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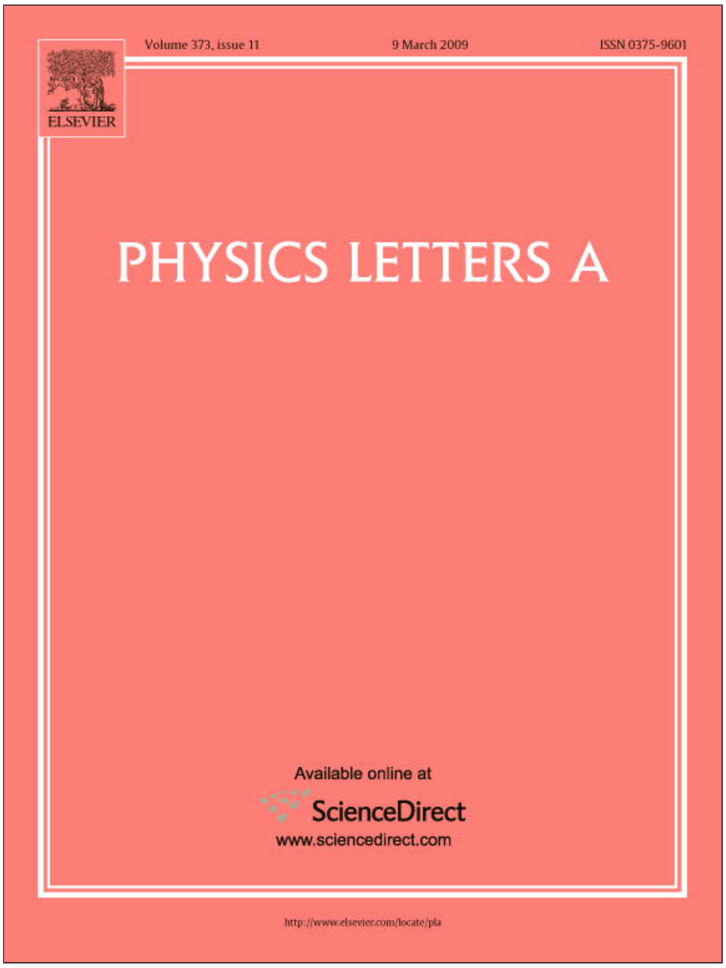

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Existence of traveling wave solutions of a high-order nonlinear acoustic wave equation

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ABSTRACT

In this Letter, we present an analytical study of a high-order acoustic wave equation in one dimension, and reformulate a previously given equation in terms of an expansion of the acoustic Mach number. We search for non-trivial traveling wave solutions to this equation, and also discuss the accuracy of acoustic wave equations in terms of the range of Mach numbers for which they are valid.

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1. Background

Traveling wave solutions in the form of solitons have been studied in detail for nonlinear wave equations of the KdV type, as well as in other areas of physics. In the case of acoustic wave equations, these solutions have received considerably less attention. In a recent series of papers, Sugimoto et al. [1–3] demonstrated the existence of acoustic solitary waves in an air-filled tube containing a periodic array of Helmholtz resonators. In [1,2] the problem was studied theoretically, and then in [3] the results were confirmed by a set of laboratory experiments. In another series of papers, Jordan [4] studied diffusive soliton solutions to Kuznetsov's equation, which models weakly-nonlinear acoustic wave propagation, and then Jordan and Puri [5,6] applied similar analysis techniques to the problem of traveling wave solutions in nonlinear viscoelastic media. The governing equations in the viscoelastic case are similar to the acoustic wave equation. In another related work, Jordan [7] studied finite amplitude waves in a porous medium. Rasmussen et al. [8] derived an alternative nonlinear wave equation, and then derived an analytical traveling wave solution that allowed for studying front interaction.

The classical theory of nonlinear acoustics, as given in Refs. [9,17–20], gives the speed of wave propagation in an adiabatic fluid as

$$c = c_0 \pm \frac{1}{2}(\gamma - 1)u \tag{1}$$

where c_0 is the small-signal speed of sound in a linear fluid, γ is the ratio of specific heats, and u is the particle velocity of the fluid. We note that u varies with both space and time. Since u in Eq. (1) varies with amplitude of the wave, the areas of higher amplitude in a wave will propagate with a faster speed than those of lower amplitude. This will lead to shock formation.

Eq. (1) shows that initially smooth waves (with smooth input signals) in a nonlinear lossless fluid will eventually steepen to form shocks, and thus cannot propagate as traveling waves, since the speed of the wave always depends on position in the waveform. The interesting question is then to consider the lossy terms in the equations of motion, along with the nonlinear terms, and to assess if traveling wave solutions are possible in the presence of both lossy and nonlinear terms.

In this Letter, we extend recent results by Jordan [4], who studied traveling wave solutions to the Kuznetsov equation, which models nonlinear acoustic waves in lossy fluids up to second order. In our approach, we use a higher-order equation [9–11], which is valid up to higher acoustic Mach numbers than Kuznetsov's equation. Since the speed of the traveling wave depends on the acoustic Mach number, this high-order equation allows for a more accurate assessment of traveling wave velocities.

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In the case of linear acoustic theory, it is easy to see that traveling wave solutions exist, since there are no dissipative or nonlinear effects that would distort the waves. The more interesting case is whether these solutions can exist when nonlinear and dissipative terms are included. In most cases, it depends on the physical constants involved, as we will show here.

It is instructive to begin the discussion with the linear wave equation, which in one dimension is given as

$$c_0^2\phi_{xx} - \phi_{tt} = 0 \tag{2}$$

where ϕ is the velocity potential, and c_0 is the linear speed of sound. Traveling waves exist for this equation, and are of the form given by the d'Alembert solution

$$\phi(x \pm vt) \tag{3}$$

where $v = c_0$ is the wave speed.

In 1971, Kuznetsov [12] derived a nonlinear acoustic wave equation that extended the linear wave equation to include dissipative and nonlinear effects. The equation takes the form

$$c_0^2\phi_{xx} - \phi_{tt} + \nu(\gamma)\phi_{txx} = \frac{\partial}{\partial t}\left[(\phi_x)^2 + \frac{1}{c_0^2}(\beta-1)(\phi_t)^2\right] \tag{4}$$

where β is the coefficient of nonlinearity, $\gamma = \frac{C_p}{C_v}$ is the ratio of specific heats, and $\nu(\gamma)$ is the diffusivity of sound. We note that the first two terms of Kuznetsov's equation are the same as the linear wave equation, (2). The additional terms account for dissipation and nonlinear effects. However, Kuznetsov's equation is only a second order equation in terms of the nonlinearities, which means that it is only valid for certain ranges of acoustic Mach numbers. More on this will be given later in this section, when we re-write this equation in a nondimensional form.

A higher order acoustic wave equation (HOAWE) exists for gases [9–11]. This equation uses a more accurate equation of state, rather than the Taylor series expansion used in Kuznetsov's equation. Consequently, it is valid for larger values of acoustic Mach number, and thus represents a more accurate model of acoustic wave propagation than Kuznetsov's equation. The equation as given by Söderholm [10,11] is as follows

$$c_0^2\phi_{xx} - \phi_{tt} + \nu(\gamma)\phi_{txx} = \left((\phi_x)^2\right)_t + \frac{1}{2}\phi_x\left((\phi_x)^2\right)_x + (\gamma-1)\left[\phi_t + \frac{1}{2}(\phi_x)^2\right]\phi_{xx} \tag{5}$$

where, for gases, γ and β are related by $\beta = \frac{\gamma+1}{2}$. We note that this equation is a generalization of the relation for lossless gases given by Eq. (3.26) in [9], the difference being the dissipative term $\nu(\gamma)\phi_{txx}$. Also, as with Kuznetsov's equation, this equation includes the linear wave equation as a special case.

Although it is clear that the two wave equations, (4) and (5), reduce to the linear wave equation (2) when nonlinear and dissipative effects are neglected, the various physical constants make the relative magnitudes of the terms difficult to interpret. Hence, we show here how all three can be written in a nondimensional form, thus facilitating their comparison and analysis. The dimensional analysis procedure follows one that was originally given by Wojcik [13], and was followed on by Jordan [4]. Defining a characteristic flow speed V and characteristic length scale L, we can define a nondimensional velocity potential as $\Phi = \frac{\phi}{VL}$. We also define the nondimensional time $T = \frac{c_0 t}{L}$, and nondimensional position $X = \frac{x}{L}$. Then, the following relations can be derived between the first time and spatial derivatives

$$\phi_t = V c_0 \Phi_T, \qquad \phi_x = V\Phi_X. \tag{6}$$

Using these relations, we can also derive the following relations for the higher derivatives and nonlinear terms

$$\phi_{xx} = \frac{V}{L}\Phi_{XX}, \qquad \phi_{tt} = \frac{Vc_0^2}{L}\Phi_{TT}, \qquad \left((\phi_x)^2\right)_t = \frac{V^2c_0}{L}\left((\Phi_X)^2\right)_T, \qquad \left((\phi_t)^2\right)_t = \frac{V^2c_0^3}{L}\left((\Phi_T)^2\right)_T, \qquad \phi_{xxt} = \frac{Vc_0}{L^2}\Phi_{XXT}. \tag{7}$$

Substituting Eqs. (6) and (7) into the original wave equations (2), (4), and (5), we obtain the following nondimensional equations

$$\Phi_{TT} - \Phi_{XX} = 0, \tag{8}$$

$$\Phi_{XX} - \Phi_{TT} + \frac{1}{Re}\Phi_{XXT} = \epsilon\frac{\partial}{\partial T}\left[(\Phi_X)^2 + (\beta-1)(\Phi_T)^2\right], \tag{9}$$

$$\Phi_{XX} - \Phi_{TT} + \frac{1}{Re}\Phi_{XXT} = \epsilon\left[\left((\Phi_X)^2\right)_T + (\gamma-1)\Phi_T\Phi_{XX}\right] + \epsilon^2\left[(\Phi_X)^2\Phi_{XX} + \frac{(\gamma-1)}{2}(\Phi_X)^2\Phi_{XX}\right], \tag{10}$$

where $\epsilon = \frac{V}{c_0}$ is the acoustic Mach number, and $Re = \frac{c_0 L}{\nu(\gamma)}$ is the Reynold's number.

Eqs. (8), (9), and (10) represent the nondimensional forms of the linear, Kuznetsov, and HOAWE wave equation, respectively. Derivations of the Kuznetsov and HOAWE equations from fundamental principles can be found in Refs. [9–11,14]. We note that the linear wave equation neglects all nonlinear effects, the Kuznetsov equation represents nonlinear effects to the first power in ϵ, and the HOAWE includes both linear and quadratic terms in ϵ.

A recent study [4] focused on searching for traveling wave solutions to the Kuznetsov equation. Although traveling wave solutions were shown to exist, the wave speed depended on the acoustic Mach number, and in fact had a bifurcation depending on the physical constants. No traveling wave solutions were possible above a certain critical acoustic Mach number. However, since the Kuznetsov equation itself is restricted to small values of the acoustic Mach number, it was not clear if the critical Mach number obtained in [4] exceeded its inherent limitations. For example, Makarov and Ochmann [14] suggest that the Kuznetsov equation is only applicable for $\epsilon < 0.1$.

In our approach, we will search for traveling wave solutions to the HOAWE equation. Since this equation is valid for larger values of the acoustic Mach number, it will allow for traveling wave solutions with a wider range of wave speeds than was obtained in [4]. In the limit of small Mach numbers, we will show that the traveling wave speeds determined from the HOAWE equation are identical to those obtained in [4] for the Kuznetsov equation. This makes sense, since the two equations model the same physics for small ϵ.

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2. Existence of traveling waves solutions for HOAWE

In this section we show the existence of traveling wave solutions to the HOAWE, Eq. (10). Our results depend on the value of γ which is in the range of 1.1 to 1.7 for most monoatomic and polyatomic gases.

We seek traveling wave solutions to Eq. (10), which take the form

$$\Phi(X,T) = \Phi(X - vT) = \Phi(\xi) \tag{11}$$

where $\xi = X - vT$, and $v > 0$ is the speed of the wave. At this point, v is unknown, since it is not known a priori what the speed of the traveling waves will be. Substituting Eq. (11) into (10), we obtain

$$\Phi'' - v^2\Phi'' - \frac{v}{Re}\Phi''' = \varepsilon\left[-2v\Phi'\Phi'' - (\gamma-1)v\Phi'\Phi''\right] + \varepsilon^2\left[(\Phi')^2\Phi'' + \frac{(\gamma-1)}{2}(\Phi')^2\Phi''\right]. \tag{12}$$

The following relations

$$\Phi'\Phi'' = \frac{1}{2}\frac{d}{d\xi}(\Phi')^2, \qquad (\Phi')^2\Phi'' = \frac{1}{3}\frac{d}{d\xi}(\Phi')^3$$

lead to the equation

$$\left(1 - v^2\right)\Phi'' - \frac{v}{Re}\Phi''' + \varepsilon\left[\frac{(\gamma+1)v}{2}\frac{\partial}{\partial\xi}(\Phi')^2\right] - \varepsilon^2\left[\frac{(\gamma+1)}{6}\frac{\partial}{\partial\xi}(\Phi')^3\right] = 0. \tag{13}$$

Also, we note that an integration in (13) can be performed in ξ, with

$$\int_0^{\xi} \Phi'\,d\xi = \Phi(\xi) - \Phi(0), \tag{14}$$

and similarly for Φ'' and Φ'''. Collecting all integration constants in the constant c on the right-hand side of (13) we obtain

$$\left(1 - v^2\right)\Phi' - \frac{v}{Re}\Phi'' + \frac{\varepsilon v(\gamma+1)}{2}(\Phi')^2 - \frac{\varepsilon^2(\gamma+1)}{6}(\Phi')^3 = c. \tag{15}$$

If we make the substitution $w = \Phi'$, and multiply through by $-\frac{Re}{v}$, we obtain

$$w' - \frac{Re(1-v^2)}{v}w - \frac{\varepsilon\,Re(\gamma+1)}{2}w^2 + \frac{\varepsilon^2 Re(\gamma+1)}{6v}w^3 = c. \tag{16}$$

Eq. (16) is an Abel equation of the first kind [15]. It is a generalization of the Ricatti equation (see (41) in the next section) which appears when searching for traveling wave solutions of Kuznetsov's equation. In this case, the w^3 term is a direct consequence of the terms of the type $(\Phi)^3$ in Eq. (13). These terms are not present in Kuznetsov's equation.

By denoting

$$a_1 := \frac{Re(1-v^2)}{v}, \qquad a_2 := \frac{\varepsilon\,Re(\gamma+1)}{2} > 0, \qquad a_3 := -\frac{\varepsilon^2 Re(\gamma+1)}{6v} < 0, \tag{17}$$

Eq. (16) becomes

$$w' = a_1 w + a_2 w^2 + a_3 w^3 + c = p(w). \tag{18}$$

In order to show the existence of solutions to Abel's equation (16) we denote one of the three roots of the cubic polynomial $p(w)$ as w_1. Given w_1, we define $h_{\gamma,\varepsilon}(v, w_1)$ as

$$h_{\gamma,\varepsilon}(v, w_1) := (3\gamma - 5)v^2 + 2\varepsilon(\gamma+1)w_1 v + 8 - \varepsilon^2(\gamma+1)w_1^2. \tag{19}$$

The following theorem then holds.

Theorem 2.1. *Let $\varepsilon > 0$ be the Mach number and $\gamma > 1$. If v, w_1 are such that $h_{\gamma,\varepsilon}(v, w_1) > 0$ then there are one or two bounded traveling wave solutions of (16) with v denoting the speed of the traveling wave. In particular, this is true for any $v > 0$ and*

$$\frac{1}{\varepsilon}\left(v - \sqrt{\frac{4v^2(\gamma-1)+8}{\gamma+1}}\right) < w_1 < \frac{1}{\varepsilon}\left(v + \sqrt{\frac{4v^2(\gamma-1)+8}{\gamma+1}}\right). \tag{20}$$

Proof. Let w be the solution of (16) which approaches to w_1 at ∞ (or $-\infty$). We compute

$$c = -a_1 w_1 - a_2 w_1^2 - a_3 w_1^3,$$

and hence (18) becomes

$$w' = a_1(w - w_1) + a_2\left(w^2 - w_1^2\right) + a_3\left(w^3 - w_1^3\right) = (w - w_1)\left[a_1 + a_2(w + w_1) + a_3\left(w^2 + ww_1 + w_1^2\right)\right] = (w - w_1)g(w),$$

where

$$g(w) = a_3 w^2 + (a_3 w_1 + a_2)w + a_1 + a_2 w_1 + a_3 w_1^2. \tag{21}$$

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If the discriminant of this quadratic form, Δ, is positive where

$$\Delta := (a_3 w_1 + a_2)^2 - 4a_3\left(a_1 + a_2 w_1 + a_3 w_1^2\right) = a_3^2 w_1^2 + 2a_2 a_3 w_1 - 4a_3\left(a_1 + a_2 w_1 + a_3 w_1^2\right) + a_2^2 = -a_3\left[4a_1 + 2a_2 w_1 + 3a_3 w_1^2\right] + a_2^2,$$

it follows that

$$w' = a_3(w - w_1)(w - w_2)(w - w_3) = p(w), \tag{22}$$

where $w_2 \neq w_3$ are the real roots of (21). Thus, the cubic polynomial $p(w)$ can have three or two different real roots. In the first case, there exist two different bounded solutions of (18) and in the second case, $w_2 = w_1$ or $w_3 = w_1$ meaning that w_1 is also a root of $g(w)$, there is only one bounded solution w.

In the first case, by relabeling the roots if necessary, we can write $w_2 < w_1 < w_3$. Since the constant functions w_1, w_2 and w_3 solve the equation $w' = p(w)$, the theory of ODE [16] implies that there exist two different bounded solutions w and $\tilde{w}$ of (18). Since $a_3 < 0$, we have $w' = p(w) < 0$ if $w_2 < w < w_1$ and hence

$$\lim_{\xi \to -\infty} w(\xi) = w_1 \quad \text{and} \quad \lim_{\xi \to +\infty} w(\xi) = w_2.$$

Also, $w' = p(w) > 0$ if $w_1 < w < w_3$ which yields

$$\lim_{\xi \to -\infty} \tilde{w}(\xi) = w_1 \quad \text{and} \quad \lim_{\xi \to +\infty} \tilde{w}(\xi) = w_3.$$

In the second case, by relabeling the roots if necessary, we can write $w_2 < w_1$. If $w' = a_3(w - w_1)^2(w - w_2)$ then $w' < 0$ for $w_2 < w < w_1$ and the only bounded solution w satisfies $\lim_{\xi \to -\infty} w(\xi) = w_1$ and $\lim_{\xi \to +\infty} w(\xi) = w_2$. If $w' = a_3(w - w_1)(w - w_2)^2$ then $w' > 0$ for $w_2 < w < w_1$ and w satisfies $\lim_{\xi \to -\infty} w(\xi) = w_2$ and $\lim_{\xi \to +\infty} w(\xi) = w_1$.

If $\Delta = 0$ then the cubic polynomial $p(w)$ can have three equal roots or only two different real roots. In the first case there is no bounded solution to (18) while in the second case we have again only one bounded solution w.

We now proceed to characterize the condition $\Delta > 0$. Since

$$\frac{\Delta}{Re^2} = \frac{\varepsilon^2(\gamma + 1)}{6v}\left[\frac{4(1 - v^2)}{v} + \varepsilon(\gamma + 1)w_1 - w_1^2\frac{\varepsilon^2(\gamma + 1)}{2v}\right] + \frac{\varepsilon^2(\gamma + 1)^2}{4},$$

we have

$$\begin{aligned}
\frac{6v^2}{Re^2}\Delta &= \varepsilon^2(\gamma + 1)\left[4\left(1 - v^2\right) + \varepsilon(\gamma + 1)w_1 v - \frac{\varepsilon^2(\gamma + 1)}{2}w_1^2\right] + \frac{3}{2}\varepsilon^2 v^2(\gamma + 1)^2 \\
&= \varepsilon^2(\gamma + 1)\left[\frac{3}{2}v^2(\gamma + 1) + 4\left(1 - v^2\right) + \varepsilon(\gamma + 1)w_1 v - w_1^2\varepsilon^2\frac{(\gamma + 1)}{2}\right] \\
&= \varepsilon^2(\gamma + 1)\left[4 - \frac{5}{2}v^2 + \frac{3}{2}v^2\gamma + \varepsilon(\gamma + 1)w_1 v - w_1^2\varepsilon^2\frac{(\gamma + 1)}{2}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{12v^2}{Re^2}\Delta &= \varepsilon^2(\gamma + 1)\left[8 - 5v^2 + 3\gamma v^2 + 2\varepsilon(\gamma + 1)w_1 v - \varepsilon^2(\gamma + 1)w_1^2\right] \\
&= \varepsilon^2(\gamma + 1)\left[(3\gamma - 5)v^2 + 2\varepsilon(\gamma + 1)w_1 v + 8 - \varepsilon^2(\gamma + 1)w_1^2\right] \\
&= \varepsilon^2(\gamma + 1)h_{\gamma,\varepsilon}(v, w_1).
\end{aligned} \tag{23}$$

From (23) it follows that $\Delta > 0$ if

$$h_{\gamma,\varepsilon}(v, w_1) = (3\gamma - 5)v^2 + 2\varepsilon(\gamma + 1)w_1 v + 8 - \varepsilon^2(\gamma + 1)w_1^2 > 0;$$

that is,

$$\varepsilon^2 w_1^2 - 2v\varepsilon w_1 - \frac{8 + (3\gamma - 5)v^2}{\gamma + 1} < 0. \tag{24}$$

If we define $\theta = \varepsilon w_1$, then $\Delta > 0$ is true if

$$\theta_1 < \theta = \varepsilon w_1 < \theta_2,$$

where $\theta_{1,2}$ are the roots of

$$\theta^2 - 2v\theta - \frac{8 + (3\gamma - 5)v^2}{\gamma + 1} = 0.$$

Using that $\gamma > 1$, and writing $\theta_{1,2}$ explicitly we obtain

$$\theta_{1,2} = \frac{2v \pm \sqrt{4v^2 + \frac{4(8 + (3\gamma - 5)v^2)}{\gamma + 1}}}{2} = v \pm \sqrt{v^2 + \frac{(8 + (3\gamma - 5)v^2)}{\gamma + 1}} = v \pm \sqrt{\frac{4v^2(\gamma - 1) + 8}{\gamma + 1}}$$

which proves the theorem. □

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From the proof of Theorem 2.1 we see that it is also possible to prescribe two roots of the polynomial $p(w)$, say w_1 and w_2, and solve for the values $v(w_1, w_2, \varepsilon)$. One can then test the existence of non-trivial traveling wave solutions with Theorem 2.1. We now describe the procedure in detail.

Let $w = \Phi'(x - vt)$ be a solution of (16) obtained in this way with

$$\lim_{\xi \to -\infty} w(\xi) = w_1 \quad \text{and} \quad \lim_{\xi \to \infty} w(\xi) = w_2. \tag{25}$$

Thus, from (16) and (25) we obtain

$$-\frac{Re(1-v^2)}{v} w_1 - \frac{\varepsilon\, Re(\gamma+1)}{2} w_1^2 + \frac{\varepsilon^2 Re(\gamma+1)}{6v} w_1^3 = -\frac{Re(1-v^2)}{v} w_2 - \frac{\varepsilon\, Re(\gamma+1)}{2} w_2^2 + \frac{\varepsilon^2 Re(\gamma+1)}{6v} w_2^3. \tag{26}$$

This expression can be simplified as follows

$$-\frac{1-v^2}{v}(w_1 - w_2) - \frac{\varepsilon(\gamma+1)}{2}\left(w_1^2 - w_2^2\right) + \frac{\varepsilon^2(\gamma+1)}{6v}\left(w_1^3 - w_2^3\right) = 0. \tag{27}$$

Since $w_1 \neq w_2$ ($w_1 = w_2$ implies constant solution), we can factor the term $w_1 - w_2$ out of the previous expression. Upon doing this, and multiplying through by $-v$, we arrive at

$$\left(1 - v^2\right) + \frac{\varepsilon(\gamma+1)}{2}(w_1 + w_2)v - \frac{\varepsilon^2}{6}(\gamma+1)\left(w_1^2 + w_1 w_2 + w_2^2\right) = 0. \tag{28}$$

Multiplying through by -1, we obtain

$$v^2 - \frac{\varepsilon(\gamma+1)}{2}(w_1 + w_2)v + \frac{\varepsilon^2}{6}(\gamma+1)\left(w_1^2 + w_1 w_2 + w_2^2\right) - 1 = 0. \tag{29}$$

Therefore we can compute v from the equation

$$v = \frac{\varepsilon(\gamma+1)}{4}(w_1 + w_2) \pm \frac{1}{2}\sqrt{\frac{\varepsilon^2(\gamma+1)^2}{4}(w_1 + w_2)^2 - \frac{2}{3}\varepsilon^2(\gamma+1)\left(w_1^2 + w_1 w_2 + w_2^2\right) + 4}. \tag{30}$$

It is interesting to note that the solutions for v are independent of the acoustic Reynolds number, Re. Since the HOAWE is a generalization of Kuznetsov's equation, we should expect the traveling wave velocity v to reduce to the one obtained in [4] in the limit of small ϵ. Indeed, using that $\sqrt{1+x} \approxeq 1 + \frac{1}{2}x$ for small x in the previous formula for v and choosing the positive sign we obtain:

$$v \approxeq 1 + \frac{\varepsilon(\gamma+1)}{4}(w_1 + w_2) + O\left(\varepsilon^2\right), \tag{31}$$

which shows that, when $\varepsilon \to 0$, our wave speed $v(w_1, w_2)$ agrees with the wave speed $v(w_1, w_2)$ of Kuznetsov's equation given in [4, Eq. (23)] up to the order $O(\varepsilon)$.

We consider the discriminant δ of the quadratic form under the square root in (30) as a function of the Mach number ε:

$$\delta(\epsilon) := a\epsilon^2 + 4 \tag{32}$$

where

$$a = \frac{(\gamma+1)^2}{4}(w_1 + w_2)^2 - \frac{2}{3}(\gamma+1)\left(w_1^2 + w_1 w_2 + w_2^2\right) = (w_1 + w_2)^2(\gamma+1)\left[\frac{(\gamma+1)}{4} - \frac{2}{3}\left(\frac{w_1^2 + w_1 w_2 + w_2^2}{(w_1 + w_2)^2}\right)\right]. \tag{33}$$

When $a > 0$, there is always at least one positive solution v for any value of ϵ. When $a < 0$, there is a critical Mach number, ϵ_c, above which there are no real-valued velocities v corresponding to traveling wave solutions to the HOAWE. This critical Mach number is given by

$$\delta(\varepsilon_c) = a\varepsilon_c^2 + 4 = 0;$$

that is,

$$\varepsilon_c = \frac{2}{\sqrt{-a}}. \tag{34}$$

We note that if $\gamma < \frac{5}{3}$ and $w_1 w_2 < 0$ then $a < 0$ because $\frac{w_1^2 + w_1 w_2 + w_2^2}{(w_1 + w_2)^2} > 1$ and $a < (w_1 + w_2)^2(\gamma+1)[\frac{(\gamma+1)}{4} - \frac{2}{3}] < 0$. Specifically, if $a < 0$, which is the case when $\gamma < \frac{5}{3}$ and $w_1 w_2 < 0$, there is no solution when $\varepsilon > \varepsilon_c$.

3. Traveling wave solutions of Kuznetsov's equation

We also seek traveling wave solutions to Eq. (9), which take the form

$$\Phi(X, T) = \Phi(X - vT) = \Phi(\xi). \tag{35}$$

Proceeding as in Section 2, and letting $w(\xi) = \Phi'(\xi)$, we obtain the equation

$$w' - b_1 w - b_2 w^2 - c = 0, \tag{36}$$

where c is the constant of integration as in (14) and (15), and

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$$b_1 := Re\frac{1-v^2}{v}, \tag{37}$$

$$b_2 := \varepsilon\, Re\big[1+v^2(\beta-1)\big]. \tag{38}$$

If w is a solution of (36), by defining

$$w_1 := \lim_{\xi\to-\infty} w(\xi) \quad \text{and} \quad w_2 := \lim_{\xi\to+\infty} w(\xi), \tag{39}$$

and assuming $\lim_{\xi\to\pm\infty} w' = 0$, then from (36) it follows that

$$c = -b_1 w_1 - b_2 w_1^2 = -b_1 w_2 - b_2 w_2^2. \tag{40}$$

Therefore, (36) becomes

$$w' - b_1 w - b_2 w^2 + b_1 w_1 + b_2 w_1^2 = 0. \tag{41}$$

For solutions of (41) of the form

$$w(\xi) = A + B\tanh(\lambda\xi),$$

we have the following theorem.

Theorem 3.1. *If the following equations hold*

$$w_1 + w_2 = -\frac{b_1}{b_2}, \tag{42}$$

$$\lambda = \frac{b_2}{2}(w_1 - w_2), \tag{43}$$

for some constants w_1, w_2, λ, b_1 and b_2, then

$$w(\xi) = -\frac{b_1}{2b_2} - \frac{\lambda}{b_2}\tanh(\lambda\xi) \tag{44}$$

satisfies Eq. (41) *and*

$$w_1 := \lim_{\xi\to-\infty} w(\xi) \quad \textit{and} \quad w_2 := \lim_{\xi\to\infty} w(\xi). \tag{45}$$

Proof. We look for solutions of the form $w = A + B\tanh(\lambda\xi)$ and thus

$$w' = B\lambda\big[1-\tanh^2(\lambda\xi)\big], \tag{46}$$

$$w^2 = A^2 + 2AB\tanh(\lambda\xi) + B^2\tanh^2(\lambda\xi). \tag{47}$$

If we substitute (46) and (47) into (41) and use the notation $y = \tanh(\lambda\xi)$ we obtain

$$B\lambda - B\lambda y^2 - b_1(A+By) - b_2\big(A^2 + 2ABy + B^2y^2\big) + b_1 w_1 + b_2 w_1^2 = 0.$$

From this and (40) we get the system of equations:

$$B\lambda - b_1 A - b_2 A^2 + b_1 w_1 + b_2 w_1^2 = 0, \tag{48}$$

$$b_1 w_1 + b_2 w_1^2 = b_1 w_2 + b_2 w_2^2, \tag{49}$$

$$-b_1 B - 2b_2 AB = 0, \tag{50}$$

$$B\lambda + b_2 B^2 = 0, \tag{51}$$

and we add to the system the two equations

$$A = \frac{1}{2}(w_1 + w_2), \tag{52}$$

$$B = -\frac{1}{2}(w_1 - w_2). \tag{53}$$

Since we want $B \neq 0$ and $\lambda \neq 0$, we must have $w_1 \neq w_2$. Using (52) and (53), Eqs. (50) and (51) simplify to:

$$w_1 + w_2 = -\frac{b_1}{b_2}, \tag{54}$$

$$\lambda = \frac{b_2}{2}(w_1 - w_2). \tag{55}$$

We note that Eq. (49) reduces to exactly Eq. (54), and thus it does not add any information.

Thus, since $\lambda = -b_2 B$, $A = -\frac{b_1}{2b_2}$ and $B = -\frac{b_1}{2b_2} - w_1$ we only need to show that Eq. (48) is satisfied:

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$$B\lambda - b_1 A - b_2 A^2 + b_1 w_1 + b_2 w_1^2$$
$$= -b_2 B^2 - b_1 A - b_2 A^2 + b_1 w_1 + b_2 w_1^2 = -b_2\left[w_1^2 + \frac{b_1}{b_2} w_1 + \frac{b_1^2}{4b_2^2}\right] + b_1 \frac{b_1}{2b_2} - b_2 \frac{b_1^2}{4b_2^2} + b_1 w_1 + b_2 w_1^2$$
$$= -b_2 w_1^2 - b_1 w_1 - \frac{b_1^2}{4b_2} + \frac{b_1^2}{2b_2} - \frac{b_1^2}{4b_2} + b_1 w_1 + b_2 w_1^2 = 0,$$

which implies our desired result. □

Remark 3.1. From the conditions (42) and (43) we conclude that:

(i) There is a traveling wave solution for any prescribed values of $(v, \varepsilon, \gamma, \lambda)$, but once these parameters are prescribed they imply certain w_1 and w_2.
(ii) If we look for a solution with prescribed behavior at $-\infty$ (or $+\infty$); that is, if we prescribe w_1 (or w_2) then there is a solution for any (v, ε, γ).

Remark 3.2. If we look for a solution with prescribed behavior both at $-\infty$ and $+\infty$; that is, if we prescribe both w_1 and w_2 then the possible values of v and ε for which there is solution are restricted. This situation was analyzed in [4] where solutions to Kuznetsov's equation were shown to exist only for Mach numbers less than a critical value ε_c.

4. Conclusions

In this Letter, we presented an analysis of traveling wave solutions to a high-order acoustic wave equation. We showed that there exist non-trivial traveling wave solutions for any wave speed v and any value of the acoustic Mach number. We also showed that if the values of the solution at $\pm\infty$ are prescribed, then there is a critical Mach number above which there is no traveling wave solutions.

Although traveling wave solutions were shown to exist for Kuznetsov's equation, waves with higher Mach number may not be accurately modeled by this equation. Since the Kuznetsov equation itself is restricted to small values of the acoustic Mach number, the authors invoked the high-order acoustic wave equation as a more accurate equation to model nonlinear acoustical waves.

We also showed that in the limit of small acoustic Mach number, the traveling wave speeds obtained here reduce to those obtained in [4]. This makes sense, since the high-order wave equation is a generalization of the Kuznetsov equation considered in [4].

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