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LONG-TIME ASYMPTOTIC BEHAVIOR OF TWO-DIMENSIONAL DISSIPATIVE BOUSSINESQ SYSTEMS

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Abstract. In this article, we consider the two-dimensional dissipative Boussinesq systems which model surface waves in three space dimensions. The long time asymptotics of the solutions for a large class of such systems are obtained rigorously for small initial data.

1. Introduction.

1.1. **Damped Boussinesq systems.** There are three important factors associated with wave propagation: dispersion, dissipation and nonlinearity. In many real physical situations, it is observed that the effect of damping (which is always present in reality) is at least comparable to the effects of dispersion and nonlinearity [5]. In such cases, a damping term (or terms) should be included in the equation. Following the pioneering work of Kakutani and Matsuuchi ([13]), Dias-Dutykh [10], Liu-Orfila [16] and H. Le Meur [15] have derived dissipation terms, which involve local and non-local terms, for Boussinesq systems under the small amplitude and long wavelength assumptions from Navier-Stokes equations.

In this article, attention is given to the two-dimensional Boussinesq systems, derived in [2], for three-dimensional water waves supplemented with various local dissipative terms. Similar to the corresponding one-dimensional dissipative Boussinesq systems, studied in [7], these systems are evolution partial differential equations involving two unknown functions, the vertical deviation of the water surface with respect to its equilibrium, $\eta(\mathbf{x}, t)$, and the horizontal velocity of the fluid, which is a two dimensional vector field, at certain depth of the water, $\mathbf{u}(\mathbf{x}, t)$. We address here two separate cases, one is when the dissipation acts both on η and $\mathbf{u}$ (strong dissipation) and the other is when the dissipation acts only on $\mathbf{u}$ (weak dissipation). The study of nonlocal dissipative terms will be carried out in a separate paper.

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1.2. **A class of dissipative Boussinesq system.** Without dissipative mechanism, the four parameter family of Boussinesq systems derived in [2, 3, 8] reads

$$\begin{aligned} &\eta_t + \nabla \cdot \mathbf{u} + \nabla \cdot \eta \mathbf{u} + a\Delta \nabla \cdot \mathbf{u} - b\Delta \eta_t = 0,\\ &\mathbf{u}_t + \nabla \eta + \frac{1}{2}\nabla |\mathbf{u}|^2 + c\Delta \nabla \eta - d\Delta \mathbf{u}_t = \mathbf{0}, \end{aligned} \tag{1.1}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the horizontal velocity of the fluid, a mapping from $\mathbb{R}^2_{\mathbf{x}} \times \mathbb{R}_t$ into $\mathbb{R}^2$, and $\eta(\mathbf{x}, t)$ is a scalar field from $\mathbb{R}^2_{\mathbf{x}} \times \mathbb{R}_t$ into $\mathbb{R}$. For the systems to model water wave with no surface tension, the constants a, b, c, d must satisfy the consistent conditions (see [2] for detail)

$$a + b + c + d = \frac{1}{3} \quad \text{and} \quad c + d \geq 0. \tag{C0}$$

Furthermore, in order for the systems to be wellposed for the initial value problems, it is relevant to assume either

$$b \geq 0, \quad d \geq 0, \quad a \leq 0, \quad c \leq 0, \tag{C1}$$

or

$$b \geq 0, \quad d \geq 0, \quad a = c > 0, \tag{C2}$$

according to the results presented in [1] and [2]. Therefore, our investigation is going to be restricted to the cases where a, b, c, d satisfy (C0)-(C1) or (C0)-(C2).

We now introduce the dissipative mechanisms interested in this article. The systems under consideration are of the form

$$\begin{aligned} &\eta_t + \nabla \cdot \mathbf{u} + \nabla \cdot \eta \mathbf{u} + a\Delta \nabla \cdot \mathbf{u} - b\Delta \eta_t = \nu \Delta \eta,\\ &\mathbf{u}_t + \nabla \eta + \frac{1}{2}\nabla |\mathbf{u}|^2 + c\Delta \nabla \eta - d\Delta \mathbf{u}_t = \Delta \mathbf{u}, \end{aligned} \tag{1.2}$$

where $\nu = 1$ or $\nu = 0$. The case with $\nu = 1$ will be called *complete dissipation* and the case with $\nu = 0$ will be called *partial dissipation.*

The following decay results

$$\|\mathbf{v}(\mathbf{x}, t)\|_{L^2_{\mathbf{x}}} \leq C(1 + t)^{-1/2} \quad \text{and} \quad \|\mathbf{v}(\mathbf{x}, t)\|_{L^\infty_{\mathbf{x}}} \leq C(1 + t)^{-1}, \tag{1.3}$$

are going to be proved rigorously for some of the systems in (1.2) where $\mathbf{v}$ is related to $(\eta, \mathbf{u})$ or one of its derivative, up to a suitable change of variables. These decay rate are faster those that in one-dimensional case, which are expected since the solution of the corresponding heat equation decays faster in two-dimensional case.

The proof will follow the method presented in our previous article [7] where the corresponding one-dimensional systems were investigated. We begin with analyzing the linearized system and then extend the results to the nonlinear system for small initial data. It is worth to note that if we use the notations in [12] which classify dissipative systems accordingly to the decay properties, our two-dimensional systems belong to the class of weak nonlinearities (this classification was also introduced in [9]; see also [4]), namely the same decay rate for solutions to systems with or without nonlinear terms. It is worth to mention that the general methods presented in [12] do not work here straightforwardly (see Remark 2.5 for details) and our proof will follow the guidelines in [7].

2. **Linear system.**

2.1. **Preliminary computations.** Following [2] and [7], we introduce the Fourier multipliers

$$\omega_1 = \frac{1-a|\boldsymbol{\xi}|^2}{1+b|\boldsymbol{\xi}|^2}, \quad \omega_2 = \frac{1-c|\boldsymbol{\xi}|^2}{1+d|\boldsymbol{\xi}|^2},$$
$$\alpha = \frac{|\boldsymbol{\xi}|^2}{1+b|\boldsymbol{\xi}|^2} \quad \text{and} \quad \varepsilon = \frac{|\boldsymbol{\xi}|^2}{1+d|\boldsymbol{\xi}|^2},$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2)$ is the Fourier variable associated to $\mathbf{x}$ and $|\boldsymbol{\xi}|$ is the Euclidean norm of $\boldsymbol{\xi}$.

Since a, b, c, d satisfy (C1) or (C2), $\omega_1\omega_2$ is non-negative and we denote

$$\widehat{H} = \left(\frac{\omega_1}{\omega_2}\right)^{1/2} \quad \text{and} \quad \sigma = (\omega_1\omega_2)^{1/2},$$

with the conventional notation $\frac{0}{0} = 1$.

Remark 2.1. *For a system satisfying (C2) assumption, ω_1 and ω_2 do change signs, but $\omega_1\omega_2 \geq 0$.*

By recalling the definition

$$\text{order}(H) = \{a\} + \{d\} - \{b\} - \{c\}, \text{ where } \{a\} = 1 \text{ iff } a \neq 0 \text{ and } \{0\} = 0,$$

it turns out that H behaves like a Bessel potential of order $m = \text{order}(H)$, i.e like $(I_d - \Delta)^{\frac{m}{2}}$. In the sequel we shall use without notice that H is an isomorphism from $W^{m,p}(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$, if $1 < p < +\infty$ (see Theorem 5.3.3 in [17]).

For the linearized system of (1.2), it is natural to study it in Fourier variables which reads

$$\begin{aligned} (1+b|\boldsymbol{\xi}|^2)\widehat{\eta}_t + \nu|\boldsymbol{\xi}|^2\widehat{\eta} + i(1-a|\boldsymbol{\xi}|^2)\boldsymbol{\xi}\cdot\widehat{\mathbf{u}} &= 0, \\ (1+d|\boldsymbol{\xi}|^2)\widehat{\mathbf{u}}_t + |\boldsymbol{\xi}|^2\widehat{\mathbf{u}} + i(1-c|\boldsymbol{\xi}|^2)\widehat{\eta}\boldsymbol{\xi} &= \mathbf{0}. \end{aligned} \tag{2.1}$$

2.2. **Helmholtz decomposition.** A well-known fact about vector fields in $\mathbb{R}^2$ is that they split into a potential part and a rotating part. Let $\boldsymbol{\xi}_\perp = (-\xi_2, \xi_1)$, one has, for $\boldsymbol{\xi} \neq \mathbf{0}$

$$\widehat{\mathbf{u}} = \widehat{q}\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} + \widehat{\psi}\frac{\boldsymbol{\xi}_\perp}{|\boldsymbol{\xi}|}, \tag{2.2}$$

where $\widehat{q}$ and $\widehat{\psi}$ are scalar functions. Using this new set of variables, the system (2.1) reads

$$\begin{aligned} (1+b|\boldsymbol{\xi}|^2)\widehat{\eta}_t + \nu|\boldsymbol{\xi}|^2\widehat{\eta} + i(1-a|\boldsymbol{\xi}|^2)|\boldsymbol{\xi}|\widehat{q} &= 0, \\ (1+d|\boldsymbol{\xi}|^2)\widehat{q}_t + |\boldsymbol{\xi}|^2\widehat{q} + i(1-c|\boldsymbol{\xi}|^2)|\boldsymbol{\xi}|\ \widehat{\eta} &= 0, \\ (1+d|\boldsymbol{\xi}|^2)\widehat{\psi}_t + |\boldsymbol{\xi}|^2\widehat{\psi} &= 0. \end{aligned} \tag{2.3}$$

Therefore, for the linear system, the dynamic decouples into a “rotating” wave ψ that decays to 0 and a problem similar to the one in one-dimensional case which has been studied (for the spectral analysis) in [7]. The coupling between ψ and other variables will show up when the nonlinear terms are included.

Following the notations of [7] and [1], the balanced system that is equivalent to (2.3) reads

$$\begin{aligned} \widehat{\eta}_t + \nu\alpha\widehat{\eta} + i\text{sgn}(\omega_1)\sigma|\boldsymbol{\xi}|\widehat{Hq} &= 0, \\ \widehat{Hq}_t + \varepsilon\widehat{Hq} + i\text{sgn}(\omega_1)\sigma|\boldsymbol{\xi}|\widehat{\eta} &= 0, \\ \widehat{H\psi}_t + \varepsilon\widehat{H\psi} &= 0, \end{aligned} \tag{2.4}$$

and we now study the decay of the rotating part ψ and the two coupled equations separately. The generic constant C is used which may change its value in each appearance.

2.3. **Decay of the rotating wave.**

Lemma 2.2. *Assume that $\psi_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, then*

$$||\psi(t)||_{L^2_{\mathbf{x}}} \leq C(1+t)^{-1/2}.$$

Proof. For $d \geq 0$,

$$\begin{aligned}||\psi(t)||^2_{L^2_{\mathbf{x}}} &= \int_{\mathbb{R}^2} |\widehat{\psi}_0|^2 e^{-2t\frac{|\boldsymbol{\xi}|^2}{1+d|\boldsymbol{\xi}|^2}} d\boldsymbol{\xi} \\ &= \int_{|\boldsymbol{\xi}|\leq 1} |\widehat{\psi}_0|^2 e^{-2t\frac{|\boldsymbol{\xi}|^2}{1+d|\boldsymbol{\xi}|^2}} d\boldsymbol{\xi} + \int_{|\boldsymbol{\xi}|>1} |\widehat{\psi}_0|^2 e^{-2t\frac{|\boldsymbol{\xi}|^2}{1+d|\boldsymbol{\xi}|^2}} d\boldsymbol{\xi} \\ &\leq ||\widehat{\psi}_0||^2_{L^\infty_{\boldsymbol{\xi}}} \int_{\mathbb{R}^2} e^{-2t\frac{|\boldsymbol{\xi}|^2}{1+d}} d\boldsymbol{\xi} + ||\widehat{\psi}_0||^2_{L^2_{\boldsymbol{\xi}}} e^{-\beta t} \\ &\leq C\|\psi_0\|^2_{L^1_{\mathbf{x}}} t^{-1} + C||\psi_0||^2_{L^2_{\mathbf{x}}} e^{-\beta t} \leq C(\psi_0)t^{-1},\end{aligned} \tag{2.5}$$

where $\beta = \frac{2}{1+d}$, which yields the conclusion. □

2.4. **Decay of η and q.**

Let the matrix

$$A(\boldsymbol{\xi}) = \begin{pmatrix} \nu\alpha & i\,\mathrm{sgn}(\omega_1)|\boldsymbol{\xi}|\sigma \\ i\,\mathrm{sgn}(\omega_1)|\boldsymbol{\xi}|\sigma & \varepsilon \end{pmatrix},$$

the decay rate of the linear operator $||e^{-tA}||$ as a function of $|\boldsymbol{\xi}|$ is studied in [7]. The relevant results (Propositions 1-4) are recalled here, where

$$\mathrm{order}(\sigma) = \{a\} + \{c\} - \{b\} - \{d\}.$$

Lemma 2.3. *If order(σ) ≥ 1, and either ($\nu = 1$) or ($\nu = 0$ and $d = 0$), then there exist constants $C > 0$ and $\beta > 0$ such that*

$$||e^{-tA}|| \leq Ce^{-\beta t|\boldsymbol{\xi}|^2}.$$

Hence the system behaves like KdV-Burgers equation for t large.

Lemma 2.4. *If $b, d > 0$ (the corresponding systems were called weakly dispersive systems in [2]) and $\nu = 1$, then there exist constants $C > 0$ and $\beta > 0$ such that*

$$||e^{-tA}|| \leq Ce^{-\beta t\frac{|\boldsymbol{\xi}|^2}{1+|\boldsymbol{\xi}|^2}}.$$

Hence the system behaves like BBM-Burgers equation for t large.

Remark 2.5. *For complete results concerning this linear system according to the parameters ν, a, b, c, d we refer to [7]. It is worth to remark that some of our results here overlap with the results for systems in [12] and some of them do not. The results in [12] are proved under the assumptions that the matrix $A(\boldsymbol{\xi})$, where A is the linear operator in (2.1) when it is written in the form of $\mathbf{u}_t + A\mathbf{u} = 0$, is diagonalisable, and that the norms of the eigenprojectors $P_j(\boldsymbol{\xi})$ and their first derivatives are bounded as functions of $|\boldsymbol{\xi}|$. For our systems there are a broad class of parameters a, b, c, d such that the matrix $A(\boldsymbol{\xi})$ has a double eigenvalue for some values $\boldsymbol{\xi}_0 \neq \mathbf{0}$. Since $A(\boldsymbol{\xi})$ is not a scalar matrix, when $\boldsymbol{\xi}$ converges towards $\boldsymbol{\xi}_0$ the* *norm of the eigenprojectors blows up. This can be seen more clearly using a simple case with two equations.*

Let $A(\xi)$ be a 2×2 matrix that depends on ξ. Assume that $A(\xi)$ has two different eigenvalues except at $\xi=\xi_0$ and assume that $A(\xi_0)$ is not a scalar matrix. For $\xi\neq\xi_0$ denote the eigenvectors by e_1 and e_2, where (e_1,e_2) is a basis, $\|e_1\|=\|e_2\|=1$ and $e_1.e_2=\cos\theta$, where θ is the angle between e_1 and e_2. Then the eigenprojector P_1 is

$$P_1(xe_1+ye_2)=xe_1,$$

and

$$\|P_1\|^2=\sup_{(x,y)\neq(0,0)}\frac{|x|^2}{(x^2+y^2+2xy\cos\theta)}=\Big(\inf_{t=y/x,t\neq 0}(1+t^2+2t\cos\theta)\Big)^{-1}=\frac{1}{\sin^2\theta}.$$

Therefore, if $\xi\to\xi_0$, then both e_1 and e_2 converge to the unique eigenvector of norm 1 of $A(\xi_0)$ (up to a multiplication by -1), and $\theta\to 0$.

3. **Decay rate for the nonlinear system.** We now consider the full nonlinear system which reads

$$\begin{pmatrix}\eta\\ \widehat{H}q\\ \widehat{H}\psi\end{pmatrix}_t+\begin{pmatrix}A(\boldsymbol{\xi}) & \mathbf{0}\\ \mathbf{0} & \varepsilon\end{pmatrix}\begin{pmatrix}\eta\\ \widehat{H}q\\ \widehat{H}\psi\end{pmatrix}=-|\boldsymbol{\xi}|\begin{pmatrix}\frac{1}{1+b|\boldsymbol{\xi}|^2}\frac{i\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\eta\mathbf{u}}\\ \frac{\widehat{H}}{1+d|\boldsymbol{\xi}|^2}\frac{i}{2}\widehat{|\mathbf{u}|^2}\\ 0\end{pmatrix}. \tag{3.1}$$

3.1. **A general theorem.** Consider a nonlinear evolution equation that reads

$$\mathbf{v}_t+L\mathbf{v}=F(\mathbf{v}) \tag{3.2}$$

where $\mathbf{v}(\mathbf{x},t)$ maps $\mathbb{R}^2\times\mathbb{R}$ into $\mathbb{R}^n$, L is a linear unbounded operator and $F(\mathbf{v})$ is a bilinear operator that might involves some derivatives of $\mathbf{v}$ (actually $F(\mathbf{v})$ is a convection term; see assumption (3.4) below) and has the structure as in (3.1), namely the last element is zero. Let $S(t)=e^{-Lt}$ which is the linear semi-group and denote its symbol as $\widehat{S}(t,\boldsymbol{\xi})$, namely $\mathbf{y}=S(t)\mathbf{y}_0$ if and only if $\widehat{\mathbf{y}}=\widehat{S}(t,\boldsymbol{\xi})\widehat{\mathbf{y}}_0$ in the Fourier space. In this section, $\widehat{S}(t,\boldsymbol{\xi})$ is in the form of

$$\widehat{S}(t,\boldsymbol{\xi})=\begin{pmatrix}e^{-A(\boldsymbol{\xi})t} & \mathbf{0}\\ \mathbf{0} & e^{-\varepsilon t}\end{pmatrix}.$$

Then the following theorem is valid.

Theorem 3.1. *Assume that there exist $\delta>0$ ($\delta=+\infty$ is allowed) and $\beta>0$ such that*

$$\|e^{-A(\boldsymbol{\xi})t}\|\le\begin{cases}Ce^{-\beta t|\boldsymbol{\xi}|^2} & \text{if}\quad |\boldsymbol{\xi}|<\delta,\\ Ce^{-\beta t} & \text{if}\quad |\boldsymbol{\xi}|\ge\delta.\end{cases} \tag{3.3}$$

Assume that the nonlinear operator satisfies

$$|\widehat{F(\mathbf{v})}|\le C|\boldsymbol{\xi}|\ |\widehat{B(\mathbf{v})}| \tag{3.4}$$

where B is a bilinear operator that satisfies, if Q_δ is the projector onto the large-frequencies $\{|\boldsymbol{\xi}|>\delta\}$ and $P_\delta=I_d-Q_\delta$ the complementary projector,

$$\begin{aligned}||P_\delta B(\mathbf{v})||_{L^2_{\mathbf{x}}}+||\,|\boldsymbol{\xi}|\widehat{B(\mathbf{v})}||_{L^{\frac43}_{\{|\boldsymbol{\xi}|\ge\delta\}}}&\le C||\mathbf{v}||^2_{L^4_{\mathbf{x}}},\\ ||P_\delta B(\mathbf{v})||_{L^{\frac43}_{\mathbf{x}}}+||Q_\delta B(\mathbf{v})||_{H^1_{\mathbf{x}}}&\le C||\mathbf{v}||_{L^2_{\mathbf{x}}}||\mathbf{v}||_{L^4_{\mathbf{x}}},\end{aligned} \tag{3.5}$$

then for initial data $\mathbf{v}(0)$ *in* $L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ *with small* $L^2(\mathbb{R}^2)$ *norm, and such that* $\widehat{\mathbf{v}}_0$ *is in* $L^1_{\boldsymbol{\xi}}(\mathbb{R}^2)$ *with small norm if* $\delta\neq+\infty$*, there exists a solution of (3.2) that satisfies the decay property*

$$||\mathbf{v}(t)||_{L^2(\mathbb{R}^2)}\leq C(1+t)^{-\frac{1}{2}}. \tag{3.6}$$

Remark 3.2. *For the assumption in (3.5), it would be more natural to have all the assumptions in physical variable* $\mathbf{x}$*. Unfortunately we had to assume a bound for a quantity in* $L^{\frac{4}{3}}_{\boldsymbol{\xi}}$*-norm, that is stronger than the "natural assumption" using* $L^4_{\mathbf{x}}$*-norm since*

$$||Q_\delta B(\mathbf{v})||_{W^{1,4}_{\mathbf{x}}}\leq C||\,|\boldsymbol{\xi}|\widehat{B(\mathbf{v})}||_{L^{\frac{4}{3}}_{\{|\boldsymbol{\xi}|\geq\delta\}}}$$

Remark 3.3. *Our method is indebted to the famous so called Kato's method for solving initial-value-problem of semi-linear partial differential equations in the "critical case". Following Kato's method (see* [11], [14]*), we first construct a solution using a fixed point argument in the class of functions* $C_b([0,+\infty),L^2(\mathbb{R}^2))\cap C((0,+\infty),L^4(\mathbb{R}^2))$*, and then prove the decay estimate.*

Proof. The proof is divided into two steps. The first step is devoted to prove that if $\mathbf{v}_0$ is small enough in $L^2(\mathbb{R}^2)$, and $\widehat{\mathbf{v}}_0$ is in $L^1(\mathbb{R}^2_{\boldsymbol{\xi}})$ and with small norm if $\delta\neq+\infty$, then there exists a unique solution of (3.2) in a small ball of the Banach space E whose norm is $||\mathbf{v}||_E=\sup_{t>0}(t^{\frac{1}{4}}||\mathbf{v}(t)||_{L^4(\mathbb{R}^2)})$.

We first write (3.2) in its Duhamel's form that reads

$$\mathbf{v}(t)=S(t)\mathbf{v}_0+\int_0^t S(t-s)F(\mathbf{v}(s))ds. \tag{3.7}$$

The analysis of the linear operator starts by recalling that the inverse Fourier transform $\mathcal{F}^{-1}$ is a bounded mapping from $L^{\frac{4}{3}}_{\boldsymbol{\xi}}$ into $L^4_{\mathbf{x}}$ (this is valid by noticing first that the inverse Fourier transform maps $L^1_{\boldsymbol{\xi}}\cap L^2_{\boldsymbol{\xi}}$ into $L^\infty_{\mathbf{x}}\cap L^2_{\mathbf{x}}$ and then applying the Riesz-Thorin interpolation theorem), and denoting $\mathbf{v}_0=(\mathbf{u}_0,w_0)^T$,

$$||S(t)\mathbf{v}_0||_{L^4_{\mathbf{x}}}\leq C||e^{-A(\boldsymbol{\xi})t}\widehat{\mathbf{u}}_0||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}+C||\mathcal{F}^{-1}(e^{-\varepsilon t}\widehat{w}_0)||_{L^4_{\mathbf{x}}}. \tag{3.8}$$

The first integral on the right-hand side of (3.8) can be split into two parts according to the magnitudes of the frequencies. For small frequency part, Hölder inequality

$$\|fg\|_{L_1}\leq\|f\|_{L_p}\|g\|_{L_q},\quad \frac{1}{p}+\frac{1}{q}=1,\quad 1\leq p,q\leq+\infty$$

and Plancherel theorem yields

$$\begin{aligned}\int_{|\boldsymbol{\xi}|<\delta}|e^{-A(\boldsymbol{\xi})t}\widehat{\mathbf{u}}_0|^{\frac{4}{3}}d\boldsymbol{\xi}&\leq C\Big(\int_{|\boldsymbol{\xi}|<\delta}e^{-4\beta t|\boldsymbol{\xi}|^2}d\boldsymbol{\xi}\Big)^{\frac{1}{3}}||\widehat{\mathbf{u}}_0||^{\frac{4}{3}}_{L^2_{\boldsymbol{\xi}}}\\&\leq Ct^{-\frac{1}{3}}||\mathbf{u}_0||^{\frac{4}{3}}_{L^2_{\mathbf{x}}}.\end{aligned} \tag{3.9}$$

For large frequency part, using (3.3) and interpolation inequality

$$\|u\|_{L_r}\leq\|u\|^{\theta}_{L_s}\|u\|^{1-\theta}_{L_t},\quad \frac{1}{r}=\frac{\theta}{s}+\frac{1-\theta}{t},\quad 1\leq s\leq r\leq t\leq+\infty$$

yields

$$\int_{|\boldsymbol{\xi}|\geq\delta}|e^{-A(\boldsymbol{\xi})t}\widehat{\mathbf{u}}_0|^{\frac{4}{3}}d\boldsymbol{\xi}\leq Ce^{-\frac{4}{3}\beta t}||\widehat{\mathbf{u}}_0||^{\frac{4}{3}}_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}\leq Ce^{-\frac{4}{3}\beta t}||\widehat{\mathbf{u}}_0||^{\frac{2}{3}}_{L^1_{\boldsymbol{\xi}}}||\mathbf{u}_0||^{\frac{2}{3}}_{L^2_{\mathbf{x}}}. \tag{3.10}$$

Therefore, by combining (3.9)-(3.10), one obtains for t large,

$$||e^{-A(\boldsymbol{\xi})t}\widehat{\mathbf{u}}_0||_{L^{4/3}_{\boldsymbol{\xi}}} \leq \begin{cases} Ct^{-\frac{1}{4}}||\mathbf{u}_0||^{\frac{1}{2}}_{L^2_{\mathbf{x}}}(||\mathbf{u}_0||^{\frac{1}{2}}_{L^2_{\mathbf{x}}} + ||\widehat{\mathbf{u}}_0||^{\frac{1}{2}}_{L^1_{\boldsymbol{\xi}}}) & \text{when} \quad \delta < +\infty, \\ Ct^{-\frac{1}{4}}\|\mathbf{u}_0\|_{L^2_{\mathbf{x}}} & \text{when} \quad \delta = +\infty. \end{cases} \tag{3.11}$$

For the second integral on the right-hand-side of (3.8), from Lemma 2.2, we have

$$||\mathcal{F}^{-1}(e^{-\varepsilon t}\widehat{w}_0)||_{L^4_{\mathbf{x}}} \leq C(1+t)^{-\frac{1}{4}}. \tag{3.12}$$

Then, for a small γ that will be chosen subsequently, if $\mathbf{v}_0$ is in $L^2(\mathbb{R}^2_{\mathbf{x}}) \cap L^1(\mathbb{R}^2_{\mathbf{x}})$ with small L^2-norm and such that $\widehat{\mathbf{v}}_0$ is in $L^1_{\boldsymbol{\xi}}(\mathbb{R}^2)$ with small norm if $\delta \neq +\infty$, then from (3.8)-(3.11)-(3.12)

$$\sup_{t>0}(t^{\frac{1}{4}}||S(t)\mathbf{v}_0||_{L^4_{\mathbf{x}}}) \leq \gamma. \tag{3.13}$$

The boundedness of $||e^{-\varepsilon t}\widehat{w}_0||_{L^{4/3}_{\boldsymbol{\xi}}}$ which is stronger than (3.12) will be useful in the proof of the decay rate with L^∞-norm. It is given here for convenience. With exactly the same proof, where $d = 0$ corresponding to $\delta = +\infty$ and $d > 0$ corresponds to $\delta < +\infty$, it shows

$$||e^{-\varepsilon t}\widehat{w}_0||_{L^{4/3}_{\boldsymbol{\xi}}} \leq \begin{cases} Ct^{-\frac{1}{4}}||w_0||^{\frac{1}{2}}_{L^2_{\mathbf{x}}}(||w_0||^{\frac{1}{2}}_{L^2_{\mathbf{x}}} + ||\widehat{w}_0||^{\frac{1}{2}}_{L^1_{\boldsymbol{\xi}}}) & \text{when} \quad d \neq 0, \\ Ct^{-\frac{1}{4}}\|w_0\|_{L^2_{\mathbf{x}}} & \text{when} \quad d = 0. \end{cases} \tag{3.14}$$

We now move to the nonlinear estimate. We first split the norm into small and large frequency parts as follows

$$\begin{aligned} &|| \int_0^t S(t-s)F(\mathbf{v}(s))ds||_{L^4_{\mathbf{x}}} \\ &\leq || \int_0^t S(t-s)P_\delta F(\mathbf{v}(s))ds||_{L^4_{\mathbf{x}}} + || \int_0^t S(t-s)Q_\delta F(\mathbf{v}(s))ds||_{L^4_{\mathbf{x}}} \\ &\leq C\int_0^t ||\widehat{S}(t-s)\ \widehat{P_\delta F(\mathbf{v}(s))}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}ds + C\int_0^t ||\widehat{S}(t-s)\ \widehat{Q_\delta F(\mathbf{v}(s))}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}ds. \end{aligned} \tag{3.15}$$

Notice that the last element in F is zero and therefore

$$|\widehat{S}(t-s)\widehat{F(\mathbf{v}(s))}| \leq C\|e^{-A(\boldsymbol{\xi})t}\|\,|\widehat{F(\mathbf{v}(s))}|.$$

For the small-frequency part, using (3.3), (3.4), Hölder inequality, Plancherel Theorem and (3.5), one obtains

$$\begin{aligned} \int_{\{|\boldsymbol{\xi}|<\delta\}} |\widehat{S}(t-s)\widehat{F(\mathbf{v}(s))}|^{\frac{4}{3}}d\boldsymbol{\xi} &\leq C\int_{\{|\boldsymbol{\xi}|<\delta\}} e^{-\frac{4}{3}\beta(t-s)|\boldsymbol{\xi}|^2}|\boldsymbol{\xi}|^{4/3}|\widehat{B(\mathbf{v})}|^{\frac{4}{3}}d\boldsymbol{\xi} \\ &\leq C\Big(\int_{\boldsymbol{\xi}} e^{-4\beta(t-s)|\boldsymbol{\xi}|^2}|\boldsymbol{\xi}|^4 d\boldsymbol{\xi}\Big)^{\frac{1}{3}}||P_\delta\widehat{B(\mathbf{v})}||^{\frac{4}{3}}_{L^2_{\boldsymbol{\xi}}} \\ &\leq C(t-s)^{-1}||P_\delta B(\mathbf{v})||^{\frac{4}{3}}_{L^2_{\mathbf{x}}} \leq C(t-s)^{-1}||\mathbf{v}(s)||^{\frac{8}{3}}_{L^4_{\mathbf{x}}}. \end{aligned} \tag{3.16}$$

For the large frequency part, due to (3.3), (3.4) and (3.5)

$$\|\widehat{S}(t-s)\widehat{Q_\delta F(\mathbf{v})}\|_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}} \leq Ce^{-\beta(t-s)}\|\,|\boldsymbol{\xi}|\widehat{B(\mathbf{v})}\|_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}(|\boldsymbol{\xi}|>\delta)} \leq e^{-\beta(t-s)}||\mathbf{v}(s)||^2_{L^4_{\mathbf{x}}}. \tag{3.17}$$

Therefore, since $e^{-\beta(t-s)} \leq C(t-s)^{-\frac{3}{4}}$ for $t-s>0$, we infer from (3.15), (3.16) and (3.17) that

$$\begin{aligned} &\|\int_0^t S(t-s)F(\mathbf{v}(s))ds\|_{L^4_{\mathbf{x}}} \leq \int_0^t \frac{C}{(t-s)^{\frac{3}{4}}}\|\mathbf{v}(s)\|^2_{L^4_{\mathbf{x}}}ds \\ &\leq C(\sup_{0<s<t} s^{\frac{1}{4}}\|\mathbf{v}(s)\|_{L^4_{\mathbf{x}}})^2\int_0^t \frac{ds}{s^{\frac{1}{2}}(t-s)^{\frac{3}{4}}} = Ct^{-\frac{1}{4}}(\sup_{0<s<t} s^{\frac{1}{4}}\|\mathbf{v}(s)\|_{L^4_{\mathbf{x}}})^2. \end{aligned} \tag{3.18}$$

We then infer from (3.7), (3.13) and (3.18) that

$$(\sup_{0<s<t} s^{\frac{1}{4}}\|\mathbf{v}(s)\|_{L^4_{\mathbf{x}}}) \leq \gamma + C(\sup_{0<s<t} s^{\frac{1}{4}}\|\mathbf{v}(s)\|_{L^4_{\mathbf{x}}})^2. \tag{3.19}$$

Let $M(t) = \sup_{0<s<t}\{s^{\frac{1}{4}}\|\mathbf{v}(s)\|_{L^4_{\mathbf{x}}}\}$, (3.19) implies that

$$CM(t)^2 - M(t) + \gamma \leq 0$$

for any t. Using the facts that $M(0)=0$ and $M(t)$ is continuous nondecreasing function with respect to t, one obtains $M(t)$ is bounded, i.e.

$$\sup_{0<s<t} s^{\frac{1}{4}}\|\mathbf{v}(s)\|_{L^4_{\mathbf{x}}} \leq 2\gamma \tag{3.20}$$

if the two roots of the quadratic form $Cx^2 - x + \gamma = 0$ are real and positive $0 < r_1 < r_2$, where $r_1 \leq 2\gamma$. Therefore if γ is small enough (i.e $\gamma < \frac{1}{4C}$), the mapping $\mathcal{T}(\mathbf{v}) = S(t)\mathbf{v}_0 + \int_0^t S(t-s)F(\mathbf{v}(s))ds$ maps the ball of radius r_1 in E into itself. The proof for showing $\mathcal{T}$ is a contraction is similar and the Banach fixed point theorem applies and therefore these exists a unique solution in E.

Remark 3.4. *In fact, using (3.14), one can prove under the additional condition $\widehat{\mathbf{v}}_0$ is in $L^1_{\boldsymbol{\xi}}(\mathbb{R}^2)$ with small norm when $d>0$, that $t^{\frac{1}{4}}\|\widehat{\mathbf{v}}\|_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}$ is bounded by 2γ which is stronger than (3.20).*

We now move to the second step of the proof. We prove that the solution defined in the previous step satisfies the desired L^2 decay estimate (3.6). The estimate on the linear term reads, due to Plancherel Theorem

$$\begin{aligned} \|\widehat{S}(t,\boldsymbol{\xi})\widehat{\mathbf{v}}_0\|^2_{L^2_{\boldsymbol{\xi}}} &\leq C\int_{\{|\boldsymbol{\xi}|<\delta\}} e^{-2\beta t|\boldsymbol{\xi}|^2}|\widehat{\mathbf{v}}_0|^2 d\boldsymbol{\xi} + C\int_{\{|\boldsymbol{\xi}|\geq\delta\}} e^{-2\beta t}|\widehat{\mathbf{v}}_0|^2 d\boldsymbol{\xi} \\ &\leq C(t^{-1}\|\mathbf{v}_0\|^2_{L^1_{\mathbf{x}}} + e^{-2\beta t}\|\widehat{\mathbf{v}}_0\|^2_{L^2_{\boldsymbol{\xi}}}) \leq C(\mathbf{v}_0)t^{-1}. \end{aligned} \tag{3.21}$$

The estimate on the nonlinear term starts by observing

$$\|\int_0^t S(t-s)F(\mathbf{v}(s))ds\|_{L^2_{\mathbf{x}}} \leq C\int_0^t \|\widehat{S}(t-s)\widehat{F(\mathbf{v}(s))}\|_{L^2_{\boldsymbol{\xi}}}ds. \tag{3.22}$$

We split the norm into small frequency part and large frequency part. On the small frequency part, due to (3.4), (3.3) and (3.5)

$$\begin{aligned} \int_{\{|\boldsymbol{\xi}|<\delta\}} |\widehat{S}(t-s)\widehat{F(\mathbf{v}(s))}|^2 d\boldsymbol{\xi} &\leq C(\int_{\boldsymbol{\xi}} e^{-4\beta(t-s)|\boldsymbol{\xi}|^2}|\boldsymbol{\xi}|^4 d\boldsymbol{\xi})^{\frac{1}{2}}\|\widehat{P_\delta B(\mathbf{v})}\|^2_{L^4_{\boldsymbol{\xi}}} \\ &\leq C(t-s)^{-\frac{3}{2}}\|\mathbf{v}(s)\|_{L^4_{\mathbf{x}}}\|\mathbf{v}(s)\|_{L^2_{\mathbf{x}}}. \end{aligned} \tag{3.23}$$

Similarly, on the large frequency part,

$$\begin{aligned} \int_{\{|\boldsymbol{\xi}|\geq\delta\}} |\widehat{S}(t-s)\widehat{F(\mathbf{v}(s))}|^2 d\boldsymbol{\xi} &\leq Ce^{-2\beta(t-s)}\int_{\{|\boldsymbol{\xi}|\geq\delta\}} |\boldsymbol{\xi}|^2|\widehat{B(\mathbf{v})}|^2 d\boldsymbol{\xi} \\ &\leq Ce^{-2\beta(t-s)}\|\mathbf{v}(s)\|_{L^4_{\mathbf{x}}}\|\mathbf{v}(s)\|_{L^2_{\mathbf{x}}}. \end{aligned} \tag{3.24}$$

We then conclude, by using $e^{-2\beta(t-s)} \leq C(t-s)^{-\frac{3}{4}}$, that

$$
\begin{aligned}
||\int_0^t S(t-s)F(\mathbf{v}(s))ds||_{L^2_{\mathbf{x}}} &\leq \int_0^t \frac{C}{(t-s)^{\frac{3}{4}}}||\mathbf{v}(s)||_{L^4_{\mathbf{x}}}||\mathbf{v}(s)||_{L^2_{\mathbf{x}}}ds \\
&\leq C(\sup_{s>0} s^{\frac{1}{4}}||\mathbf{v}(s)||_{L^4_{\mathbf{x}}})(\sup_{s>0} s^{\frac{1}{2}}||\mathbf{v}(s)||_{L^2_{\mathbf{x}}})\int_0^t \frac{ds}{s^{\frac{3}{4}}(t-s)^{\frac{3}{4}}} \\
&\leq C(\sup_{s>0} s^{\frac{1}{4}}||\mathbf{v}(s)||_{L^4_{\mathbf{x}}})(\sup_{s>0} s^{\frac{1}{2}}||\mathbf{v}(s)||_{L^2_{\mathbf{x}}})t^{-1/2}.
\end{aligned}
\tag{3.25}
$$

Therefore, due to (3.7)-(3.21)-(3.20)-(3.25)

$$
t^{\frac{1}{2}}||\mathbf{v}(t)||_{L^2_{\mathbf{x}}} \leq C(\mathbf{v}_0) + C\gamma(\sup_{s>0} s^{\frac{1}{2}}||\mathbf{v}(s)||_{L^2_{\mathbf{x}}}). \tag{3.26}
$$

Then, if γ is small enough, by moving $(\sup_{s>0} s^{\frac{1}{2}}||\mathbf{v}(s)||_{L^2_{\mathbf{x}}})$ to the left hand side of (3.26), the proof is completed. $\square$

3.2. **Application to KdV-Burgers-type systems.** A system is called KdV-Burgers type if (3.3) is valid with $\delta = +\infty$ in the linear estimates. From Lemma 2.3, this is the case when order(σ) ≥ 1, and either $\{\nu = 1\}$ or $\{\nu = 0$ and $d = 0\}$. Therefore For the KdV-Burgers-type systems, the following theorem holds.

Theorem 3.5. *For system (1.2) with order(σ) ≥ 1, and either $\{\nu = 1\}$ or $\{\nu = 0$ and $d = 0\}$ and for large t,*

- *if order(H) $= -1$, then for $(\nabla\eta_0, \mathbf{u}_0)$ in $L^1(\mathbb{R}^2_{\mathbf{x}}) \cap L^2(\mathbb{R}^2_{\mathbf{x}})$ and small enough in $L^2(\mathbb{R}^2_{\mathbf{x}})$,*
$$
||\mathbf{u}(t)||_{L^2_{\mathbf{x}}} + ||\nabla\eta(t)||_{L^2_{\mathbf{x}}} \leq Ct^{-\frac{1}{2}}; \tag{3.27}
$$
- *if order(H) $= 0$, then for $(\eta_0, \mathbf{u}_0)$ in $L^1(\mathbb{R}^2_{\mathbf{x}}) \cap L^2(\mathbb{R}^2_{\mathbf{x}})$ and small enough in $L^2(\mathbb{R}^2_{\mathbf{x}})$,*
$$
||\mathbf{u}(t)||_{L^2_{\mathbf{x}}} + ||\eta(t)||_{L^2_{\mathbf{x}}} \leq Ct^{-\frac{1}{2}}; \tag{3.28}
$$
- *if order(H) $= 1$, then for $(\eta_0, \nabla\mathbf{u}_0)$ in $L^1(\mathbb{R}^2_{\mathbf{x}}) \cap L^2(\mathbb{R}^2_{\mathbf{x}})$ and small enough in $L^2(\mathbb{R}^2_{\mathbf{x}})$,*
$$
||\nabla\mathbf{u}(t)||_{L^2_{\mathbf{x}}} + ||\eta(t)||_{L^2_{\mathbf{x}}} \leq Ct^{-\frac{1}{2}}. \tag{3.29}
$$

Proof. The theorem is going to be proved by splitting cases according to the constants a, b, c, d. The conditions order(σ) ≥ 1 and (C0)-(C2) implies that there are three possibilities:

- $b = d = 0,\ \ a = c = \frac{1}{6}$;
- $b = 0,\ d > 0,\ a$ and c do not vanish;
- $d = 0,\ b > 0,\ a = c > 0$.

Case I: $b = d = 0$, $a = c = \frac{1}{6}$, so $H = 1$ and order(H) $= 0$. Applying the Helmholtz splitting to (1.1), the nonlinear system reads in a balanced form as

$$
\begin{aligned}
\widehat{\eta}_t + \nu\alpha\widehat{\eta} + i\mathrm{sgn}(\omega_1)\sigma|\boldsymbol{\xi}|\widehat{q} &= |\boldsymbol{\xi}|(-\frac{i\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\eta\mathbf{u}}), \\
\widehat{q}_t + \varepsilon\widehat{q} + i\mathrm{sgn}(\omega_1)\sigma\widehat{\eta}|\boldsymbol{\xi}| &= |\boldsymbol{\xi}|(-\frac{i}{2}\widehat{|\mathbf{u}|^2}), \\
\widehat{\psi}_t + \varepsilon\widehat{\psi} &= 0.
\end{aligned}
\tag{3.30}
$$

The theorem is valid if (3.5) is true with

$$\widehat{B(\mathbf{v})} = \begin{pmatrix} -\frac{i\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\eta\mathbf{u}} \\ -\frac{i}{2}\widehat{|\mathbf{u}|^2} \\ 0 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} \eta \\ q \\ \psi \end{pmatrix},$$

$P_\delta = I_d$ and $Q_\delta = 0$.

The first inequality is straightforward due to Plancherel theorem

$$\begin{aligned} ||\widehat{B(\mathbf{v})}||^2_{L^2_{\boldsymbol{\xi}}} &\le C\int(|\widehat{\eta\mathbf{u}}|^2 + |\widehat{|\mathbf{u}|^2}|^2)d\boldsymbol{\xi} \le C(||\eta||^4_{L^4_{\mathbf{x}}} + ||\mathbf{u}||^4_{L^4_{\mathbf{x}}}) \\ &\le C(||\eta||^4_{L^4_{\mathbf{x}}} + ||q||^4_{L^4_{\mathbf{x}}} + ||\psi||^4_{L^4_{\mathbf{x}}}). \end{aligned} \tag{3.31}$$

For the second inequality, we use that the operator that has symbol $-\frac{i\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$, which is a vector valued Riesz transform, is bounded on any $L^p_{\mathbf{x}}$, $1 < p < +\infty$, and then by Hölder inequality to obtain

$$||B(\mathbf{v})||_{L^{\frac{4}{3}}_{\mathbf{x}}} \le C(||\eta\mathbf{u}||_{L^{\frac{4}{3}}_{\mathbf{x}}} + ||\,|\mathbf{u}|^2||_{L^{\frac{4}{3}}_{\mathbf{x}}}) \le C\|\mathbf{v}\|_{L^2_{\mathbf{x}}}\|\mathbf{v}\|_{L^4_{\mathbf{x}}}. \tag{3.32}$$

Case II: $b = 0$ and $d > 0$, a and c do vanish. In this case, order$(H) = 1$ and the system reads

$$\begin{aligned} \widehat{\eta}_t + \nu\alpha\widehat{\eta} + i\text{sgn}(\omega_1)\sigma|\boldsymbol{\xi}|\widehat{Hq} &= |\boldsymbol{\xi}|(-\frac{i\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\eta\mathbf{u}}), \\ \widehat{Hq}_t + \varepsilon\widehat{Hq} + i\text{sgn}(\omega_1)\sigma\widehat{\eta}|\boldsymbol{\xi}| &= \frac{\widehat{H}|\boldsymbol{\xi}|}{1+d|\boldsymbol{\xi}|^2}(-\frac{i}{2}\widehat{|\mathbf{u}|^2}), \\ \widehat{H\psi}_t + \varepsilon\widehat{H\psi} &= 0. \end{aligned} \tag{3.33}$$

Again, the proof amounts to verify that (3.5) is valid with

$$\widehat{B(\mathbf{v})} = \begin{pmatrix} -\frac{i\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\eta\mathbf{u}} \\ -\frac{i\widehat{H}}{2(1+d|\boldsymbol{\xi}|^2)}\widehat{|\mathbf{u}|^2} \\ 0 \end{pmatrix} \quad \text{where} \quad \mathbf{v} = \begin{pmatrix} \eta \\ Hq \\ H\psi \end{pmatrix}.$$

Since the operator whose symbol is $(1 + d|\boldsymbol{\xi}|^2)^{-1}\widehat{H}$ is bounded on $L^p_{\mathbf{x}}$ for $1 < p < +\infty$, we just have to prove that $\mathbf{v} \mapsto \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\eta\mathbf{u}}$ and $\mathbf{v} \mapsto \widehat{|\mathbf{u}|^2}$ satisfy the assumptions. This can be demonstrated by using (3.31), (3.32) and the fact that H^{-1} is bounded on $L^p_{\mathbf{x}}$ for $1 < p < +\infty$.

Case III: $d = 0$, $b > 0$ and $a = c > 0$. In this case, order$(H) = -1$ and the system reads

$$\begin{aligned} \widehat{H^{-1}\eta}_t + \nu\alpha\widehat{H^{-1}\eta} + i\text{sgn}(\omega_1)\sigma|\boldsymbol{\xi}|\widehat{q} &= \frac{H^{-1}|\boldsymbol{\xi}|}{1+b|\boldsymbol{\xi}|^2}(-\frac{i\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\cdot\widehat{\eta\mathbf{u}}) \\ \widehat{q}_t + \varepsilon\widehat{q} + i\text{sgn}(\omega_1)\sigma\widehat{H^{-1}\eta}|\boldsymbol{\xi}| &= |\boldsymbol{\xi}|(-\frac{i}{2}\widehat{|\mathbf{u}|^2}) \\ \widehat{\psi}_t + \varepsilon\widehat{\psi} &= 0. \end{aligned} \tag{3.34}$$

The bilinear term involved here can be handled exactly as in case II with $\mathbf{v} = (H^{-1}\eta,\ q,\ \psi)^T$. □

3.3. **Application to weakly dispersive systems.** In this case, $b > 0$, $d > 0$ and (3.3) is valid with $\delta < +\infty$.

Theorem 3.6. *For system (1.2) with $b, d > 0$, and either $\{\nu = 1\}$ or $\{\nu = 0$ and order$(\sigma) = -1\}$ and for large t,*

- *if order$(H) = -1$, then for $(\nabla\eta_0, \mathbf{u}_0)$ in $L^1(\mathbb{R}^2_{\mathbf{x}})$, and $(\widehat{\nabla\eta_0}, \widehat{\mathbf{u}}_0)$ in $L^2(\mathbb{R}^2_{\boldsymbol{\xi}}) \cap L^1(\mathbb{R}^2_{\boldsymbol{\xi}})$ and small enough in $L^2(\mathbb{R}^2_{\boldsymbol{\xi}}) \cap L^1(\mathbb{R}^2_{\boldsymbol{\xi}})$,*
$$||\mathbf{u}(t)||_{L^2_{\mathbf{x}}} + ||\nabla\eta(t)||_{L^2_{\mathbf{x}}} \leq Ct^{-\frac{1}{2}}; \tag{3.35}$$
- *if order$(H) = 0$, then for $(\eta_0, \mathbf{u}_0)$ in $L^1(\mathbb{R}^2_{\mathbf{x}})$, and $(\widehat{\eta}_0, \widehat{\mathbf{u}}_0)$ in $L^2(\mathbb{R}^2_{\boldsymbol{\xi}}) \cap L^1(\mathbb{R}^2_{\boldsymbol{\xi}})$ and small enough in $L^2(\mathbb{R}^2_{\boldsymbol{\xi}}) \cap L^1(\mathbb{R}^2_{\boldsymbol{\xi}})$,*
$$||\mathbf{u}(t)||_{L^2_{\mathbf{x}}} + ||\eta(t)||_{L^2_{\mathbf{x}}} \leq Ct^{-\frac{1}{2}}; \tag{3.36}$$
- *if order$(H) = 1$, then for $(\eta_0, \nabla\mathbf{u}_0)$ in $L^1(\mathbb{R}^2_{\mathbf{x}})$, and $(\widehat{\eta}_0, \widehat{\nabla\mathbf{u}_0})$ in $L^2(\mathbb{R}^2_{\boldsymbol{\xi}}) \cap L^1(\mathbb{R}^2_{\boldsymbol{\xi}})$ and small enough in $L^2(\mathbb{R}^2_{\boldsymbol{\xi}}) \cap L^1(\mathbb{R}^2_{\boldsymbol{\xi}})$,*
$$||\nabla\mathbf{u}(t)||_{L^2_{\mathbf{x}}} + ||\eta(t)||_{L^2_{\mathbf{x}}} \leq Ct^{-\frac{1}{2}}. \tag{3.37}$$

Proof. We only need to check that system (3.1) fits into the abstract framework of Theorem 3.1. We again split the study to the small frequency part and large frequency part.
Small frequencies: the P_δ part of (3.5) needs to be checked and the proof is the same as that for Theorem 3.5.

Large frequencies: the Q_δ part of (3.5), i.e. for $|\boldsymbol{\xi}| \geq \delta$, needs to be checked and we again separate our investigation according to the order of H.
Case I: order(H)=0. The proof for the first inequality in (3.5) amounts to show
$$||\frac{|\boldsymbol{\xi}|\widehat{H}}{1+d|\boldsymbol{\xi}|^2}\widehat{|\mathbf{u}|^2}||_{L^{\frac{4}{3}}_{\{|\boldsymbol{\xi}|\geq\delta\}}} \leq C||\mathbf{v}||^2_{L^4_{\mathbf{x}}}. \tag{3.38}$$
with $\mathbf{v} = (\eta, q, \psi)^T$. The proof is obtained by using order$(H) = 0$, the Hölder inequality and Plancherel theorem which yield
$$\begin{aligned}
||\frac{|\boldsymbol{\xi}|\widehat{H}}{1+d|\boldsymbol{\xi}|^2}\widehat{|\mathbf{u}|^2}||^{\frac{4}{3}}_{L^{\frac{4}{3}}_{\{|\boldsymbol{\xi}|\geq\delta\}}} &\leq C\int_{\{|\boldsymbol{\xi}|\geq\delta\}} \frac{1}{|\boldsymbol{\xi}|^{\frac{4}{3}}}|\,\widehat{|\mathbf{u}|^2}|^{\frac{4}{3}}d\boldsymbol{\xi} \\
&\leq C(\int_{\{|\boldsymbol{\xi}|\geq\delta\}} \frac{d\boldsymbol{\xi}}{|\boldsymbol{\xi}|^4})^{\frac{1}{3}}(\int_{\boldsymbol{\xi}} |\,\widehat{|\mathbf{u}|^2}|^2 d\boldsymbol{\xi})^{\frac{2}{3}} \leq C(\delta)||\,|\mathbf{u}|^2||^{\frac{4}{3}}_{L^2_{\mathbf{x}}} \\
&= C(\delta)||\mathbf{u}||^{\frac{8}{3}}_{L^4_{\mathbf{x}}} \leq C(\delta)||\mathbf{v}||^{\frac{8}{3}}_{L^4_{\mathbf{x}}}.
\end{aligned} \tag{3.39}$$
We skip the proof of the analogous estimate on the bilinear term $\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\widehat{\eta\mathbf{u}}$ because it is very similar to this one.

The second inequality to prove in (3.5) amounts to prove
$$||\mathcal{F}^{-1}\left(\frac{|\boldsymbol{\xi}|}{1+d|\boldsymbol{\xi}|^2}\widehat{|\mathbf{u}|^2}\right)||_{L^2_{\mathbf{x}}} \leq C||\mathbf{v}||_{L^4_{\mathbf{x}}}||\mathbf{v}||_{L^2_{\mathbf{x}}}. \tag{3.40}$$
Since $H^1_{\mathbf{x}} \subset L^4_{\mathbf{x}}$, one has by duality
$$||\,|\mathbf{u}|^2||_{H^{-1}_{\mathbf{x}}} \leq C||\,|\mathbf{u}|^2||_{L^{\frac{4}{3}}_{\mathbf{x}}}.$$

Therefore, with the use of interpolation inequality $[L^2, L^4]_{\frac{1}{2}} = L^{\frac{8}{3}}$,

$$\begin{aligned} ||\mathcal{F}^{-1}\left(\frac{|\boldsymbol{\xi}|}{1+d|\boldsymbol{\xi}|^2}\widehat{|\mathbf{u}|^2}\right)||_{L^2_{\mathbf{x}}} &\le C||\,|\mathbf{u}|^2||_{L^{\frac{4}{3}}_{\mathbf{x}}} \le C||\mathbf{u}||^2_{L^{\frac{8}{3}}_{\mathbf{x}}} \\ &\le C||\mathbf{u}||_{L^4_{\mathbf{x}}}||\mathbf{u}||_{L^2_{\mathbf{x}}} \le C||\mathbf{v}||_{L^4_{\mathbf{x}}}||\mathbf{v}||_{L^2_{\mathbf{x}}}. \end{aligned} \tag{3.41}$$

We again skip the proof of analogous estimate on the bilinear term $\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\widehat{\eta\mathbf{u}}$ because it is very similar to this one.

Case II: order$(H) = 1$. We first prove the $L^{\frac{4}{3}}_{\boldsymbol{\xi}}$ estimate, namely the first inequality in (3.5). This amounts to prove that

$$||\mathcal{F}(|\mathbf{u}|^2)||_{L^{\frac{4}{3}}_{\{|\boldsymbol{\xi}|\ge\delta\}}} + ||\frac{1}{|\boldsymbol{\xi}|}\mathcal{F}(\eta\mathbf{u})||_{L^{\frac{4}{3}}_{\{|\boldsymbol{\xi}|\ge\delta\}}} \le C||\mathbf{v}||^2_{L^4_{\mathbf{x}}} \tag{3.42}$$

where $\mathbf{v} = (\eta, Hq, H\psi)^T$.

The second term in the left-hand-side of this inequality can be handled exactly as in the previous case, since H^{-1} is a bounded operator on $L^p_{\mathbf{x}}$. For the first term in the left-hand-side of (3.42) we use the following trick:

$$\Omega = \left\{\boldsymbol{\xi}\Big||\boldsymbol{\xi}| \ge \delta\right\} \subset \Omega_1 \cup \Omega_2$$

where

$$\Omega_1 = \left\{\boldsymbol{\xi}\Big||\xi_1| \ge \frac{\delta}{\sqrt{2}},\ |\xi_1| \ge |\xi_2|\right\}, \quad \Omega_2 = \left\{\boldsymbol{\xi}\Big||\xi_2| \ge \frac{\delta}{\sqrt{2}},\ |\xi_2| \ge |\xi_1|\right\}.$$

One then obtains

$$||\mathcal{F}(|\mathbf{u}|^2)||_{L^{\frac{4}{3}}_{\Omega_1}} \le ||\frac{2}{\xi_1}\mathcal{F}(\mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial x_1})||_{L^{\frac{4}{3}}_{\Omega_1}} \le ||\frac{2\sqrt{2}}{|\boldsymbol{\xi}|}\mathcal{F}(\mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial x_1})||_{L^{\frac{4}{3}}_{\{\sqrt{2}|\boldsymbol{\xi}|\ge\delta\}}}. \tag{3.43}$$

Using the same argument as in (3.39) and the fact that H^{-1} is a bounded operator on $L^p_{\mathbf{x}}$,

$$||\mathcal{F}(|\mathbf{u}|^2)||_{L^{\frac{4}{3}}_{\Omega_1}} \le C||\mathbf{v}||^2_{L^4_{\mathbf{x}}}$$

Since the similar estimate is true for $||\mathcal{F}(|\mathbf{u}|^2)||_{L^{\frac{4}{3}}_{\Omega_2}}$, it yields

$$||\mathcal{F}(|\mathbf{u}|^2)||_{L^{\frac{4}{3}}_{\{|\boldsymbol{\xi}|\ge\delta\}}} \le C(||\mathcal{F}(|\mathbf{u}|^2)||_{L^{\frac{4}{3}}_{\Omega_1}} + ||\mathcal{F}(|\mathbf{u}|^2)||_{L^{\frac{4}{3}}_{\Omega_2}}) \le C||\mathbf{v}||^2_{L^4_{\mathbf{x}}}.$$

We now prove the $L^2_{\mathbf{x}}$ estimate, namely the second inequality in (3.5). By Plancherel inequality, Sobolev embedding $W^{1,\frac{8}{3}}_{\mathbf{x}} \subset L^4_{\mathbf{x}}$ associated with order(H)=1, and interpolation inequality $[L^2, L^4]_{\frac{1}{2}} = L^{\frac{8}{3}}$,

$$||\mathcal{F}(|\mathbf{u}|^2)||_{L^2_{\boldsymbol{\xi}}} = C||\mathbf{u}||^2_{L^4_{\mathbf{x}}} \le C||\mathbf{v}||^2_{L^{\frac{8}{3}}_{\mathbf{x}}} \le C||\mathbf{v}||_{L^2_{\mathbf{x}}}||\mathbf{v}||_{L^4_{\mathbf{x}}}. \tag{3.44}$$

Together with the following inequality

$$||\eta\mathbf{u}||_{H^{-1}_{\mathbf{x}}} \le C||\eta\mathbf{u}||_{L^{\frac{4}{3}}_{\mathbf{x}}} \le C||\eta||_{L^2_{\mathbf{x}}}||\mathbf{u}||_{L^4_{\mathbf{x}}} \le C||\mathbf{v}||_{L^2_{\mathbf{x}}}||\mathbf{v}||_{L^4_{\mathbf{x}}}, \tag{3.45}$$

the theorem in proved for the case with order(H)=1.

Case III: order$(H) = -1$. In this case, the system reads (3.34), and with $\widehat{\mathbf{v}} = (\widehat{H}^{-1}\widehat{\eta}, \widehat{\mathbf{u}})$ the nonlinearity can be handled exactly as in the cases where order$(H) =$ 0 or 1. $\square$

Corollary 3.7. *For the Bona-Smith system ($a = 0, b > 0, c < 0, d > 0$) the decay rates are valid if we are either in the partial dissipation case or in the complete dissipation case. Solutions to BBM-BBM systems ($a = c = 0, b > 0, d > 0$ satisfy the decay rates in the case of complete dissipation $\nu = 1$.*

3.4. L^∞ **decay rate.** In this section we prove the following abstract result

Theorem 3.8. *With the same assumptions as in Theorem 3.1 and assume also $\widehat{\mathbf{v}}_0$ is in $L^1_{\boldsymbol{\xi}}(\mathbb{R}^2)$ with small norm when $d > 0$, and*

$$||\widehat{B(\mathbf{v})}||_{L^{\frac{4}{3}}_{\{|\boldsymbol{\xi}|\le\delta\}}} + ||\,|\boldsymbol{\xi}|\widehat{B(\mathbf{v})}||_{L^1_{\{|\boldsymbol{\xi}|\ge\delta\}}} \le C||\widehat{\mathbf{v}}||_{L^1_{\boldsymbol{\xi}}}||\widehat{\mathbf{v}}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}. \tag{3.46}$$

Then the solution defined in Theorem 3.1 satisfies

$$||\mathbf{v}||_{L^\infty_{\mathbf{x}}} \le C(1+t)^{-1}. \tag{3.47}$$

Proof. We are going to prove that

$$||\widehat{\mathbf{v}}||_{L^1_{\boldsymbol{\xi}}} \le C(1+t)^{-1}, \tag{3.48}$$

which will complete the proof of the theorem, since the inverse Fourier transform maps $L^1_{\boldsymbol{\xi}}$ into $L^\infty_{\mathbf{x}}$. From (3.7) and (3.4),

$$\begin{aligned} &||\widehat{\mathbf{v}}(t)||_{L^1_{\boldsymbol{\xi}}} \le ||\widehat{S(t)\mathbf{v}_0}||_{L^1_{\boldsymbol{\xi}}} \\ &+ \int_0^t ||\widehat{S}(t-s)|\boldsymbol{\xi}|\widehat{B(\mathbf{v})}||_{L^1_{\{|\boldsymbol{\xi}|\le\delta\}}}ds + \int_0^t ||\widehat{S}(t-s)|\boldsymbol{\xi}|\widehat{B(\mathbf{v})}||_{L^1_{\{|\boldsymbol{\xi}|\ge\delta\}}}ds. \end{aligned} \tag{3.49}$$

For the linear part

$$\begin{aligned} ||\widehat{S(t)\mathbf{v}_0}||_{L^1_{\boldsymbol{\xi}}} &\le (\int e^{-\beta t|\boldsymbol{\xi}|^2}d\boldsymbol{\xi})||\widehat{\mathbf{v}}_0||_{L^\infty_{\{|\boldsymbol{\xi}|\le\delta\}}} + e^{-\beta t}||\widehat{\mathbf{v}}_0||_{L^1_{\{|\boldsymbol{\xi}|\ge\delta\}}} \\ &\le Ct^{-1}||\mathbf{v}_0||_{L^1_{\mathbf{x}}} + Ce^{-\beta t}||\widehat{\mathbf{v}}_0||_{L^1_{\boldsymbol{\xi}}}, \end{aligned} \tag{3.50}$$

which provides the desired decay rate since both $\mathbf{v}_0$ and $\widehat{\mathbf{v}}_0$ are integrable.

For the nonlinear part with small frequencies, using Hölder's inequality and the assumptions above

$$\begin{aligned} &\int_0^t ||e^{-\beta(t-s)|\boldsymbol{\xi}|^2}|\boldsymbol{\xi}|\widehat{B(\mathbf{v})}||_{L^1_{\{|\boldsymbol{\xi}|\le\delta\}}}ds \\ &\le C\int_0^t ||e^{-\beta(t-s)|\boldsymbol{\xi}|^2}|\boldsymbol{\xi}|\,||_{L^4_{\boldsymbol{\xi}}}||\widehat{B(\mathbf{v})}||_{L^{\frac{4}{3}}_{\{|\boldsymbol{\xi}|\le\delta\}}}ds \le \int_0^t \frac{C}{(t-s)^{\frac{3}{4}}}||\widehat{\mathbf{v}}||_{L^1_{\boldsymbol{\xi}}}||\widehat{\mathbf{v}}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}ds \\ &\le C(\int_0^t \frac{ds}{(t-s)^{\frac{3}{4}}s^{\frac{1}{4}}})(\sup_{s\le t}||\widehat{\mathbf{v}}(s)||_{L^1_{\boldsymbol{\xi}}})(\sup_{s>0}s^{\frac{1}{4}}||\widehat{\mathbf{v}}(s)||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}) \\ &\le C\gamma\sup_{s\le t}(||\widehat{\mathbf{v}}(s)||_{L^1_{\boldsymbol{\xi}}}), \end{aligned} \tag{3.51}$$

by recalling Remark 3.4 that $\sup_{s>0}(s^{\frac{1}{4}}||\widehat{\mathbf{v}}(s)||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}) \le 2\gamma$ which is small. Therefore the upper bound in (3.51) moves into the left-hand-side of (3.49).

For the nonlinear term with high frequencies

$$\int_0^t ||\widehat{S}(t-s)|\boldsymbol{\xi}|\widehat{B(\mathbf{v})}||_{L^1_{\{|\boldsymbol{\xi}|\ge\delta\}}}ds \le C\int_0^t e^{-\beta(t-s)}||\,|\boldsymbol{\xi}|\widehat{B(\mathbf{v})}||_{L^1_{\{|\boldsymbol{\xi}|\ge\delta\}}}ds, \tag{3.52}$$

and we proceed exactly as above using $e^{-\beta(t-s)} \le C(t-s)^{-\frac{3}{4}}$ which completes the proof. $\square$

We now apply the abstract theorem to the KdV-Burger-type systems and weakly dispersive systems and obtain the following theorem.

Theorem 3.9. *For the systems listed in Theorem 3.5 and in Theorem 3.6,*

$$||\mathbf{v}||_{L^\infty_{\mathbf{x}}} \le C(1+t)^{-1} \tag{3.53}$$

where $\mathbf{v}$ *is defined according to the order of* H *and the initial data satisfies the corresponding conditions as in those theorems and also* $\widehat{\mathbf{v}}_0$ *is in* $L^1_{\boldsymbol{\xi}}(\mathbb{R}^2)$ *with small norm when* $d > 0$.

Proof. For KdV-Burgers systems and $b = d = 0$, $H = 1$, we essentially have to check that for a pair of function f, g

$$||\widehat{fg}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}} \le C||\widehat{f}||_{L^1_{\boldsymbol{\xi}}}||\widehat{g}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}, \tag{3.54}$$

which is true since $L^1_{\boldsymbol{\xi}} * L^{\frac{4}{3}}_{\boldsymbol{\xi}} \subset L^{\frac{4}{3}}_{\boldsymbol{\xi}}$. For the cases $b = 0$, $d > 0$, it is necessary to check that

$$||\eta\widehat{\mathbf{u}}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}} + ||\frac{\widehat{H}}{1+d|\boldsymbol{\xi}|^2}\widehat{|\mathbf{u}|^2}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}} \le C||\widehat{\mathbf{v}}||_{L^1_{\boldsymbol{\xi}}}||\widehat{\mathbf{v}}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}. \tag{3.55}$$

This is straightforward by using (3.54) and the facts that $\widehat{H^{-1}}$ and $\frac{\widehat{H}}{1+d|\boldsymbol{\xi}|^2}$ are in $L^\infty_{\boldsymbol{\xi}}$. The only case remaining is with $b > 0$ and $d = 0$ which can be handled exactly in the same manner.

For weakly dispersive systems, we again split the discussion according to the magnitude of frequencies. For small frequencies, the proofs are analogous to the KdV-Burgers case. For large frequencies, considering first the case with order$(H) = 0$, so the problem becomes to check that

$$||\frac{1}{|\boldsymbol{\xi}|}\widehat{fg}||_{L^1_{\{|\boldsymbol{\xi}|\ge\delta\}}} \le C(||\widehat{f}||_{L^1_{\boldsymbol{\xi}}}||\widehat{g}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}} + ||\widehat{g}||_{L^1_{\boldsymbol{\xi}}}||\widehat{f}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}). \tag{3.56}$$

With Hölder inequality

$$||\frac{1}{|\boldsymbol{\xi}|}\widehat{fg}||_{L^1_{\{|\boldsymbol{\xi}|\ge\delta\}}} \le (\int_{\{|\boldsymbol{\xi}|\ge\delta\}} |\boldsymbol{\xi}|^{-4} d\boldsymbol{\xi})^{\frac{1}{4}} ||\widehat{f} * \widehat{g}||_{L^{\frac{4}{3}}_{\boldsymbol{\xi}}}, \tag{3.57}$$

and we conclude as in (3.54).

The other cases order$(H) = 1$ or order$(H) = -1$ can be handled exactly in the same way.

4. **Numerical simulations.** In this section, the decay rates of the solutions are computed numerically for the BBM-BBM system with full and partial dissipations.

The numerical code is based on a Legendre-Fourier spectral discretization in space and a leap-frog Crank-Nicholson scheme in time (see [6] for detail). The initial data is taken to be

$$\begin{aligned} &\eta(x, y) = 0.5e^{-0.1((x-120)^2+(y-120)^2)} \\ &u = v = 0, \end{aligned} \tag{4.1}$$

on the computation domain $[0, 240] \times [0, 240]$ and the solution is computed for $t \in [0, 80]$. The number of modes used in both x and y directions is 1024 and $\Delta t = 0.05$. Since the solution is axisymmetric about the point $(120, 120)$, the norms on u and v are the same and therefore only the norms on η and u are presented. The decay rate r for function $f(x, y, t)$, where f is η or u, in

$$\|f\| \sim Ct^{-r}, \text{ as } t \to \infty$$

is calculated by first computing

$$r(t_n) := -\frac{\log \frac{\|f(t_n)\|}{\|f(t_{n-1})\|}}{\log \frac{t_n}{t_{n-1}}},$$

where the norm is either $\|\cdot\|_\infty$ or $\|\cdot\|_{L^2}$, and then calculating the mean using a constant least square fitting for the last 50 data which corresponds to t between 77.5 to 80.

The first case is for the full dissipation, namely $\nu = 1$, on the BBM-BBM system ($b = d = \frac{1}{6}, a = c = 0$). The L_∞ norm and L_2 norms of the solution η and u with respect to t are plotted in Figure 1. It is clear that after an initial transition period, namely after wave is generated from initial water displacement, the solutions decay monotonically. The corresponding decay rate functions $r(t)$ with $f = \eta$ and $f = u$ with L_∞-norms and L_2-norms are plotted respectively in Figure 2. By calculating the decay rate for t large, as described above, one obtains

$$\begin{aligned} \|\eta\|_{L^\infty} \sim Ct^{-1.20} \quad &\text{and} \quad \|u\|_{L^\infty} \sim Ct^{-1.24}, \\ \|\eta\|_{L^2} \sim Ct^{-0.48} \quad &\text{and} \quad \|u\|_{L^2} \sim Ct^{-0.49}. \end{aligned} \tag{4.2}$$

The Figures and the decay rate of r confirm the theoretical results in Theorem 3.6 and in Theorem 3.9 and the small data requirement might not be necessary if other methods are employed.

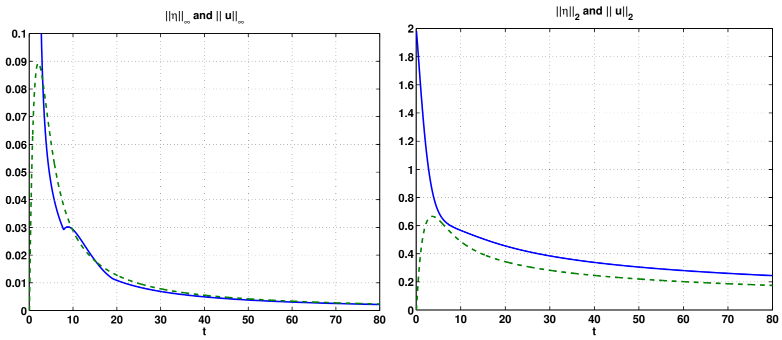

FIGURE 1. The left figure is for $\|\eta\|_\infty$ (solid line) and $\|u\|_\infty$ (dash line) with respect to t and the figure on the right is for $\|\eta\|_{L^2}$ (solid line) and $\|u\|_{L^2}$ (dash line).

The second case in for the partial dissipation $\nu = 0$ on the BBM-BBM system. This is a case which we do not have the theoretical proof. In fact, for the linearized system, one can show, just as in the corresponding one-dimensional case (see [7]), that the solution can decay arbitrarily slow depends on the initial data. But for this initial data, which consists every frequency, we observe the decay rates, which is almost identical to the case of full dissipation,

$$\begin{aligned} \|\eta\|_{L^\infty} \sim Ct^{-1.18} \quad &\text{and} \quad \|u\|_{L^\infty} \sim Ct^{-1.21}, \\ \|\eta\|_{L^2} \sim Ct^{-0.47} \quad &\text{and} \quad \|u\|_{L^2} \sim Ct^{-0.47}. \end{aligned} \tag{4.3}$$

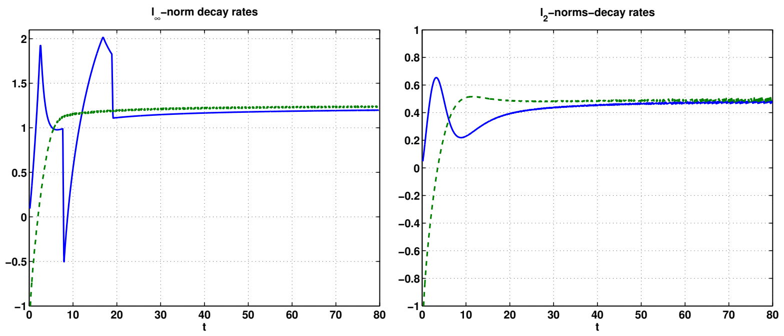

FIGURE 2. The plots of $r(t_n)$ with $\|\eta\|_\infty$: solid line on the left; $\|u\|_\infty$: dash line on the left; $\|\eta\|_{L^2}$: solid line on the right and $\|u\|_{L^2}$: dash line one the right.

We also tested numerically the case where we only apply the dissipation on the first equation. The numerical results read

$$\begin{aligned} \|\eta\|_{L^\infty} \sim Ct^{-1.18} \quad &\text{and} \quad \|u\|_{L^\infty} \sim Ct^{-1.21}, \\ \|\eta\|_{L^2} \sim Ct^{-0.47} \quad &\text{and} \quad \|u\|_{L^2} \sim Ct^{-0.48}. \end{aligned} \tag{4.4}$$

In summary, our numerical simulations confirm the theoretical results and demonstrated the theoretical results are sharp. Furthermore, for the systems we were unable to prove rigorously the decay rates, a prediction is given.

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