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Exact Solutions of Various Boussinesq Systems

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Abstract—It was shown in [1,2] that surface water waves in a water tunnel can be described by systems of the form

$$\begin{aligned} \eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} &= 0, \\ u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} &= 0, \end{aligned} \tag{1}$$

where a, b, c, and d are real constants. In this paper, we show that to find an exact traveling-wave solution of the system, it is suffice to find a solution of an ordinary differential equation, and the solution of the ordinary differential equation in a prescribed form can be found by solving a system of nonlinear algebraic equation. The exact solutions for some of the systems are presented at the end of the paper. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

To describe small amplitude and long waves in a water channel, systems in the form of (1), which include the classical Boussinesq system (cf. [3]), were derived by Bona, Saut and Toland in [1], where a, b, c, and d are real constants and determined by three parameters λ, μ, and $0 \leq \theta \leq 1$ in the following way:

$$a = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda, \qquad b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\lambda),$$
$$c = \frac{1}{2}\left(1-\theta^2\right)\mu, \qquad d = \frac{1}{2}\left(1-\theta^2\right)(1-\mu).$$

The dimensionless variables x and t are scaled, respectively, by h and $(h/g)^{1/2}$ where h denotes the undisturbed water depth and g denotes the acceleration of gravity. The variable $\eta(x,t)$ is the nondimensional deviation of the water surface (scaled by h) from its undisturbed position and $u(x,t)$ is the nondimensional horizontal velocity (scaled by $\sqrt{gh}$) at a height θh with $0 \leq \theta \leq 1$ above the bottom of the channel. These three parameter family of systems are formally equivalent and correct through first order with regard to the small parameter $\epsilon = \sup\{\eta(x,t)\}$. In this paper, we concentrate on finding exact traveling-wave solutions of (1) which approach constants at infinities. The existence of these solutions is useful in the theoretical and numerical studies of

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the model systems. In fact, one of the exact solution we found here for the regularized Boussinesq system ($a = c = 0$, $b = d = 1/6$) has been used in [2] to demonstrate the convergence rate of a numerical algorithm.

2. MAIN RESULTS

Denoting $\xi = x + x_0 - C_s t$ with x_0 and C_s being constants, we first present the result on the existence of traveling-wave solution

$$\eta(x,t) = \eta(\xi), \qquad u(x,t) = u(\xi), \tag{2}$$

that $\eta(\xi)$ and $u(\xi)$ are asymptotically small at large ξ and proportional to each other, so

$$\lim_{\xi\to\pm\infty} (\eta(\xi), u(\xi)) = 0, \qquad \eta(x,t) = Bu(x,t), \tag{3}$$

with B being a constant. Substituting (2) and (3) into system (1) and using the fact that the resulting two equations are consistent, one can prove the following.

THEOREM. *For a given system in the form of (1), if the constants a, b, c, d satisfy one of the following conditions:*

(i) $a - b + 2d \neq 0$, $p = (-b + c + 2d)/(a - b + 2d) > 0$, *and* $(p - 1/2)((b - a)p - b) > 0$;
(ii) $a = b = c > 0$, $d = 0$;
(iii) $a = b = c < 0$, $d = 0$;
(iv) $a - b + 2d = 0$, $a = c$, $d > 0$;
(v) $a - b + 2d = 0$, $a = c$, $d < 0$;

then the given system has solitary-wave solutions. Moreover, the exact solitary-wave solutions are of the form

$$\eta(x,t) = \eta_0 \operatorname{sech}^2(\lambda(x + x_0 - C_s t)),$$
$$u(x,t) = \pm\sqrt{\frac{3}{\eta_0 + 3}}\,\eta_0 \operatorname{sech}^2(\lambda(x + x_0 - C_s t)),$$

where

$$C_s = \frac{3 + 2\eta_0}{\pm\sqrt{3(3+\eta_0)}}, \qquad \lambda = \frac{1}{2}\sqrt{\frac{2\eta_0}{3(a-b) + 2b(\eta_0 + 3)}},$$

and η_0 can be any constant satisfies

- *in Case (i),* $\eta_0 = (3(1 - 2p))/2p$;
- *in Case (ii),* $0 < \eta_0 < +\infty$;
- *in Case (iii),* $-3 \le \eta_0 < 0$;
- *in Case (iv),* $\eta_0 > -3$ *and* $3/(\eta_0 + 3)$ *is not in the closed interval between* 1 *and* b/d;
- *in Case (v),* $\eta_0 > -3$ *and* $3/(\eta_0 + 3)$ *is in the closed interval between* 1 *and* b/d.

With a more general approach, one can find exact solutions where $u(\xi)$ and $\eta(\xi)$ are not proportional to each other and they do not approaches zero at infinity. Assuming that the traveling-wave solution $(u(\xi), \eta(\xi))$ tends to (u_∞, η_∞) as ξ tends to $\pm\infty$. Substituting functions

$$h(\xi) = \eta(\xi) - \eta_\infty, \qquad v(\xi) = u(\xi) - u_\infty, \tag{4}$$

into (1) and integrating the system once, one obtains

$$\begin{aligned} -C_s h + v + vh + \eta_\infty v + u_\infty h + av'' + bC_s h'' &= 0, \\ -C_s v + h + \frac{1}{2}v^2 + u_\infty v + ch'' + dC_s v'' &= 0. \end{aligned} \tag{5}$$

Eliminating one of the dependent variables, one can find that $v(\xi)$ (or $h(\xi)$) satisfies a forth-order ordinary differential equations (cf. [4]). For instance, in the case that $c \neq 0$, one can eliminate $h(\xi)$ and obtain an ordinary differential equation on $v(\xi)$ as follows. Notice from (5) that h and h'' can be expressed as a function of $v(\xi)$,

$$h = \frac{g_1(v)}{f(v)} \quad \text{and} \quad h'' = \frac{g_2(v)}{f(v)}, \tag{6}$$

where

$$\begin{aligned} f(v) &= c(-C_s + v + u_\infty) - bC_s, \\ g_1(v) &= c\left(-v - av'' - \eta_\infty v\right) - bC_s\left(C_s v - \frac{1}{2}v^2 - dC_s v'' - u_\infty v\right), \\ g_2(v) &= v + av'' + \eta_\infty v + \left(-C_s + v + u_\infty\right)\left(C_s v - \frac{1}{2}v^2 - dC_s v'' - u_\infty v\right). \end{aligned}$$

Differentiating the first equation in (6) twice with respect to ξ and using the second equation, one finds

$$f^2 g_2 = g_1'' f^2 - g_1 f'' f - 2g_1' f f' + 2g_1 \left(f'\right)^2, \tag{7}$$

which is an ordinary differential equation with dependent variable $v(\xi)$. One can therefore established the fact that in order to find a traveling-wave solution of (1), it is suffice to find a solution $v(\xi)$ satisfying the ordinary differential equation.

Notice again that the ordinary differential equation (7) involves only

$$v'''', \quad v'v''', \quad \left(v''\right)^2, \quad v'', \quad \left(v'\right)^2$$

terms, the Ansatz equation

$$\left(v'\right)^2 = \rho v^2 + \sigma v^3, \qquad \rho \geq 0, \tag{8}$$

can be used to find solutions in the form of

$$v(\xi) = -\frac{\rho}{\sigma} \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right) \tag{9}$$

(cf. [6]). Substituting (8) into (9) yields a polynomial equation on $v(\xi)$ where the coefficients depend on ρ, σ, C_s, u_∞, and η_∞. By requiring the coefficients to be zero, one obtains a system of algebraic equations and the solution ρ, σ, C_s, u_∞, and η_∞ provides the solution of ordinary differential equation in the form of (9), which in turn yields the exact traveling-wave solution of the system with the help of (6) and (4).

The method described above is used on a large class of the systems in (1) which includes the system in [5] (formula (13.101)), the systems in [6], regularized Boussinesq system in [1], Boussinesq's original system (cf. [3]), and the integrable version of the Boussinesq system (cf. [7]). The exact traveling-wave solutions founded are listed in next section. The method presented in this paper is quite general and it recovered the solutions founded in [8,9], where a homogeneous balance method was used.

Other Ansatz equations can be used to find solutions in different forms [10].

3. EXACT TRAVELING-WAVE SOLUTIONS FOR SYSTEMS IN (1)

Denote $\xi = x + x_0 - C_s t$, where x_0 and C_s are arbitrary constants, one can find the exact traveling-wave solutions for the following systems ($\rho \geq 0$ is an arbitrary constant).

- $a = 0$:

$$\begin{aligned} u(\xi) &= (1 - d\rho)\, C_s + 3\, dC_s \rho \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right), \\ \eta(\xi) &= -1. \end{aligned}$$

- $a = 0,\ c \neq 0$:

$$u(\xi) = \frac{C_s}{2c}\left(-b + 2c + 2d - bc\rho\right) + \frac{3}{2}C_s b\rho \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right),$$

$$\eta(\xi) = -1 + \frac{C_s^2}{4c^2}\left(b^2 - 4bd + 4d^2 - b^2c\rho + 2bcd\rho\right) + \frac{3C_s^2}{4c}b\left(b - 2d\right)\rho \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\,\xi\right).$$

- $a = c = 0$:

$$u(\xi) = \frac{C_s}{3}\left(3 - 5b\rho\right) + 5C_s b\rho \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right),$$

$$\eta(\xi) = -1 + \frac{C_s^2}{9}b\left(10b - 6d\right)\rho^2 + \frac{5}{6}C_s^2 b\left(5b - 3d\right)\rho^2\left(2\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right) - 3\operatorname{sech}^4\left(\frac{1}{2}\sqrt{\rho}\xi\right)\right).$$

- $b = c = 0,\ d \neq 0$:

$$u(\xi) = \frac{-a + 2dC_s^2 - 2d^2C_s^2\rho}{2dC_s} + 3dC_s\rho \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right),$$

$$\eta(\xi) = -1 + \frac{a}{4d^2C_s^2}\left(a - 2d^2C_s^2\rho\right) + \frac{3}{2}a\rho \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right).$$

- $c = d = 0,\ b \neq 0$:

$$u(\xi) = \frac{-a}{4bC_s} + C_s - \frac{5}{3}C_s b\rho + 5C_s b\rho \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right),$$

$$\eta(\xi) = -1 + \frac{a^2}{16b^2C_s^2} - \frac{5}{12}a\rho + \frac{10}{9}C_s^2b^2\rho^2 + \left(\frac{5}{4}a\rho + \frac{25}{3}C_s^2b^2\rho^2\right)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right) - \frac{25}{2}b^2k^2\rho^2 \operatorname{sech}^4\left(\frac{1}{2}\sqrt{\rho}\xi\right).$$

- $b = c = d = 0,\ a > 0$:

$$u(\xi) = C_s \pm \sqrt{a\rho}\tanh\left(\frac{1}{2}\sqrt{\rho}\xi\right),$$

$$\eta(\xi) = -1 + \frac{1}{2}a\rho \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right).$$

- $b = c = d = 0,\ a < 0$:

$$u(\xi) = C_s \pm \sqrt{-a\rho}\operatorname{sech}\left(\frac{1}{2}\sqrt{\rho}\xi\right),$$

$$\eta(\xi) = -1 - \frac{1}{4}a\rho + \frac{1}{2}a\rho \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right).$$

- $b = d = 0,\ a = c$:

$$u(\xi) = \frac{\mp\sqrt{2}\left(1 + c\rho\right) + 2C_s}{2} \pm \frac{3c\rho}{\sqrt{2}}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right),$$

$$\eta(\xi) = -\frac{1 + c\rho}{2} + \frac{3c\rho}{2}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\rho}\xi\right).$$

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