HIGHER-ORDER BOUSSINESQ SYSTEMS FOR TWO-WAY PROPAGATIONS OF WATER WAVES

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Abstract. A class of higher-order approximations for the two-way propagation of surface water waves in a channel is derived. These systems, which generalize the classical Boussinesq equations, are formally equivalent and correct through second-order for small-amplitude, long wavelength surface motions. Preliminary analysis reveals significant mathematical differences between these model equations.

1. Introduction.

This work is inspired by the recent work of Bona, Saut and Toland [2] and has as its aim a substantial extension of their results. In [2], a class of systems of partial differential equations that approximate the two-way propagation of surface water waves was derived and analyzed in various respects. These models, which include the classical Boussinesq system of equations, have the same formal status as the Kortweg-deVries equation for the unidirectional propagation of waves in a channel, which means that they are correct through first order with regard to a small parameter characterizing the typical amplitude to depth ratio encountered in the small-amplitude long waves being considered.

Our specific purpose here is to extend this class of models to second order in the relevant small parameter. This is a project that has independent interest, but it also proves helpful to have such accurate model systems available in conjunction with some laboratory experiments on two-way propagation of waves in a channel. While the resulting second-order correct systems of equations are more involved than the class derived in [2], the level of complexity that arises is not an order of magnitude more as regards either mathematical analysis or the potential for associated accurate algorithms for the numerical approximation of waves. Especially this latter aspect appears useful for comparison of theory with the aforementioned laboratory experiments.

The plan of the paper is as follows. Section 2 contains the derivation of the model systems from the full Euler equations governing exactly the propagation of wave motion on the surface of an inviscid liquid. The linearized initial-value problem for these systems is analyzed in Section 3. Section 4 is concerned with some theory about the full, nonlinear systems.

2. Model Systems.

Let Ω_t be the domain in $I\!R^3$ which is occupied by an inviscid, incompressible fluid at time t. The system describing gravity waves on the free surface of an ideal fluid is the Euler equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} + \nabla p = -g\vec{k}, \quad \text{in } \Omega_t, \tag{1}$$

$$\nabla \cdot \vec{v} = 0, \quad \text{in } \Omega_t, \tag{2}$$

where g denotes the acceleration of gravity, $\vec{v} = u\vec{i} + v\vec{j} + w\vec{k}$ denotes the velocity field, $\vec{i}, \vec{j}$, and $\vec{k}$ are the unit vectors along each axis in $I\!R^3$, p denotes the pressure field, and $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^T$. Assuming the initial velocity field is irrotational ($\nabla \times \vec{v} = 0$), Helmholtz's vorticity theorem implies that the velocity field remains irrotational for all time, and hence that $\vec{v} = \nabla\phi$ for some potential function $\phi = \phi((x,y,z),t)$. It follows from (2) that for each t ϕ satisfies Laplace's equation,

$$\Delta\phi = 0. \tag{3}$$

View the boundary of Ω_t as consisting of two parts, the surface which is fixed located at $z = -h(x,y)$, and the free surface $z = \eta(x,y,t)$. Note that $\eta(x,y,t)$ is a fundamental unknown of the problem. On the former, the impermeable boundary condition $\vec{v} \cdot \vec{n} = 0$ is imposed where $\vec{n}$ is the normal direction of the surface, which is to say

$$\phi_x h_x + \phi_y h_y + \phi_z = 0, \text{ on } z = -h(x,y). \tag{4}$$

Since the free surface is a material surface, it satisfies the kinematic condition $\frac{D(\eta - z)}{Dt} = 0$, where $\frac{D}{Dt}$ is the usual material derivative $\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$. In consequence, we have

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0, \quad \text{on } z = \eta(x,y,t). \tag{5}$$

Assuming the pressure on the free surface is equal to the air pressure, it follows that the Bernoulli condition (cf. Whitham [5], pp. 432)

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gz = 0, \quad \text{on } z = \eta(x,y,t), \tag{6}$$

is satisfied on the free surface as well.

Consider now the simpler case of an open channel in which the fluid motion is irrotational, inviscid and uniform in the cross-channel direction and the bottom of the channel is taken to be flat and horizontal. Let h denote the depth of the liquid in its undisturbed state. Suppose the deviation of the free surface to have small amplitude and long wavelength, which is to say that compared to h, $a = \sup|\eta|$ is small and a typical wave length λ is large. Introducing the scaled, dimensionless variables

$$(7) \qquad x = \lambda\tilde{x}, \quad z = h(\tilde{z}-1), \quad \eta = a\tilde{\eta}, \quad t = \lambda\tilde{t}/c_0, \quad \phi = ga\lambda\tilde{\phi}/c_0,$$

and denoting $c_0 = \sqrt{gh}$, $\alpha = a/h$, $\beta = h^2/\lambda^2$, equations (3)–(6) become

$$(8) \qquad \beta\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{z}\tilde{z}} = 0, \qquad \text{in } 0 < \tilde{z} < 1 + \alpha\tilde{\eta}(\tilde{x},\tilde{t}),$$

$$(9) \qquad \tilde{\phi}_{\tilde{z}} = 0, \qquad \text{on } \tilde{z} = 0,$$

$$(10) \qquad \tilde{\eta}_{\tilde{t}} + \alpha\tilde{\phi}_{\tilde{x}}\tilde{\eta}_{\tilde{x}} - \frac{1}{\beta}\tilde{\phi}_{\tilde{z}} = 0, \qquad \text{on } \tilde{z} = 1 + \alpha\tilde{\eta}(\tilde{x},\tilde{t}),$$

$$(11) \qquad \tilde{\eta} + \tilde{\phi}_{\tilde{t}} + \frac{1}{2}\alpha\tilde{\phi}_{\tilde{x}}^2 + \frac{1}{2}\frac{\alpha}{\beta}\tilde{\phi}_{\tilde{z}}^2 = 0, \qquad \text{on } \tilde{z} = 1 + \alpha\tilde{\eta}(\tilde{x},\tilde{t}),$$

for $-\infty < x < \infty$, $t > 0$.

Consideration is further restricted to wave motion for which the classical Stokes number $S = \frac{\alpha}{\beta}$ is of order one. In this circumstance, the two small parameters α and β may be treated on an equal footing. Choosing β to be the primary parameter, we seek to write solutions of (8)-(11) in a Taylor series with respect to β, and thereby to obtain approximate equations corresponding to orders of accuracy characterized by β^n for $n = 1, 2, \cdots$.

This program, which is a standard one (cf. Whitham [5], Ch.13 and [3]), begins by writing $\tilde{\phi}$ in an expansion of the form

$$\tilde{\phi}(\tilde{x},\tilde{z},\tilde{t}) = \sum_{m=0}^{\infty} f_m(\tilde{x},\tilde{t})\tilde{z}^m.$$

Demanding $\tilde{\phi}$ satisfy (8)-(11) and letting $F = f_0$, the velocity potential at the bottom $\tilde{z} = 0$, leads to

$$(12) \qquad \begin{aligned} &\tilde{\eta}_{\tilde{t}} + \alpha\tilde{\eta}_{\tilde{x}}\sum_{k=0}^{\infty}\{\frac{(-1)^k}{(2k)!}\frac{\partial^{2k+1}F}{\partial\tilde{x}^{2k+1}}(1+\alpha\tilde{\eta})^{2k}\}\beta^k \\ &\qquad + \sum_{k=0}^{\infty}\{\frac{(-1)^k}{(2k+1)!}\frac{\partial^{2k+2}F}{\partial\tilde{x}^{2k+2}}(1+\alpha\tilde{\eta})^{2k+1}\}\beta^k = 0, \end{aligned}$$

and

$$\tilde{\eta}+\sum_{k=0}^{\infty}\{\frac{(-1)^k}{(2k)!}\frac{\partial^{2k+1}F}{\partial\tilde{x}^{2k}\partial\tilde{t}}(1+\alpha\tilde{\eta})^{2k}\}\beta^k+\frac{1}{2}\alpha\{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\frac{\partial^{2k+1}F}{\partial\tilde{x}^{2k+1}}(1+\alpha\tilde{\eta})^{2k}\beta^k\}^2$$

$$+\frac{1}{2}\alpha\beta\{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{\partial^{2k+2}F}{\partial\tilde{x}^{2k+2}}(1+\alpha\tilde{\eta})^{2k+1}\beta^k\}^2=0. \tag{13}$$

Account is now taken of the formal order of the various terms appearing in (12) and (13). Keeping only the terms in (12) and (13) which are of order α and β, differentiating the second equation with respect to $\tilde{x}$, and letting $\tilde{u}(\tilde{x},\tilde{t})=\frac{\partial F(\tilde{x},\tilde{t})}{\partial\tilde{x}}$, which is the scaled horizontal velocity at the bottom of the channel, yields a second-order Boussinesq system of equations, namely

$$\begin{aligned}&\tilde{\eta}_{\tilde{t}}+\tilde{u}_{\tilde{x}}+\alpha\tilde{\eta}_{\tilde{x}}\tilde{u}+\alpha\tilde{\eta}\tilde{u}_{\tilde{x}}-\frac{1}{6}\beta\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}-\frac{1}{2}\alpha\beta\tilde{\eta}_{\tilde{x}}\tilde{u}_{\tilde{x}\tilde{x}}\\&\qquad-\frac{1}{2}\alpha\beta\tilde{\eta}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}+\frac{1}{120}\beta^2\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}}=\text{terms cubic in }\alpha,\beta,\end{aligned} \tag{14}$$

$$\begin{aligned}&\tilde{\eta}_{\tilde{x}}+\tilde{u}_{\tilde{t}}-\frac{1}{2}\beta\tilde{u}_{\tilde{x}\tilde{x}\tilde{t}}+\alpha\tilde{u}\tilde{u}_{\tilde{x}}-\alpha\beta\tilde{\eta}\tilde{u}_{\tilde{x}\tilde{x}\tilde{t}}-\alpha\beta\tilde{\eta}_{\tilde{x}}\tilde{u}_{\tilde{x}\tilde{t}}\\&\qquad+\frac{1}{2}\alpha\beta\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{x}\tilde{x}}-\frac{1}{2}\alpha\beta\tilde{u}\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}+\frac{1}{24}\beta^2\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{t}}=\text{terms cubic in }\alpha,\beta.\end{aligned} \tag{15}$$

Our purpose now is to derive a class of systems all of which are formally equivalent to the system displayed in (14)-(15). This will be accomplished by considering changes in the dependent variables and by making use of lower-order relations in higher-order terms. Letting $\tilde{w}$ be the scaled horizontal velocity corresponding to the depth $z=-(1-\theta)h$ below the undisturbed surface, a formal use of Taylor's theorem with remainder shows that

$$\tilde{u}=\tilde{w}+\frac{1}{2}\beta\theta^2\tilde{w}_{\tilde{x}\tilde{x}}+\frac{5}{24}\beta^2\theta^4\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}+O(\beta^3) \tag{16}$$

as $\beta\to 0$. Substituting this relation into (14) and (15) and using the lower order approximations to alter the higher-order terms, one finds by neglecting the cubic terms that

$$\begin{aligned}\tilde{\eta}_{\tilde{t}}-c_1\beta\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{t}}+c_2\beta^2\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{t}}&=-\tilde{w}_{\tilde{x}}-\alpha(\tilde{\eta}\tilde{w})_{\tilde{x}}+c_3\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}}+c_1\alpha\beta(\tilde{\eta}\tilde{w})_{\tilde{x}\tilde{x}\tilde{x}}\\&\qquad-\frac{1}{2}(\theta^2-1)\alpha\beta(\tilde{\eta}\tilde{w}_{\tilde{x}\tilde{x}})_{\tilde{x}}-c_4\beta^2\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}},\end{aligned}$$

$$\begin{aligned}\tilde{w}_{\tilde{t}}-c_5\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}}+c_6\beta^2\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{t}}&=-\tilde{\eta}_{\tilde{x}}+c_7\beta\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}}-\alpha\tilde{w}\tilde{w}_{\tilde{x}}+c_7\alpha\beta(\tilde{w}\tilde{w}_{\tilde{x}})_{\tilde{x}\tilde{x}}\\&\quad-\alpha\beta\tilde{\eta}(\tilde{\eta}_{\tilde{x}\tilde{x}})_{\tilde{x}}-\frac{1}{2}(\theta^2+1)\alpha\beta\tilde{w}_{\tilde{x}}\tilde{w}_{\tilde{x}\tilde{x}}-\frac{1}{2}(\theta^2-1)\alpha\beta\tilde{w}\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}}-c_8\beta^2\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}},\end{aligned}$$

where $0\le\theta^2\le 1$ and the constants $c_1,\cdots,c_8$ are given by

$$
\begin{aligned}
c_1 &= \frac{1}{2}(\theta^2-\frac{1}{3})(1-\lambda_1), & c_2 &= -\frac{5}{24}(\theta^2-\frac{1}{5})^2(1-\lambda_2),\\
c_3 &= -\frac{1}{2}(\theta^2-\frac{1}{3})\lambda_1 & c_4 &= -\frac{1}{4}(\theta^2-\frac{1}{3})^2(1-\lambda_1)+\frac{5}{24}(\theta^2-\frac{1}{5})^2\lambda_2,\\
c_5 &= -\frac{1}{2}(\theta^2-1)\lambda_3, & c_6 &= -\frac{1}{4}(\theta^2-1)^2(1-\lambda_3)+\frac{5}{24}(\theta^2-1)(\theta^2-\frac{1}{5})\lambda_4,\\
c_7 &= \frac{1}{2}(\theta^2-1)(1-\lambda_3), & c_8 &= -\frac{5}{24}(\theta^2-1)(\theta^2-\frac{1}{5})(1-\lambda_4).
\end{aligned}
$$

with $\lambda_i \in I\!R$ for $i = 1, \cdots, 4$.

An appraisal of the scales of the various terms above suggests making the following change of variables:

$$\tilde{x} = \beta^{\frac{1}{2}}\hat{x}, \quad \tilde{t} = \beta^{\frac{1}{2}}\hat{t}, \quad \tilde{\eta} = \beta^{-1}N, \quad \tilde{w} = \beta^{-1}W.$$

In the new variables, the second-order correct system has the form

$$
\begin{aligned}
N_{\hat{t}} - c_1 N_{\hat{x}\hat{x}\hat{t}} + c_2 N_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{t}} &= -W_{\hat{x}} - S(NW)_{\hat{x}} + c_3 W_{\hat{x}\hat{x}\hat{x}}\\
&+ c_1 S(NW)_{\hat{x}\hat{x}\hat{x}} - \frac{1}{2}(\theta^2-1)S(NW_{\hat{x}\hat{x}})_{\hat{x}} - c_4 W_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{x}},
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
W_{\hat{t}} - c_5 W_{\hat{x}\hat{x}\hat{t}} + c_6 W_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{t}} &= -N_{\hat{x}} + c_7 N_{\hat{x}\hat{x}\hat{x}} - SWW_{\hat{x}} + c_7 S(WW_{\hat{x}})_{\hat{x}\hat{x}}\\
-S(NN_{\hat{x}\hat{x}})_{\hat{x}} &- \frac{1}{2}(\theta^2+1)SW_{\hat{x}}W_{\hat{x}\hat{x}} - \frac{1}{2}(\theta^2-1)SWW_{\hat{x}\hat{x}\hat{x}} - c_8 N_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{x}},
\end{aligned}
\tag{18}
$$

where $S = \frac{\alpha}{\beta}$ is the Stokes number defined earlier.

It is worth listing some interesting examples of the systems in (17)-(18).

KdV type: Let $c_1 = c_2 = c_5 = c_6 = 0$ in (17)-(18), a situation obtained by choosing

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 0, \quad \lambda_4 = \frac{6}{5}\frac{(\theta^2-1)}{(\theta^2-\frac{1}{5})} \ (\lambda_4 \text{ is arbitrary if } \theta^2 = \frac{1}{5}).$$

Then the system of equations specializes to

$$
\begin{aligned}
N_{\hat{t}} =& - W_{\hat{x}} - S(NW)_{\hat{x}} - \frac{1}{2}(\theta^2-\frac{1}{3})W_{\hat{x}\hat{x}\hat{x}}\\
&-\frac{1}{2}(\theta^2-1)S(NW_{\hat{x}\hat{x}})_{\hat{x}} - \frac{5}{24}(\theta^2-\frac{1}{5})^2 W_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{x}},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
W_{\hat{t}} =& - N_{\hat{x}} + \frac{1}{2}(\theta^2-1)N_{\hat{x}\hat{x}\hat{x}} - SWW_{\hat{x}} - S(NN_{\hat{x}\hat{x}})_{\hat{x}}\\
&+ (\theta^2-2)SW_{\hat{x}}W_{\hat{x}\hat{x}} - \frac{1}{24}(\theta^2-1)(\theta^2-5)N_{\hat{x}\hat{x}\hat{x}\hat{x}\hat{x}}.
\end{aligned}
\tag{20}
$$

Lower-degree system: Let $c_2 = c_4 = c_6 = c_8 = 0$, which is obtained for $\theta^2 \neq \frac{1}{3}$ by letting

$$\lambda_2 = \lambda_4 = 1, \quad \lambda_1 = 1 - \frac{5}{6}\frac{(\theta^2 - \frac{1}{5})^2}{(\theta^2 - \frac{1}{3})^2}, \quad \lambda_3 = \frac{(\theta^2 - 5)}{6(\theta^2 - 1)} \; (\lambda_3 \text{ is arbitrary if } \theta^2 = 1),$$

for which values one finds

$$\begin{aligned} c_1 &= \frac{5}{12}\frac{(\theta^2 - \frac{1}{5})^2}{(\theta^2 - \frac{1}{3})}, \quad c_3 = -\frac{1}{2}(\theta^2 - \frac{1}{3}) + \frac{5}{12}\frac{(\theta^2 - \frac{1}{5})^2}{(\theta^2 - \frac{1}{3})}, \\ c_5 &= -\frac{1}{12}(\theta^2 - 5), \quad c_7 = \frac{5}{12}(\theta^2 - \frac{1}{5}). \end{aligned} \tag{21}$$

The corresponding system of equations, which has degree three, is

$$\begin{aligned} N_{\hat{t}} - c_1 N_{\hat{x}\hat{x}\hat{t}} &= -W_{\hat{x}} - S(NW)_{\hat{x}} \\ &\quad + c_3 W_{\hat{x}\hat{x}\hat{x}} + c_1 S(NW)_{\hat{x}\hat{x}\hat{x}} - \frac{1}{2}(\theta^2 - 1)S(NW_{\hat{x}\hat{x}})_{\hat{x}}, \\ W_{\hat{t}} - c_5 W_{\hat{x}\hat{x}\hat{t}} &= -N_{\hat{x}} + c_7 N_{\hat{x}\hat{x}\hat{x}} - SWW_{\hat{x}} - S(NN_{\hat{x}\hat{x}})_{\hat{x}} \\ &\quad + \frac{3}{4}(\theta^2 - 1)SW_{\hat{x}}W_{\hat{x}\hat{x}} - \frac{1}{12}(\theta^2 - 5)SWW_{\hat{x}\hat{x}\hat{x}}. \end{aligned}$$

Regularized Long-Wave type: There are many options for obtaining systems of equations which correspond to the regularized long-wave equation in the modeling of unidirectional waves. To make the linearized equation well posed (see Section 3), it suffices to require

$$c_1 \geq 0, \quad c_2 > 0, \quad c_5 \geq 0, \quad c_6 > 0, \quad c_4 = 0, \quad c_8 = 0,$$

which is equivalent as requiring

$$\frac{1}{3} < \theta^2 < 1, \quad \lambda_1 = 1 - \frac{5}{6}(\frac{\theta^2 - 1/5}{\theta^2 - 1/3})^2\lambda_2, \quad \lambda_2 > 1, \quad \lambda_3 > \frac{5 - \theta^2}{6(1 - \theta^2)}, \quad \lambda_4 = 1.$$

An example of these systems is discussed in Section 4.

3. Linearized Systems.

In this section, consideration is given to the linear part of the system (17), (18). To simplify the notation, the circumflexes are dropped over the independent variables. The system under consideration is

$$N_t - c_1 N_{xxt} + c_2 N_{xxxxt} = -W_x + c_3 W_{xxx} - c_4 W_{xxxxx}, \tag{22}$$

$$W_t - c_5 W_{xxt} + c_6 W_{xxxxt} = -N_x + c_7 N_{xxx} - c_8 N_{xxxxx}, \tag{23}$$

$$N(x,0) = \phi(x), \quad W(x,0) = \psi(x), \tag{24}$$

for $x \in \mathbb{R}$, where ϕ and ψ are selected from a class of functions which vanish at $\pm\infty$ and belong to $H^s(\mathbb{R})$ for some $s \geq 0$.

Since the nonlinear terms have been dropped, the system can be analyzed through Fourier analysis. Taking the Fourier transform of (22) and (23) with respect to the spatial variable x, we see that

$$\frac{d}{dt}\begin{pmatrix} \widehat{N} \\ \widehat{W} \end{pmatrix} = A(k)\begin{pmatrix} \widehat{N} \\ \widehat{W} \end{pmatrix}, \quad \text{where} \quad A(k) = \begin{pmatrix} 0 & \omega_1(k) \\ \omega_2(k) & 0 \end{pmatrix}$$

and

$$\omega_1(k) = -ik\frac{1 + c_3k^2 + c_4k^4}{1 + c_1k^2 + c_2k^4}, \quad \omega_2(k) = -ik\frac{1 + c_7k^2 + c_8k^4}{1 + c_5k^2 + c_6k^4}.$$

The solution of the system is

$$\begin{pmatrix} \widehat{N} \\ \widehat{W} \end{pmatrix}(k,t) = e^{A(k)t}\begin{pmatrix} \widehat{\phi} \\ \widehat{\psi} \end{pmatrix}(k). \tag{25}$$

The eigenvalues of the matrix $A(k)$ are $\pm\lambda(k)$ where

$$\lambda(k) = [\omega_1(k)\omega_2(k)]^{\frac{1}{2}} \quad \text{when } \omega_1(k)\omega_2(k) \geq 0,$$
$$\lambda(k) = i[-\omega_1(k)\omega_2(k)]^{\frac{1}{2}} \quad \text{when } \omega_1(k)\omega_2(k) < 0.$$

Since a function $m(k)$ is a Fourier multiplier in L_2 if and only if $m(k)$ is in L_∞ (see [4]), it follows that (22)-(24) is L_2-well posed if and only if $Re(\lambda(k))$ is a bounded function of k.

Theorem 3.1. *For a fixed θ, the linear problem (22)-(24) is L_2-well posed if and only if $\omega_1\omega_2$ is bounded from above. That is, there exists a positive constant M, such that for all k, $\omega_1(k)\omega_2(k) \leq M$.*

Applying the theorem to the sample systems listed in the last section yields the following corollaties.

Corollary 3.2. *For the KdV-type system, the linearized problem is L_2-well posed for any value of θ in $[0,1]$.*

Corollary 3.3. *For the lower-degree system, the linearized problem is L_2-well posed if and only if $\frac{1}{3} < \theta^2 \leq 1$ or $\theta^2 = \frac{1}{5}$.*

Corollary 3.4. *For the regularized long-wave type systems, the linearized problem is L_2-well posed for any θ in $[0,1]$.*

4. Local Wellposedness of a Sample System.

We take a system of equations from the class of regularized long-wave type in Section 2. If we set $\theta^2 = 2/3,\ \lambda_1 = -34/15, \lambda_2 = 2, \lambda_3 = 19/5, \lambda_4 = 1$, the full system of equations is

$$
\begin{aligned}
N_t - \frac{49}{90}N_{xxt} + \frac{49}{1080}N_{xxxxt} &= -W_x - (NW)_x \\
&\quad + \frac{17}{45}W_{xxx} + \frac{49}{90}(NW)_{xxx} + \frac{1}{6}(NW_{xx})_x,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
W_t - \frac{19}{30}W_{xxt} + \frac{49}{1080}W_{xxxxt} &= -N_x - WW_x + \frac{7}{15}N_{xxx} \\
&\quad - (NN_{xx})_x + \frac{19}{30}(WW_{xx})_x + \frac{52}{15}W_xW_{xx},
\end{aligned}
\tag{27}
$$

$$
N(x,0) = \phi(x), \quad W(x,0) = \psi(x).
$$

in which we have taken $S = 1$ for simplicity of notation.

Uniqueness and existence result over a short time interval can be obtained by first converting (26)-(27) into a coupled system of integral equations. The details follow the line already appearing in Benjamin, Bona, and Mahony [1].

Theorem 4.1. *(Uniqueness) Corresponding to given initial data $(\phi, \psi) \in H^1 \times H^1$, there is at most one solution pair (N, W) defined on $\mathbb{R} \times [0, T]$ to the system of integral equations such that $(N, W) \in C(0, T; H^1) \times C(0, T; H^1)$.*

Theorem 4.2. *(Local existence) Let $(\phi, \psi) \in H^1 \times H^1$ and let $b = \|\phi\|_1 + \|\psi\|_1$. Then there exists $T = T(b) > 0$ such that the integral equations have a solution pair $(N, W) \in C(0, T; H^1) \times C(0, T; H^1)$.*

Remark 4.3. Solutions (N, W) of the integral equation with the regularity indicated in Theorem 4.2 are straightforwardly inferred to satisfy the differential sysytem (26) and (27) (see [3]).

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