefore κ maps L^2_x into $H_x^{-\text{order}(\kappa)}$ (or H^n_x into $H_x^{n-\text{order}(\kappa)}$).

Since (4) is a linear system, it is convenient to use the Fourier transform. Let $(\widehat{\eta}, \widehat{u})$ denote the Fourier transform of (η, u) and set $\widehat{Y} = (\widehat{\eta}, \widehat{w})$ with $\widehat{w} = \widehat{H}\widehat{u}$, then (4) reads

$$\widehat{Y}_t + A\widehat{Y} = 0, \tag{5}$$

where

$$A(\xi) = \begin{pmatrix} \nu\alpha & i\,\mathrm{sgn}(\omega_1)\xi\sigma \\ i\,\mathrm{sgn}(\omega_2)\xi\sigma & \varepsilon \end{pmatrix}$$

is the *symbol* of the linear (unbounded) operator in (4). Since we are dealing with a system, $A(\xi)$ is a matrix.

By multiplying $\widehat{Y}^*$ on (5), taking the real part and integrating over $\mathbb{R}$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int(|\widehat{\eta}(t,\xi)|^2 + |\widehat{w}(t,\xi)|^2)d\xi \\ &+ \nu\int\alpha(\xi)|\widehat{\eta}(t,\xi)|^2 d\xi + \int\varepsilon(\xi)|\widehat{w}(t,\xi)|^2 d\xi = 0. \end{aligned} \tag{6}$$

Since $\alpha(\xi)$ and $\varepsilon(\xi)$ are positive,

$$\frac{d}{dt}E(t) \leq 0$$

with

$$E(t) := \int_{\mathbb{R}} |\widehat{Y}(t,\xi)|^2 d\xi \tag{7}$$

where $|\widehat{Y}| = (|\widehat{\eta}|^2 + |\widehat{w}|^2)^{1/2}$ is the Euclidean norm on $\mathbb{C}^2$. We shall prove below that $E(t)$ decays towards 0 as $t \to \infty$ and find the decay rates of solutions to the linear systems in (4) and to the nonlinear systems in (3).

2.2. **Linear algebra.** We recall some facts from linear algebra and then apply them to the dissipative systems (4).

Definition 2. Let M be a 2×2 matrix in the complex space, the norm of M is defined by

$$\|M\| = \sup_{Y\in\mathbb{C}^2\setminus\{0\}} \frac{|MY|}{|Y|}.$$

Lemma 1. *Let $\rho(M)$ denote the spectral radius of a matrix M and $\mathrm{tr}(M)$ denote the trace of M, then*

$$\|M\| = \rho(M^*M)^{1/2} \leq \mathrm{tr}(M^*M)^{1/2}.$$

We are now going to bound $E(t)$ (see (5) and (7)) by using the pointwise estimate

$$|\widehat{Y}(t,\xi)| \leq \|e^{-tA}\|\ |\widehat{Y}_0(\xi)|. \tag{8}$$

Noticing that the matrix A can be written as $A = D + U$, where $D = \begin{pmatrix} \nu\alpha & 0 \\ 0 & \varepsilon \end{pmatrix}$ represents the dissipation terms and $U = \begin{pmatrix} 0 & i\mathrm{sgn}(\omega_1)\xi\sigma \\ i\mathrm{sgn}(\omega_2)\xi\sigma & 0 \end{pmatrix}$ is skew-symmetric. When D and U commute, the behavior of $\|e^{-tA}\|$ with respect to ξ is characterized by the behaviors of ε and α via

$$\|e^{-tA}\| \leq e^{-t\min\{\nu\alpha(\xi),\varepsilon(\xi)\}}. \tag{9}$$

But when D and U do not commute, a more accurate estimate than (9) can be obtained by studying e^{-tA} in detail.

We now recall the following lemma (Theorem 9.28 from [13]).

Lemma 2. *There exists a unitary matrix Q (i.e. $QQ^* = Q^*Q = I$) such that*

$$A = Q^* \begin{pmatrix} \lambda_1 & z \\ 0 & \lambda_2 \end{pmatrix} Q,$$

where λ_1 and λ_2 are the eigenvalues of A, ordered by $Re(\lambda_1) \leq Re(\lambda_2)$.

As a consequence, one can prove the following lemma (which is given at the end of this subsection).

Lemma 3. *There exists $C > 0$ such that*

$$\|\exp(-tA)\| \leq C \left(1 + |z| \min\left(t, \frac{1}{|\lambda_2 - \lambda_1|}\right)\right) \exp(-tRe(\lambda_1)). \tag{10}$$

It is easy to see that λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0 \tag{11}$$

where

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 = \nu\alpha + \varepsilon \geq 0 \tag{12}$$

and

$$\det(A) = \lambda_1\lambda_2 = \nu\alpha\varepsilon + \xi^2\sigma^2 \geq 0. \tag{13}$$

We now estimate $\|\exp(-tA)\|$ by separating the cases $\Delta \leq 0$ and $\Delta > 0$ where Δ is the determinant of (11), namely

$$\Delta = \mathrm{tr}(A)^2 - 4\det(A) = (\varepsilon - \nu\alpha)^2 - 4\xi^2\sigma^2. \tag{14}$$

Lemma 4. *For any $t > 0$ and for any $\xi \in \mathbb{R}$,*
- *when $\Delta \leq 0$ (perturbation range),*

$$\begin{aligned} \|\exp(-tA)\| &\leq C(1 + \mathrm{tr}(A)t) \exp\left(-\frac{\mathrm{tr}(A)}{2}\, t\right) \\ &\leq C \exp\left(-\frac{\mathrm{tr}(A)}{4}\, t\right); \end{aligned} \tag{15}$$

- *when $\Delta > 0$ (non-perturbation range),*

$$\|\exp(-tA)\| \leq C\left(1 + 2|\xi|\sigma \min\left(t, \frac{1}{\sqrt{\Delta}}\right)\right) \exp(-t\lambda_1), \tag{16}$$

where λ_1 satisfies

$$\frac{\det(A)}{\mathrm{tr}(A)} \leq \lambda_1 \leq \min\left(\mathrm{tr}(A), \frac{2\det(A)}{\mathrm{tr}(A)}\right). \tag{17}$$

Proof. It is worthwhile to note from Lemma 2 that

$$\mathrm{tr}(A^*A) = |\lambda_1|^2 + |\lambda_2|^2 + |z|^2 = \nu^2\alpha^2 + \varepsilon^2 + 2\xi^2\sigma^2. \tag{18}$$

When $\Delta \leq 0$ (perturbation range): matrix A has two conjugate complex eigenvalues λ_1 and λ_2 with

$$|\lambda_1| = |\lambda_2|, \quad \mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \frac{\mathrm{tr}(A)}{2}, \quad |\lambda_1|^2 = |\lambda_2|^2 = \det(A).$$

Using (18) and then (13)-(12) leads to

$$\begin{aligned} |z|^2 &= \nu^2\alpha^2 + \varepsilon^2 + 2\xi^2\sigma^2 - 2|\lambda_1|^2 \\ &= \nu^2\alpha^2 + \varepsilon^2 + 2\xi^2\sigma^2 - 2\det(A) = (\nu\alpha - \varepsilon)^2 \leq \mathrm{tr}(A)^2. \end{aligned}$$

Hence (15) follows from Lemma 3.

When $\Delta > 0$ (non-perturbation range): $\mathrm{tr}(A) \geq 0$ and $\det(A) \geq 0$ imply that the matrix A features two real eigenvalues $0 \leq \lambda_1 < \lambda_2$. Then (18) leads to

$$|z|^2 = \nu^2\alpha^2 + \varepsilon^2 + 2\xi^2\sigma^2 - \mathrm{tr}(A)^2 + 2\det(A) = 4\xi^2\sigma^2$$

and (16) is proved by using Lemma 3. Since $\lambda_2 \leq \mathrm{tr}(A) \leq 2\lambda_2$, one sees immediately that

$$\frac{\det(A)}{\mathrm{tr}(A)} \leq \lambda_1 = \frac{\det(A)}{\lambda_2} \leq \frac{2\det(A)}{\mathrm{tr}(A)}$$

and (17) follows. □

It is noted that when $\Delta \leq 0$, the dissipation can be considered as a perturbation term with respect to the skew symmetric operator. More precisely, the decay is the same as pretending U and D commute, up to a linear correction.

When $\Delta > 0$, this is no longer valid. In the first case, matrix A has conjugate complex eigenvalues. In the latter case, A has real positive eigenvalues and the smallest one monitors the decay estimate. $\Delta = 0$ is the bifurcation point.

For the sake of completeness, we now give the proof of Lemma 3.

Proof of Lemma 3. Straightforward computations lead to

$$e^{-tA} = Q^* \begin{pmatrix} e^{-t\lambda_1} & z\frac{e^{-t\lambda_1}-e^{-t\lambda_2}}{\lambda_1-\lambda_2} \\ 0 & e^{-t\lambda_2} \end{pmatrix} Q,$$

where

$$\frac{e^{-t\lambda_1} - e^{-t\lambda_2}}{\lambda_1 - \lambda_2} = -te^{-t\lambda_2}, \quad \text{if} \quad \lambda_1 = \lambda_2.$$

Lemma 1 then yields

$$\begin{aligned} \|e^{-tA}\|^2 &\leq \mathrm{tr}(e^{-tA^*}e^{-tA}) \\ &= e^{-2tRe(\lambda_1)} + e^{-2tRe(\lambda_2)} + |z|^2 \left| \frac{e^{-t\lambda_1} - e^{-t\lambda_2}}{\lambda_1 - \lambda_2} \right|^2. \end{aligned} \tag{19}$$

Therefore, if $|\lambda_1 - \lambda_2| > 0$

$$\|e^{-tA}\|^2 \leq (2 + \frac{|z|^2}{|\lambda_1 - \lambda_2|^2})e^{-2tRe(\lambda_1)}$$

which proves part of the lemma.

Now, for $|\lambda_1 - \lambda_2| \geq \Lambda$, where $\Lambda > 0$ will be chosen later,

$$\frac{|e^{-t\lambda_1} - e^{-t\lambda_2}|}{|\lambda_1 - \lambda_2|} \leq \frac{2}{\Lambda} e^{-tRe(\lambda_1)}; \tag{20}$$

and for $|\lambda_1 - \lambda_2| \leq \Lambda$, using $|e^\zeta - 1| \leq |\zeta|\exp(|\zeta|)$ for any complex number ζ,

$$\begin{aligned} |e^{-t\lambda_1} - e^{-t\lambda_2}| &= e^{-t\mathrm{Re}(\lambda_1)}|e^{-t(\lambda_2-\lambda_1)} - 1| \\ &\leq e^{-t\mathrm{Re}(\lambda_1)}t|\lambda_2 - \lambda_1|e^{t\Lambda}. \end{aligned} \tag{21}$$

Therefore, choosing $\Lambda = \dfrac{1}{t}$ in (20) and (21),

$$\frac{|e^{-t\lambda_1} - e^{-t\lambda_2}|}{|\lambda_1 - \lambda_2|} \leq Cte^{-tRe(\lambda_1)}.$$

Substituting above into (19) completes the proof. □

3. **Decay rate of linear systems.** In subsections 3.1, 3.2 and 3.3, low-frequency ($|\xi|$ close to 0), high-frequency (large $|\xi|$) and middle range frequency analysis for the linear systems are performed respectively. We will identify systems for which there exist positive constants β and δ_m such that for any $t > 0$ and

$$\begin{aligned} &\bullet \text{ for } |\xi| \le \delta_m,\ \|\exp(-tA)\| \le C\exp(-\beta t\xi^2), \\ &\bullet \text{ for } |\xi| > \delta_m,\ \|\exp(-tA)\| \le C\exp(-\beta t). \end{aligned} \tag{22}$$

Here $\|\exp(-tA)\|$ is the norm of the linear operator $\exp(-tA(\xi))$ acting on $\mathbb{C}^2$. The generic constants C and β are independent of t and ξ. If $\delta_m = +\infty$ is feasible in (22), the system is in the *KdV–Burgers class*. Otherwise the system is in the *BBM–Burgers class*. A summary of decay rates for the linear systems is given in subsection 3.4.

Remark 2. It will become apparent soon that (22) is sufficient but not necessary for proving the desired decay rate. An example is given in Section 3.3 where (22) is not valid but the linear system has the desired decay rate.

3.1. **Low frequency analysis.** We now prove that for $|\xi| \to 0$, all systems are equivalent. This is to say

Proposition 1. *There exist positive constants δ_m, β and C depending on the data a, b, c, d and ν, such that for $|\xi| \le \delta_m$ and for any $t > 0$,*

$$\|\exp(-tA)\| \le C\exp(-\beta\xi^2 t). \tag{23}$$

Consequently, for any initial data Y_0 with supp$(\widehat{Y_0}) \subset [-\delta_m, \delta_m]$,

$$E(t) \le C\ t^{-1/2}\|Y_0\|^2_{L^1_x}.$$

Proof. By referring to the definitions of Δ, α and ε, one sees that as $|\xi| \to 0$,

$$\Delta \sim -4\xi^2 \quad \text{and} \quad \mathrm{tr}(A) = \nu\alpha + \varepsilon \sim (\nu+1)\xi^2.$$

Therefore, there exists $\delta_m > 0$ such that for ξ in $[-\delta_m, \delta_m]$, $\Delta \le 0$ and

$$\frac{1}{2} \le \frac{\mathrm{tr}(A)}{(\nu+1)\xi^2} \le 2.$$

(23) then follows promptly from (15).

Now, for any initial data Y_0 with supp$(\widehat{Y_0}) \subset [-\delta_m, \delta_m]$, it is obtained

$$\begin{aligned} E(t) = \int_{|\xi|\le\delta_m} |\widehat{Y}(t,\xi)|^2 d\xi &\le C\int \exp(-2\beta t\xi^2)d\xi(\sup_\xi(|\widehat{Y_0}(\xi)|^2)) \\ &\le Ct^{-1/2}\|Y_0\|^2_{L^1_x} \end{aligned}$$

by using the change of variable $\tau = \sqrt{2\beta t}\xi$. □

3.2. **High frequency analysis.** For the high frequency analysis, the complete dissipation and the partial dissipation cases have to be studied separately. In the latter case, we will give one example where the decay rate can be arbitrarily small.

Let us first introduce the notation

$$\{r\} = \begin{cases} 1, & \text{if } r \ne 0, \\ 0, & \text{if } r = 0, \end{cases}$$

for $r \in \mathbb{R}$. Then order$(\sigma) = \{a\} + \{c\} - \{b\} - \{d\}$ and order$(\varepsilon) = 2 - 2\{d\}$.

3.2.1. *The complete dissipation case ($\nu = 1$).* It is observed in the following that order(σ) dictates if the system is in the KdV-Burgers class or in the BBM-Burgers class.

Proposition 2. *Assume $\nu = 1$. For any $\delta > 0$, there exists $\beta > 0$, such that if supp$(\widehat{Y_0}) \subset \mathbb{R}\backslash[-\delta, \delta]$,*

$$E(t) \leq \exp(-2\beta t)\|Y_0\|^2_{L^2_x} \tag{24}$$

for any $t > 0$. In addition,

- *if order(σ) ≤ 0, the system is in the BBM–Burgers class. Namely, there exist positive constants δ_M, β and C, such that for $|\xi| > \delta_M$ and $t > 0$,*
$$\|\exp(-tA)\| \leq Ce^{-\beta t};$$
- *if order(σ) ≥ 1, the system is in the KdV–Burgers class. Namely, there exist positive constants δ_M, β and C, such that for $|\xi| > \delta_M$ and $t > 0$,*
$$\|\exp(-tA)\| \leq Ce^{-\beta\xi^2 t}.$$

Proof. From (6), one finds that for ξ almost everywhere,

$$\frac{1}{2}\frac{d}{dt}|\widehat{Y}(t,\xi)|^2 + \alpha(\xi)|\widehat{\eta}(t,\xi)|^2 + \varepsilon(\xi)|\widehat{w}(t,\xi)|^2 = 0.$$

This gives directly, by setting $\beta = \min\{\alpha(\delta), \varepsilon(\delta)\}$ which is positive, that (24) is valid. Furthermore, $\|\exp(-tA)\| \leq Ce^{-\beta t}$ for $|\xi| > \delta$. To figure out if the system is in the BBM-Burgers or in the KdV-Burgers classes, we separate the cases as follows.

- Assume first order(σ) ≥ 1. Then either $d = 0$ or $b = 0$. Without loss of generality, let us assume $d = 0$.
  - If $\Delta = (\alpha - \varepsilon)^2 - 4\xi^2\sigma^2 > 0$ for $|\xi|$ large enough, then order(σ) $= 1$. In that case, there exist $\beta > 0$ and $\delta_M > 0$
  $$\lambda_1 \geq \frac{\det(A)}{\operatorname{tr}(A)} = \frac{\alpha + \sigma^2}{\frac{\alpha}{\xi^2} + 1} \geq 2\beta\xi^2$$
  for $|\xi| > \delta_M$. By using (16)
  $$\|\exp(-tA)\| \leq C(1 + t\xi^2)e^{-2\beta t\xi^2} \leq Ce^{-\beta t\xi^2}$$
  for $|\xi| > \delta_M$ and the system is in the KdV-Burgers class.
  - If $\Delta \leq 0$ for $|\xi|$ large enough, then there exists $\delta_M > 0$ such that for $|\xi| > \delta_M$, $\frac{1}{2}\xi^2 \leq \operatorname{tr}(A) \leq 2\xi^2$, and (15) implies that the system is in the KdV-Burgers class.
- Assume now that order(σ) ≤ 0.
  - If $b \neq 0$ and $d \neq 0$ (weakly dispersive systems) then for $|\xi|$ large enough, $\operatorname{Re}(\lambda_1) \leq \operatorname{tr}(A) \sim \frac{1}{b} + \frac{1}{d}$ as $|\xi| \to \infty$. This shows that a damping like $e^{-\beta t\xi^2}$ is unlikely for high frequencies. Therefore the weakly dispersive systems are in the BBM-Burgers class.
  - If ($b \neq 0$ and $d = 0$) or ($b = 0$ and $d \neq 0$), without loss of generality, let us consider the case $b \neq 0$ and $d = 0$. Since $\Delta \sim \xi^4$ as $|\xi| \to \infty$, we have
  $$\lambda_1 \leq \frac{2\det(A)}{\operatorname{tr}(A)} = \frac{2(\alpha + \sigma^2)}{\frac{\alpha}{\xi^2} + 1} \sim C = O(1)$$
  as $|\xi| \to \infty$. This shows that a damping like $e^{-\beta t\xi^2}$ is again unlikely for high frequencies. Therefore the system is in the BBM-Burgers class.

□

3.2.2. *The partial dissipation case ($\nu = 0$).* We first note that when a system satisfies (C2) hypothesis, $\sigma = 0$ and therefore $\lambda_1 = 0$ at $\xi = a^{-\frac{1}{2}}$. But one can always chose δ_M large enough so for $|\xi| > \delta_M$, σ is positive, bounded from below and away from zero. Therefore the point where σ vanishes will be considered in the next subsection. We now prove that in the partial dissipation case, the decay rate is related to order(σ) and the strength of the dissipation which is characterized by order(ε).

Proposition 3. *With $\nu = 0$,*

- *if order(σ) $\geq 2 - \frac{1}{2}$order(ε), then the system is in the KdV-Burgers class;*
- *if $|$order(σ)$| < 2 - \frac{1}{2}$order(ε), then the system is in the BBM-Burgers class;*
- *if order(σ) $\leq -2 + \frac{1}{2}$order(ε), arbitrarily slow decay can occur.*

In the first two cases, when $\widehat{Y}_0$ is supported in $\mathbb{R}\backslash[-\delta_M, \delta_M]$, then for any $t > 0$,

$$E(t) \leq C \exp(-2\beta t)\|Y_0\|^2_{L^2_x}.$$

Proof. To begin, one observes that $\Delta = \varepsilon^2 - 4\xi^2\sigma^2$. When $\Delta > 0$, λ_1 satisfies

$$2\lambda_1 = \varepsilon\left(1 - \left(1 - \frac{4\xi^2\sigma^2}{\varepsilon^2}\right)^{1/2}\right) \tag{25}$$

which is a direct consequence of (11).

- *When order(ε)=0, i.e $d \neq 0$ (and order(σ) ≤ 1):*
  - if order(σ) ≥ -1, and if $\Delta = \varepsilon^2 - 4\xi^2\sigma^2 > 0$ for $|\xi|$ large enough, which is possible only for order(σ) $= -1$, we have

    $$\lambda_1 \geq \frac{\det(A)}{\operatorname{tr}(A)} \geq C\xi^2\sigma^2 = O(1)$$

    as $|\xi| \to \infty$. Then by (16) we have

    $$\|e^{-tA}\| \leq C(1 + t|\xi|\sigma)e^{-t\lambda_1} \leq Ce^{-\beta t}$$

    for $|\xi|$ large enough and the system is in the BBM-Burgers class. On the other hand, if $\Delta \leq 0$ for high frequencies, since $\operatorname{tr}(A) \sim \frac{1}{d}$ as $|\xi| \to \infty$, then (15) implies that the system is in the BBM-Burgers class;
  - if order(σ) $= -2$, $\Delta \sim \frac{1}{d^2} > 0$ as $|\xi| \to \infty$ and by (25), $\lambda_1 \sim C|\xi|^{-2}$, therefore arbitrarily slow decay could occur. An example of such case will be given below.
- *When order(ε) $= 2$ i.e $d = 0$ (and order(σ) ≥ -1):* $\Delta = \xi^2(\xi^2 - 4\sigma^2)$ has a limit Δ_0 in $[-\infty, +\infty]$ when $|\xi|$ approaches $+\infty$.
  - If Δ_0 is in $[-\infty, 0]$, then since $\operatorname{tr}(A) = \xi^2$, (15) implies that the system is in the KdV-Burgers class. This occurs when order(σ) $= 2$ and may occur when order(σ) $= 1$.
  - If Δ_0 is in $(0, +\infty]$, then since

    $$\frac{2\det(A)}{\operatorname{tr}(A)} \geq \lambda_1 \geq \frac{\det(A)}{\operatorname{tr}(A)} = \sigma^2,$$

    (16) implies for any ξ

    $$\|\exp(-tA)\| \leq C\left(1 + |\xi|\sigma \min\left(t, \frac{1}{\sqrt{\Delta}}\right)\right)\exp(-t\sigma^2). \tag{26}$$

    * If order(σ) $= 1$, then (26) implies the system is in the KdV-Burgers class.

* If order$(\sigma) = 0$, $\frac{|\xi\sigma|}{\sqrt{\Delta}} = O(1)$ as $|\xi| \to \infty$, the system is in the BBM-Burgers class. And similarly,
* if order$(\sigma) = -1$, any arbitrarily slow decay could occur.

□

Example of slow decay: Consider the linearized BBM-BBM system with partial dissipation,

$$\eta_t + u_x - b\eta_{xxt} = 0,$$
$$u_t + \eta_x - du_{xxt} = u_{xx},$$

which has order$(\varepsilon) = 0$ and order$(\sigma) = -2$. Since as $|\xi| \to +\infty$,

$$\Delta \sim \frac{1}{d} > 0, \quad |z| = 2|\xi|\sigma \sim \frac{2}{\sqrt{bd}}\ \frac{1}{|\xi|},$$
$$2\lambda_1 = \text{tr}(A)\left(1 - \left(1 - \frac{4\det(A)}{(\text{tr}(A))^2}\right)^{1/2}\right) \sim\ 2\ \frac{\det(A)}{\text{tr}(A)} \sim \frac{2}{d\xi^2}.$$

Therefore

$$\|e^{-tA}\| \le C\exp\left(-\frac{\beta t}{\xi^2}\right)$$

which shows that any arbitrary slow decay could occur.

3.3. **Middle range frequency analysis.** We first note from Lemma 4 that to get the optimal decay estimates for the cases where $\det(A)$ (and therefore λ_1) has a zero for $|\xi| > 0$, these cases need to be discussed separately. Therefore, we have the following two propositions.

Proposition 4. *Assume that $\nu = 1$, or that $\nu = 0$ and the dispersive coefficients a, b, c, d satisfy (C1). Then for any δ_m and δ_M, $0 < \delta_m \le \delta_M$, there exists $\beta > 0$ such that for $|\xi| \in [\delta_m, \delta_M]$ and for any $t > 0$,*

$$\|\exp(-tA)\| \le C\exp(-\beta t). \tag{27}$$

Moreover for any $\widehat{Y_0}$ with support included in $[\delta_m, \delta_M] \cup [-\delta_M, -\delta_m]$,

$$E(t) \le C\exp(-2\beta t)||Y_0||^2_{L^2_x}.$$

Proof. Since $\text{tr}(A)$ and $\det(A)$ cannot vanish for $|\xi| \in [\delta_m, \delta_M]$ under the assumptions, (27) is the direct consequence of (15) and (16). In addition

$$E(t) \le \sup_\xi ||e^{-tA}||^2\, ||\widehat{Y_0}||^2_{L^2_\xi} \le C\exp(-2\beta t)||Y_0||^2_{L^2_x},$$

which completes the proof of the proposition. □

Remark 3. By noticing that (27) can be replaced by

$$\|\exp(-tA)\| \le C\exp(-\beta^* t\xi^2).$$

with $\beta^* = \beta/\delta_M^2$, the middle range frequency analysis and the high frequency analysis can be combined to simplify certain calculations for these systems regardless if they are in BBM-Burgers class or KdV-Burgers class.

Proposition 5. *Assume that $\nu = 0$ and the dispersive coefficients satisfy (C2). Then for any $0 < \delta_m < \delta_M$ with $r = a^{-\frac{1}{2}} \in [\delta_m, \delta_M]$, there exist $\beta > 0$ and $C > 0$ such that for any $|\xi| \in [\delta_m, \delta_M]$ and for any $t > 0$,*

$$\|\exp(-tA)\| \le C\exp\{-\beta t(|\xi| - r)^2)\}. \tag{28}$$

Moreover for any $\widehat{Y_0}$ with support included in $[\delta_m, \delta_M] \cup [-\delta_M, -\delta_m]$ and for any $t > 0$,

$$E(t) \leq Ct^{-1/2}||Y_0||^2_{L^1_x}.$$

Remark 4. Proposition 5 shows that even when the dichotomy is not valid, the energy could decay as $O(t^{-1/4})$ when t goes to $+\infty$.

Proof. Since $\det(A)$ vanishes at $r = (\sqrt{a})^{-1}$ and, when $|\xi| \to r$, $\Delta \sim \frac{a}{a+d} > 0$, $\lambda_1 \sim \beta\det(A) \sim \beta\sigma^2 \sim \beta(|\xi| - r)^2$. Therefore, from (16),

$$\| \exp(-tA)\| \leq C(1 + \min(t, 1)\sigma)\exp(-\beta\sigma^2 t)$$

for $|\xi|$ in the neighborhood of r. Using the fact that for any $t > 0$, $\min(t, 1) \leq \sqrt{t}$, there exists $C > 0$ such that

$$\| \exp(-tA)\| \leq C(1 + \sqrt{t}\sigma)\exp(-\beta\sigma^2 t) \leq C\exp(-\frac{\beta}{2}\sigma^2 t),$$

we obtain the estimate (28) for $|\xi|$ close to r. For other $|\xi|$ in $[\delta_m, \delta_M]$, the same argument in the proof of Proposition 4 and Remark 3 applies. To prove the decay rate of $E(t)$, the same argument as in the proof of Propositions 1 can be used. In fact, $|\xi| - r$ plays the same role as $|\xi|$ in that case. □

3.4. **Decay for linear systems.** Since linear system (3) defines a semi-group e^{-tA} for $t \geq 0$ which is contracting on $L^2 \times L^2$ in the variable (η, w), the initial value problem is therefore well-posed and the L^2 norm decays.

Combining the low, middle and high frequency analysis, the decay rate for the linear system (3) can be stated as

Theorem 2. *For systems (3) with the dispersive constants a, b, c, d satisfy the constraints (C0)-(C1) or (C0)-(C2), assuming either $\{\nu = 1\}$ or, $\{\nu = 0$ and $order(\sigma) > -2 + \frac{1}{2}order(\varepsilon)\}$, then for any $(\eta_0, Hu_0) = (\eta_0, w_0) \in (L^1(\mathbb{R}) \cap L^2(\mathbb{R}))^2$ where $\widehat{H}\widehat{u}_0 = (\frac{(1-a\xi^2)(1+d\xi^2)}{(1-c\xi^2)(1+b\xi^2)})^{\frac{1}{2}}\widehat{u}_0$, there exists a constant C, such that for any $t > 0$*

$$\|(\eta, Hu)\|_{L^2_x} = \|(\eta, w)\|_{L^2_x} \leq Ct^{-1/4}.$$

Remark 5. Theorem 2 implies, with respect to physical variables (η, u), that for any $(\eta_0, u_0) \in (L^2_x \cap L^1_x) \times (H^h_x \cap W^{h,1}_x)$,

$$\|(\eta, u)\|_{L^2_x \times H^h_x} \leq Ct^{-1/4}$$

for any $t > 0$, where $h = \text{order}(\widehat{H}) = \{a\} + \{d\} - \{c\} - \{b\}$.

Proof. Combining the low, middle and high frequency analysis, we have

$$E(t) = E_{low}(t) + E_{middle}(t) + E_{high}(t) \leq C(\eta_0, u_0)(t^{-\frac{1}{2}} + e^{-2\beta t})$$

for $t > 0$ where $C(\eta_0, u_0)$ is a function of the dispersive coefficients and the norms of η_0 and u_0. □

We complete this section with the following corollaries and remark.

Corollary 1. *For any dissipation, the classical Boussinesq system, the Bona–Smith system, the coupled KdV–BBM ($b = c = 0$) system, the BBM–KdV systems ($a = d = 0$) and the weakly dispersive systems ($b > 0$ and $d > 0$) with $a < 0$ or $c < 0$ belong to the BBM–Burgers class.*

Corollary 2. *With complete dissipation, the KdV–KdV system ($b = d = 0$, $a = c > 0$) belongs to the KdV–Burgers class; the weakly dispersive systems ($b > 0$ and $d > 0$) belong to the BBM–Burgers class.*

Remark 6. When the consideration is restricted to the linear systems, a result similar to Theorem 2 (substituting α for ε in the statement) holds when one replaces $(\nu\eta_{xx}, u_{xx})$ by $(\eta_{xx}, \nu u_{xx})$ in the right-hand side of (4) since η and u play the same role.

4. Nonlinear Theory.

For convenience, we will only consider in this section (i) the complete dissipation and (ii) the partial dissipation with a, b, c, d satisfy the (C1) assumption. The partial dissipation with a, b, c, d satisfy (C2) will be considered elsewhere.

4.1. A general result.

Consider an evolution equation that reads

$$v_t + Lv + \partial_x(F(v)) = 0, \quad v(t=0) = v_0 \tag{29}$$

where L is a linear unbounded operator with symbol A and F is a nonlinear quadratic operator.

Assuming that L generates a semi–group $S(t)$ on L^2_x that satisfies the *dichotomy assumption* (22), namely there exist $\beta > 0$ and $\delta > 0$ (δ can be $+\infty$) such that for any $t > 0$ and

$$\begin{aligned} &\bullet \text{ for } |\xi| \le \delta,\ \|S(t)\| \le C\exp(-\beta t\xi^2), \\ &\bullet \text{ for } |\xi| > \delta,\ \|S(t)\| \le C\exp(-\beta t). \end{aligned} \tag{30}$$

In addition,

$$\sup_{t\ge 0}(t^{1/4}\|S(t)v_0\|_{L^2_x}) \le C_1\|v_0\|_{L^1_x\cap L^2_x} = \overline{C}. \tag{31}$$

Assuming also the nonlinear term $F(v)$ satisfies

$$\sup_{|\xi|\le\delta}|\widehat{F}| + \left(\int_{|\xi|\ge\delta}|\xi|^2|\widehat{F}|^2 d\xi\right)^{1/2} \le C\|v\|^2_{L^2_x} \tag{32}$$

for any $t > 0$, where $\widehat{F} = \mathcal{F}(F(v))$.

Let us recall that a mild solution to (29) is a solution to the integral equation

$$v(t) = S(t)v_0 - \int_0^t S(t-s)\partial_x F(v(s))ds. \tag{33}$$

Under the above assumptions, we may construct a solution to (33) by performing a fixed point argument (see [10], [14], [6], [11]) on the space

$$E = \left\{u :\ \sup_{t>0}\{t^{1/4}\|u(t)\|_{L^2_x}\} < \infty\right\}$$

which is a Banach space of functions that are continuous in time with value in L^2 that are $O(t^{-1/4})$ when t goes to $+\infty$. If $\overline{C}$ is small enough, a fixed point argument to the Duhamel's form of the equation in a ball in E centered at origin would provide the solution.

Theorem 3. *For system (29) with assumptions (30)-(31)-(32), there exists a numerical constant C such that for any mild solution to (29) starting from v_0 with*

$$\|v_0\|_{L^1_x\cap L^2_x} \le C,$$

one finds

$$\|v(t)\|_{L^2_x} \le O(t^{-1/4}) \quad \textit{as} \quad t\to\infty. \tag{34}$$

Proof. To begin, we first control the low frequency part of the nonlinear term. Let

$$\begin{aligned}\widehat{N} &:= \mathcal{F}\left(\int_0^t S(t-s)\partial_x F(v(s))ds\right)\\ &= i\int_0^t e^{-(t-s)A}\xi\widehat{F}(v(s))ds.\end{aligned}$$

Using the first inequality in (30) in combination with (32), one obtains

$$\begin{aligned}\left(\int_{|\xi|\le\delta}|\widehat{N}|^2 d\xi\right)^{1/2} &\le C\int_0^t\left[\int_{|\xi|\le\delta}\|e^{-(t-s)A}\|^2\xi^2|\widehat{F}|^2 d\xi\right]^{1/2} ds\\ \le C\int_0^t&\left[\int_{\mathbb{R}}\xi^2 e^{-2\beta(t-s)\xi^2}d\xi\right]^{1/2}\left(\sup_{|\xi|\le\delta}|\widehat{F}|\right)ds\\ \le C\int_0^t&\frac{\|v(s)\|^2_{L^2_x}}{(t-s)^{3/4}}\,ds.\end{aligned}\tag{35}$$

We now control the high frequency part of the nonlinear term, using the second inequality in (30) in combination with (32)

$$\begin{aligned}\left(\int_{|\xi|\ge\delta}|\widehat{N}|^2 d\xi\right)^{1/2} &\le C\int_0^t e^{-\beta(t-s)}\left(\int_{|\xi|>\delta}\xi^2|\widehat{F}|^2 d\xi\right)^{1/2} ds\\ &\le C\int_0^t e^{-\beta(t-s)}\|v(s)\|^2_{L^2_x}ds.\end{aligned}\tag{36}$$

Introducing the norm

$$M(t) = \sup_{s\in[0,t]}(s^{1/4}\|v(s)\|_{L^2_x}),\tag{37}$$

and if v solves (33), then due to (35)–(36),

$$\begin{aligned}t^{1/4}\|v(t)\|_{L^2_x} &\le t^{1/4}\|S(t)v_0\|_{L^2_x} + t^{1/4}\|\hat{N}(t)\|_{L^2_\xi}\\ &\le t^{1/4}\|S(t)v_0\|_{L^2_x} + CM(t)^2\int_0^t\left[\frac{t^{1/4}}{s^{1/2}(t-s)^{3/4}} + \frac{t^{1/4}}{s^{1/2}}\exp(-\beta(t-s))\right]ds.\end{aligned}$$

By applying the change of variable $s=\tau t$ in the integration, one finds

$$\begin{aligned}&\int_0^t\left[\frac{t^{1/4}}{s^{1/2}(t-s)^{3/4}} + \frac{t^{1/4}}{s^{1/2}}\exp(-\beta(t-s))\right]ds\\ \le\,& C + \int_0^1\frac{t^{3/4}}{\tau^{\frac{1}{2}}}\exp(-\beta t(1-\tau))d\tau \le C.\end{aligned}$$

Therefore, using the property (31), the positive and nondecreasing function $M(t)$ satisfies $M(0)=0$ and for any $t\ge 0$,

$$C_0 M(t)^2 - M(t) + \overline{C} \ge 0,\tag{38}$$

where C_0 is a positive constant. Choosing $\overline{C}$ such that $C_0x^2 - x + \overline{C} = 0$ has two real roots $0<r_1<r_2$, namely choosing $\overline{C} < \frac{1}{4C_0}$, then (38) holds only if $M(t)$ is trapped in the interval $[0,r_1]$. Therefore when

$$\|v_0\|_{L^1_x\cap L^2_x} \le \frac{\overline{C}}{C_1} < \frac{1}{4C_0C_1},$$

$M(t)$ is bounded and (34) is valid. □

4.2. **Applications to weakly dispersive systems with complete dissipation or partial dissipation.** Since $b > 0$ and $d > 0$, order$(\sigma) \le 0$ and the corresponding linearized system is in the BBM-Burgers class. From Proposition 2 and Proposition 3, this corresponds to consider *complete dissipation*, or *partial dissipation* with order$(\sigma) \ge -1$.

Theorem 4. *Consider a weakly dispersive two–way wave model* ($b > 0$ *and* $d > 0$) *with either the complete dissipation or the partial dissipation with* $a < 0$ *or* $c < 0$. *Then, for small initial data,*

- *if* H *is of order 0,*
$$\|\eta(t)\|^2_{L^2_x} + \|u(t)\|^2_{L^2_x} \le O(t^{-1/2}); \tag{39}$$
- *if* H *is of order 1,*
$$\|\eta(t)\|^2_{L^2_x} + \|u(t)\|^2_{H^1_x} \le O(t^{-1/2}); \tag{40}$$
- *if* H *is of order* -1,
$$\|\eta(t)\|^2_{H^1_x} + \|u(t)\|^2_{L^2_x} \le O(t^{-1/2}); \tag{41}$$

as $t \to \infty$.

Proof. Note that the theorem is proved after (32) is validated and we will do that by discussing the cases according to the order of H.

- If H is of order 0 or 1. Introducing the change of variable
$$v = \mathcal{F}^{-1}(\widehat{\eta}, \widehat{H}\widehat{u}) = \mathcal{F}^{-1}(\widehat{\eta}, \widehat{w}),$$
the full nonlinear system
$$\begin{aligned} &\eta_t + u_x + au_{xxx} - b\eta_{xxt} + (\eta u)_x = \nu\eta_{xx},\\ &u_t + \eta_x + c\eta_{xxx} - du_{xxt} + uu_x = u_{xx}, \end{aligned} \tag{42}$$
transforms to
$$v_t + Lv = -\partial_x F(v),$$
where L has symbol $A = \begin{pmatrix} \nu\alpha & i\mathrm{sgn}(\omega_1)\xi\sigma \\ i\mathrm{sgn}(\omega_2)\xi\sigma & \varepsilon \end{pmatrix}$ and F reads
$$F(v) = \begin{pmatrix} (1 - b\partial_x^2)^{-1}\eta H^{-1}w \\ \frac{1}{2}H(1 - d\partial_x^2)^{-1}(H^{-1}w)^2 \end{pmatrix}.$$
To check (32), it is natural to separate the estimate into two parts.
  - *Low frequency* ($|\xi| \le \delta$) *estimate*: Since H^{-1}, which has order 0 or -1, is bounded on L^2_x, straightforward computations lead to
$$\begin{aligned}\|\widehat{\eta H^{-1}w}\|_{L^\infty_\xi} + \|\widehat{(H^{-1}w)^2}\|_{L^\infty_\xi} &\le C(\|\eta\|^2_{L^2_x} + \|H^{-1}w\|^2_{L^2_x})\\ &\le C(\|\eta\|^2_{L^2_x} + \|w\|^2_{L^2_x}).\end{aligned}$$
Because $(1 - b\partial_x^2)^{-1}$ and $H(1 - d\partial_x^2)^{-1}$ are bounded operators,
$$\sup_{|\xi|\le\delta} |\widehat{F}| \le C\|v\|^2_{L^2_x}.$$
  - *High frequency* ($|\xi| > \delta$) *estimate*: Since $\partial_x(1 - b\partial_x^2)^{-1}$ is a smoothing operator (it is of order -1) and H^{-1} is a bounded operator on L^2_x,
$$\begin{aligned}\|\partial_x(1 - b\partial_x^2)^{-1}(\eta H^{-1}w)\|_{L^2_x} &\le C\|\eta H^{-1}w\|_{H^{-1}_x}\\ &\le C\|\eta\|_{L^2_x}\|H^{-1}w\|_{L^2_x} \le C(\|\eta\|^2_{L^2_x} + \|w\|^2_{L^2_x}),\end{aligned}$$

where Lemma 2.2(ii) in [6],

$$\|fg\|_{H^{-1}} \le C\|f\|_{L^2}\|g\|_{L^2}$$

is used. Now, consider the term $\partial_x(H(1-d\partial_x^2)^{-1})((H^{-1}w)^2)$ in F. If H is of order 0, it can be bounded in the same way. If H is of order 1, then

$$\|\partial_x H(1-d\partial_x^2)^{-1}(H^{-1}w)^2\|_{L^2_x} \le C\|(H^{-1}w)^2\|_{L^2_x}$$
$$\le C\|H^{-1}w\|^2_{H^1_x} \le C\|w\|^2_{L^2_x},$$

where Lemma 2.2(iv) in [6]

$$\|fg\|_{L^0} \le C\|f\|_{H^1}\|g\|_{H^1}$$

is used.

Combining the lower frequency and higher frequency analysis, one sees (32) is valid and Theorem 3 yields the desired result.

- If H is of order -1, introducing the change of variable

$$v = \mathcal{F}^{-1}((H^{-1}\widehat{\eta}, \widehat{u})) \tag{43}$$

and setting $\widehat{\tau} = H^{-1}\widehat{\eta}$, the full nonlinear system (42) reads

$$v_t + Lv = -\partial_x F(v)$$

where L has symbol $A = \begin{pmatrix} \nu\alpha & i\mathrm{sgn}(\omega_1)\xi\sigma \\ i\mathrm{sgn}(\omega_2)\xi\sigma & \varepsilon \end{pmatrix}$ and

$$F(v) = \begin{pmatrix} H^{-1}(1-b\partial^2 x)^{-1}(uH\tau) \\ \frac{1}{2}(1-d\partial_x^2)^{-1}(u^2) \end{pmatrix}.$$

The proof is then very similar to the previous case and therefore omitted.

□

Corollary 3. *For the following two special cases, we have*

- *solutions to Bona–Smith system* ($a=0, b>0, c<0$ *and* $d>0$) *with complete or partial dissipation satisfy (41);*
- *solutions to BBM–BBM system with complete dissipation* ($a=c=0, \nu=1$) *satisfy (39).*

4.3. **Application to systems in the KdV-Burgers class with complete dissipation.** Using Proposition 2, this implies that order$(\sigma) \ge 1$, and then that $b=0$ and/or $d=0$.

First case: Consider the case where $b=d=0$. The analysis in [6] implies that $a=c=1/6$, so the system satisfies (C2) assumptions. Since the dichotomy assumption (30) was proved in Section 3 and the linearized system is in the *KdV-Burgers class*, Theorem 3 applies when (32) with $\delta=+\infty$ is verified.

Let us observe that $H=1$ and that the full nonlinear system reads

$$v_t + Lv = -\partial_x F(v)$$

where $v=(\eta,u)$, L has symbol $A = \begin{pmatrix} \xi^2 & i\mathrm{sgn}(\omega_1)\xi\sigma \\ i\mathrm{sgn}(\omega_2)\xi\sigma & \xi^2 \end{pmatrix}$ and

$$F(v) = \begin{pmatrix} \eta u \\ \frac{u^2}{2} \end{pmatrix}.$$

Since

$$\|F(v)\|_{L^1_x} \le C(\|\eta\|^2_{L^2_x} + \|u\|^2_{L^2_x}),$$

one finds (32) with $\delta = \infty$.

Second case: Consider the case where $b > 0$ and $d = 0$. Due to (C0), $c \geq 0$. Since order(σ) ≥ 1, the system must have $a = c > 0$ and therefore order(H) $= -1$. Using the change of variable (43), the system reads as (4.3) with the following nonlinearity

$$F(v) = \begin{pmatrix} H^{-1}(1 - b\partial^2 x)^{-1}(uH\tau) \\ \frac{u^2}{2} \end{pmatrix}.$$

It is then straightforward to show (32) with $\delta = \infty$ is valid.

Third case: Consider the case where $b = 0$ and $d > 0$. Since order(σ) $= 1$, a and c can not vanish and order(H) $= 1$. In this case, with the change of variable (4.2), the system reads as (4.3) with F being

$$F(v) = \begin{pmatrix} \eta H^{-1} w \\ \frac{1}{2} H(1 - d\partial_x^2)^{-1}(H^{-1}w)^2 \end{pmatrix}.$$

One can again easily prove (32) with $\delta = \infty$.

The following is the outcome of the analysis.

Theorem 5. *For a system in the KdV-Burgers class with complete dissipation and for small initial data,*

- *if H is of order 0,*
$$\|\eta(t)\|_{L_x^2}^2 + \|u(t)\|_{L_x^2}^2 \leq O(t^{-1/2}); \tag{44}$$
- *if H is of order 1,*
$$\|\eta(t)\|_{L_x^2}^2 + \|u(t)\|_{H_x^1}^2 \leq O(t^{-1/2}); \tag{45}$$
- *if H is of order -1,*
$$\|\eta(t)\|_{H_x^1}^2 + \|u(t)\|_{L_x^2}^2 \leq O(t^{-1/2}); \tag{46}$$

as $t \to \infty$.

Remark 7. This result can be predicted for KdV-KdV system ($b = d = 0$) since if we introduce the new variables $\eta = w^1 + w^2$ and $u = w^1 - w^2$, then (42) reads as a system of two linear KdV–Burgers systems (weakly) coupled through nonlinear terms. See Section 2.3 in [6].

5. The L_x^∞–decay rate.

It is first observed that for the cases of weakly dispersive wave equations and KdV-KdV system, the nonlinear terms satisfy

$$\sup_{|\xi| \leq \delta} |\xi \widehat{F}| + \left(\int_{|\xi| > \delta} |\xi|^4 |\widehat{F}|^2 d\xi \right)^{1/2} \leq C \|v\|_{L_x^2} \|v_x\|_{L_x^2}. \tag{47}$$

We will estimate the decay rate of $\partial_x v(t)$ in L_x^2 when v solves (33). To begin, we differentiate (33) with respect to x and treat the nonlinear term of the resulting equation with a procedure similar to the one in the proof of Theorem 3, but using (47) instead of (32).

It is useful to first note that for low frequencies,

$$\left(\int_{|\xi| \leq \delta} \xi^2 |\widehat{N}|^2 d\xi \right)^{1/2} \leq C \int_0^t \frac{\|v(s)\|_{L_x^2} \|v_x(s)\|_{L_x^2}}{(t-s)^{3/4}} ds$$

and for high frequencies

$$\left(\int_{|\xi|\geq\delta} \xi^2 |\widehat{N}|^2 d\xi\right)^{1/2} \leq C \int_0^t e^{-\beta(t-s)} \|v\|_{L^2_x} \|v_x\|_{L^2_x} ds.$$

We now consider the linear part. For v_0 in $H^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the linear term can be estimated by splitting the region of integration into low frequencies and high frequencies. By using (30), it obtains

$$\begin{aligned}\|\partial_x S(t) v_0\|^2_{L^2_x} &\leq C\left[\left(\int \xi^2 e^{-\beta\xi^2 t} d\xi\right) \|v_0\|^2_{L^2_x} + e^{-2\beta t} \|\partial_x v_0\|^2_{L^2_x}\right] \\ &\leq \frac{C}{t^{3/2}} \|v_0\|^2_{L^2} + C e^{-2\beta t} \|v_0\|^2_{H^1}.\end{aligned}$$

Therefore, the linear part behaves like $O(t^{-3/4})$ as $t \to \infty$. Since $\|v\|_{L^2}$ is $O(t^{-1/4})$ as $t \to \infty$,

$$\begin{aligned}\|\partial_x v(t)\|_{L^2_x} \leq\ & C(v_0) t^{-3/4} + C \sup_{s\in[0,t]} \left(s^{1/4} \|v(s)\|_{L^2}\right) \\ &\times \int_0^t \frac{\|v_x(s)\|_{L^2_x}}{(t-s)^{3/4}} \frac{1}{s^{1/4}} + \frac{e^{-\beta(t-s)}}{s^{1/4}} \|v_x(s)\|_{L^2_x} ds.\end{aligned}$$

Simple calculations show

$$\int_0^t \frac{ds}{s^{1/4}(t-s)^{3/4}} + \int_0^t \frac{e^{-\beta(t-s)}}{s^{1/4}} \, ds \leq C.$$

Therefore

$$\begin{aligned}\|\partial_x v(t)\|_{L^2_x} &\leq C(v_0) t^{-3/4} + C \sup_{s\in[0,t]} \|\partial_x v(s)\|_{L^2_x} \sup_{s\in[0,t]} (s^{1/4} \|v(s)\|_{L^2}) \\ &\leq C(v_0) t^{-3/4} + C_2 M(t) \sup_{s\in[0,t]} \|\partial_x v(s)\|_{L^2_x}.\end{aligned} \tag{48}$$

Since $M(t)$, which is defined in (37), is bounded by the first root r_1 of

$$C_0 x^2 - x + \overline{C} = 0,$$

and $r_1 \sim \overline{C}$ as $\overline{C} \to 0$ (since $\frac{1}{C_0}(1 - (1 - 4C_0\overline{C})^{1/2}) \sim 2\overline{C}$ as $\overline{C} \to 0$), $M(t) \leq 2\overline{C}$ when $\overline{C}$ is small enough. Hence by choosing $\overline{C}$ satisfying

$$2C_2\overline{C} \leq \frac{1}{2}, \tag{49}$$

one finds from (48)

$$\|\partial_x v(t)\|_{L^2_x} \leq 2C(v_0) t^{-3/4} \tag{50}$$

and we obtain the following theorem.

Theorem 6. *For system (29) with assumptions (30)-(31)-(32), assume v_0 is in $H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\|v_0\|_{L^1\cap L^2}$ is small enough. Then*

$$\|v\|_{L^\infty_x} \leq O(t^{-1/2})$$

as $t \to \infty$.

Proof. Using (50) and (34) together with

$$\|v\|_{L^\infty_x} \leq \|v\|^{1/2}_{L^2_x} \|\partial_x v\|^{1/2}_{L^2_x}$$

yields the desired result. □

6. **Numerical result.** Numerical simulations are performed on several systems and results on BBM-BBM and Bona-Smith systems with complete or partial dissipations are reported here. The results show not only that the theoretical results on the decay rates are sharp, but also the constants involved are reasonably sized.

In these numerical computations, the initial data are taken to be

$$\eta_0 = \operatorname{sech}^2(\frac{\sqrt{2}}{2}(x - x_0)),$$
$$u_0 = \eta_0 - \eta_0^2/4,$$

where x_0 is in the spatial domain $[0, L]$, where L is taken to be large enough so the solution near the boundary is smaller than the machine roundoff error during the whole computation. The spectral method is used on the spatial domain $[0, L]$ and the leap-frog algorithm is used on the time advancing. The decay rate r and the constant C in

$$\|v\| \sim Ct^{-r}, \text{ as } t \to \infty$$

is calculated by first computing

$$r(t_n) := -\frac{\log \frac{\|v\|(t_n)}{\|v\|(t_{n-1})}}{\log \frac{t_n}{t_{n-1}}}.$$

The interval of time integration is chosen large enough so $r(t_n)$ is approaching to a constant and the value r is obtained by averaging the last five data. The constant C is then computed by averaging the last five $\|v\|(t_n)t_n^r$.

In the computations reported below, $L = 320$, $dx = 0.1$ and $dt = 0.05$, where dx and dt are the meshsizes in space and in time respectively.

BBM-BBM system ($a = c = 0, b = d = 1/6$) with complete dissipation. It is shown in Theorem 4 and 6 that for *small data*,

$$\|v\|_{L^2} \le C_1 t^{-1/4} \quad \text{and} \quad \|v\|_{L^\infty} \le C_2 t^{-1/2}.$$

The numerical computation is performed for time interval $[0, 50]$, and the result shows

$$\|v\|_{L^2} \sim 1.4232t^{-0.2470} \quad \text{and} \quad \|v\|_{L^\infty} \sim 1.4989t^{-0.4963}.$$

Therefore, it is clear that the theoretical result is sharp and the constants involved are not large. Moreover, it seems that the small data requirement is not necessary and might be removed if, for example, other methods were employed.

Bona-Smith system ($a = 0, b = -c = d = 1/3$) with complete and partial dissipation. This case is again covered by Theorem 4 and 6. By direct computation, we obtain for complete dissipation,

$$\|v\|_{L^2} \sim 1.4015t^{-0.2477}, \quad \|v\|_{L^\infty} \sim 1.4466t^{-0.4998},$$

and for partial dissipation

$$\|v\|_{L^2} \sim 0.6676t^{-0.2519}, \quad \|v\|_{L^\infty} \sim 0.6595t^{-0.5105}.$$

Acknowledgments. Part of this work was performed when the second author was enjoying the hospitality of the mathematics department in Purdue University.

REFERENCES

[1] C. J. Amick, *Regularity and uniqueness of solutions to the Boussinesq system of equations*, J. Diff. Eq., **54** (1984), 231–247.

[2] C. J. Amick, J. L. Bona, and M. E. Schonbek, *Decay of solutions of some nonlinear wave equations*, J. Differential Equations, **81** (1989), 1–49.

[3] J. T. Beale, T. Y. Hou, and J. Lowengrub, *Convergence of boundary integral methods for water waves*, SIAM J. Numer. Anal, **33** (1996), 1797–1843.

[4] J. L. Bona and M. Chen, *A Boussinesq system for two-way propagation of nonlinear dispersive waves*, Physica D, **116** (1998), 191–224.

[5] J. L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media I: Derivation and the linear theory*, J. Nonlinear Sci., **12** (2002), 283–318.

[6] ———, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media II: Nonlinear theory*, Nonlinearity, **17** (2004), 925 – 952.

[7] J. L. Bona, W. G. Pritchard, and L. R. Scott, *An evaluation of a model equation for water waves*, Philos. Trans. Royal Soc. London, Ser. A, **302** (1981), 457–510.

[8] J. L. Bona and R. Smith, *A model for the two-way propagation of water waves in a channel*, Math. Proc. Cambridge Philos. Soc., **79** (1976), 167–182.

[9] J. V. Boussinesq, *Théorie de l'intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire*, C. R. Acad. Sci. Paris, **72** (1871), 755–759.

[10] T. Cazenave and A. Haraux, "An introduction to Semilinear Evolution Equations," vol. 13 of Oxford Lecture Series in Mathematics and its Applications, The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors.

[11] T. Cazenave and F. B. Weissler, *Scattering theory and self-similar solutions for the nonlinear Schrödinger equation*, SIAM J. Math. Anal., **31** (2000), 625–650.

[12] C.C.Mei and L.F.Liu, *The damping of surface gravity waves in a bounded liquid*, J. Fluid Mech., **59** (1973), 239–256.

[13] D. T. Finkbeiner, II, "Introduction to Matrices and Linear Transformations," Second edition, W. H. Freeman and Co., San Francisco, Calif., 1966.

[14] G. Karch, *Large-time behaviour of solutions to non-linear wave equations: higher-order asymptotics*, Math. Methods Appl. Sci., **22** (1999), 1671–1697.

[15] G. Karch, *Long-time asymptotics of solutions to some nonlinear wave equations*, in "Evolution Equations: Existence, Regularity and Singularities" (Warsaw, 1998), vol. 52 of Banach Center Publ., Polish Acad. Sci., Warsaw, 2000, 133–146.

[16] J. J. Mahony and W. G. Pritchard, *Wave reflexion from beaches*, J. Fluid Mech., **101** (1980), 809–832.

[17] J. W. Miles, *Surface-wave damping in a closed basin*, Proc. Royal Soc. London A, **297** (1967), 459–475.

[18] D. H. Peregrine, *Equations for water waves and the approximation behind them*, in "Waves on Beaches and Resulting Sediment Transport," proceedings of an advanced seminar conducted by the Mathematics Research Center, Univ. Wisconsin, Academic Press: New York, 1972, 95–121.

[19] S. Rajopadhye, M. Schonbek, and M. Wiegner, *Asymptotic decay for a generalized Boussinesq system*, J. Dynam. Differential Equations, **11** (1999), 595–623.

[20] M. E. Schonbek, *Existence of solutions for the Boussinesq system of equations*, J. Diff. Eq., **42** (1981), 325–352.

[21] C. J. Tang, V. C. Patel, and L. Landweber, *Viscous effects on propagation and reflection of solitary waves in shallow channels*, J. Comput. Phys., **88** (1990), 86–113.

[22] M. H. Teng and T. Y. Wu, *A bidirectional long-wave model*, Methods Appl. Anal., **1** (1994), 108–117.

[23] G. B. Whitham, "Linear and Nonlinear Waves," John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication.

Received October 2005; revised July 2006.

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