RESTRICTIONS ON HILBERT COEFFICIENTS GIVE THE
DEPTH OF A PRIME IDEAL INSIDE THE POLYNOMIAL RING

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Abstract. In this paper, we prove that for a prime ideal $P$ of dimension $r$
inside a polynomial ring, if adjoining $s$ general linear forms to the prime ideal
changes the $r - s - 1$-th Hilbert coefficient by 1 and doesn’t change the 0th to
$r - s - 1$-th Hilbert coefficients where $s \leq r$, then the depth of $S/P$ is $n - s - 1$.
This criteria also tells us about possible restrictions on the generic initial ideal
of a prime ideal inside a polynomial ring.

1. PRELIMINARY

Let $k$ be a field. If $n$ is a positive integer, let $S(n) = k[x_1, \ldots, x_n]$ be the polynomial
ring of $n$ variables over $k$. Every element in $S(n)$ of the form $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ is called
a monomial in $S(n)$. The set of all monomials in $S(n)$ is denoted by $Mon(S(n))$. Every
monomial $f \in S(n)$ can be expressed uniquely as a $k$-linear combination of monomials;
write $f = \sum_{u \in Mon(S(n))} a_u u$, then we call the set $\{u : a_u \neq 0\}$ the support of $f$, denoted by $supp(f)$. A monomial order $<$ is a total order on $Mon(S(n))$ satisfying
the two properties: (1) $1 < u$ for any $1 \neq u \in S(n)$; (2) $u, v, w \in Mon(S(n))$,
$u < v$ implies $uw < vw$. One of the order is the graded reverse lexicographic order
$<_{rev}$: for two monomials $u = x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ and $v = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$, $u < v$ if and only if
(1) $deg(u) < deg(v)$ or (2) $deg(u) = deg(v)$ and for the largest $i$ such that $e_i \neq d_i$,
$e_i > d_i$. Given a monomial order $<$ and an element $f$, we can define the initial
monomial of $f$, denoted by $in_{<}(f)$ or $in(f)$ if the order is clear, to be the maximal
element in $in(f)$; for an ideal $I$ of $S(n)$, we define the initial ideal of $I$, $in_{<}(I)$ or
$in(I)$ if the order is clear, to be the ideal generated by $\{in_{<}(f) : f \in I\}$.

Now we recall the definition of the generic initial ideal. Suppose $k$ is an infinite
field. Let $\alpha \in GL_n(k)$ be a linear automorphism of $k^n$, we extend $\alpha$ to be a $k$-
linear algebra endomorphism by mapping $x_i$ to $\alpha(x_i)$; this gives a $k$-linear automorphism
of $S(n)$ and we still denote it by $\alpha$. Now if we fix a monomial order $<$, then for any
$\alpha$ we can define $in_{<}(\alpha(I))$ for any $\alpha \in GL_n(k)$; there exist a Zariski open set
$U$ of $GL_n(k)$ such that for all $\alpha \in U$ this initial ideal $in_{<}(\alpha(I))$ is the same. This
statement only makes sense when $k$ is infinite because otherwise $GL_n(k)$ is discrete,
so any subset of $GL_n(k)$ is open. This ideal is called the generic initial ideal with
respect to $<$ and denoted by $gin_{<}(I)$ or $gin(I)$ if the order is clear.

Then we recall some properties about a monomial ideal. Assume we have a monomial
order $<$ on $S(n)$ and $x_1 > x_2 > \cdots > x_n$ under this order. A monomial
ideal $J$ is called strongly stable if for any $1 \leq i < j \leq n$, $u \in J$ and $x_j | u$ we
have that $x_i u/x_j \in J$. A monomial ideal $J$ is called of Borel type if for any $1 \leq i < j \leq n$,
$u \in J$ and $x_j | u$ there exist an integer $t$ such that $x_i^t u/x_j \in J$. It suffices to check
these conditions for all $u$ inside a monomial generating set of $J$. Obviously being

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strongly stable implies being of Borel type. By Herzog-Hibi we know that being of Borel type is equivalent to the fact that $J : x_i^\infty = J : (x_1, \ldots, x_i)^\infty$ for any $i$. By Herzog-Hibi we know the generic initial ideal $\text{gin}_< (I)$ is always of Borel type, and it is strongly stable when $k$ has characteristic 0.

By Bayer and Stillman’s theorem [3] we know that the generic initial ideal with respect to $<_\text{rev}$ carries a lot of information about the original ideal, that is, $S(n)/I$ and $S(n)/\text{gin}_{<_\text{rev}} (I)$ have the same projective dimension and regularity as $S(n)$-module, and same depth as rings, and hence are simultaneously Cohen-Macaulay or not Cohen-Macaulay. Also if the characteristic of $k$ is 0, then $\text{gin}_{<_\text{rev}} (I)$ is strongly stable and it is easy to describe the projective dimension or the regularity of such a monomial ideal. So in order to know these data about $S(n)/I$ it is reasonable to find $\text{gin}_{<_\text{rev}} (I)$. In this paper we restrict to the case when $I$ is a prime ideal in $S(n)$ and try to find any restriction on a strongly stable monomial ideal to be the generic initial ideal of $I$.

Finally we recall the definition of Hilbert coefficients. Let $M$ be a finitely generated graded $S(n)$-module. The function $H : \mathbb{N} \to \mathbb{N}, H(n) = \dim_k (M_n)$ is called the Hilbert function. The power series $h(t) = \sum_{i \in \mathbb{N}} H(i) t^i$ is called the Hilbert series. It is well known that the Hilbert series is of the form $q(t)/(1 - t)^d$ with $d = \dim (M)$, $q(t)$ is a polynomial with integer coefficients satisfying $q(1) \neq 0$. Now we can express $q(t) = c_0 + c_1 (t - 1) + c_2 (t - 1)^2 + \ldots$, and these $c_i$’s are called the Hilbert coefficients. They are all rational numbers.

2. SETTINGS

Let $k$ be an infinite field of characteristic 0. $S(n) = k[x_1, \ldots, x_n]$ is the polynomial ring of $n$ variables over $k$. $\mathfrak{m}(n) = (x_1, \ldots, x_n)$ be the graded maximal ideal of $S$. We fix the monomial order $<$ to be the graded reverse lexicographic order from now on. When we talk about the initial monomial ideal we always use this order. Let $l = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ be a linear form in $S(n)$. We have the projection map $S(n) \to S(n)/IS(n)$. If $c_n \neq 0$, we have an isomorphism $S(n - 1) \cong S(n)/IS(n)$ which is the composition of the embedding of $S(n - 1)$ into $S(n)$ and the projection. We use this isomorphism to identify $S(n)/IS(n)$ with $S(n - 1)$; define the new projection map to be $\pi : S(n) \to S(n - 1)$. This makes sense when $c_n \neq 0$, so it makes sense when $l = x_n$ or when $l$ is a general linear form. If $l = x_n$, we also denote $\pi_{x_n}$ by $\pi_n$. We can also define $<_\text{rev}$ on $S(n - 1)$ and $\pi_{n-1} : S(n - 1) \to S(n - 2)$. So we can compose the maps $\pi_{n-1} \pi_{n-2} \ldots \pi_n$ and denote it by $\Pi_{d+1}$; it is the projection map $S(n) \to S(d)$. Also, $\pi_n (\mathfrak{m}(n)) = \mathfrak{m}(n - 1)$. If $I \subset S(n)$, $J : \mathfrak{m}(n)^\infty$ is called the saturation of $I$, denoted by $I_{\text{sat}}$.

For a monomial $u = x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n}$, we denote $\phi_i (u) = u/x_i^{d_i}$, that is, we take out all $x_i$’s from the factors of $u$. Denote the composition $\phi_{d+1} \phi_{d+2} \ldots \phi_{n}$ to be $\Phi_{d+1}$. If $J$ is a monomial ideal in $S(n)$, minimally generated by monomials $u_1, \ldots, u_s$. We denote $\phi_i (J) = \langle \phi_i (u_1), \ldots, \phi_i (u_s) \rangle = J : x_i^\infty$ and $\overline{\phi_i (J)}$ in $S(n)/x_i S(n)$. In particular $\overline{\phi_n (J)}$ is an ideal in $S(n)/x_n S(n) = S(n - 1)$. Inductively we can define $\Phi_{d+1} (J) = \phi_{d+1} \phi_{d+2} \ldots \phi_n (J)$; it is an ideal in $S(n)$. Define $\Phi_{d+1} (J) = \phi_{d+1} \phi_{d+2} \ldots \phi_n (J)$. It is an ideal in $S(d)$.

**Proposition 2.1.** Let $I$ be a homogeneous ideal in $S(n)$, $l$ be a general linear form in $S(n)$. Then $\text{gin}(\pi_{n} (I)) = \pi_{n} (\text{gin}(I))$. 
Remark 2.2. Here we view $\pi(I)$ as an ideal of $S(n-1)$ so that the generic initial ideal is well-defined. $\pi_n(gin(I))$ is also an ideal of $S(n-1)$, thus this equality makes sense because it compares two ideals in the same ring $S(n-1)$.

Proof. See [2], prop 2.14.

Proposition 2.3. Let $l$ be a general linear form in $S(n)$, then $gin(I : m(n)) = gin(I : l) = gin(I) : x_n$.

Proof. See [2].

Corollary 2.4. $gin(I : m(n)^i) = gin(I) : (x_n)^i$ and $gin(I^{sat}) = gin(I)^{sat}$.

Proof. We use the last theorem inductively to get the first statement. Then we take union on both sides. Since $gin(I)$ is of Borel type, $gin(I)^{sat} = gin(I) : x_n^{\infty}$. So we get the second statement.

3. General hyperplane section

Let $I$ be a saturated homogeneous ideal in $S(n)$. Let $l$ be a linear form such that the $x_n$-coefficient of $l$ is nonzero. We call the ideal $\pi(I) = (\pi(I))^{sat}$ the section with one hyperplane. It is an ideal in $S(n-1)$. If we have $s$ linear forms $l_1, \ldots, l_s$, then inductively we can define the section with $s$ hyperplanes: the section with one hyperplane is $I_1 = \pi_{l_1}(I)$, it is an ideal in $S(n-1)$; let $I_2$ be the image of $l_2$ inside $S(n-1)$; so define the section with two hyperplanes $I_2 = \pi_{l_1}(I_1)^{sat}$, and inductively, with $s$ hyperplanes is $I_s = \pi_{l_1}(I_s-1)^{sat}$. Let $d = n - s$, so $I_s$ is an ideal in $S(d)$.

Proposition 3.1. $gin(I_s) = \pi_{d+1}^{\pi_{d+2}} \ldots \pi_n(I)$.

Proof. We use the first theorem inductively and note that saturation commutes with $gin$.

Proposition 3.2. Let $J$ be a saturated monomial ideal of Borel type of $S(n)$. Then $(1) \pi_{d+1} \pi_{d+2} \ldots \pi_n(J) = \phi_{d+1} \ldots \phi_n(J) = \Phi_{d+1}(J)$ and $(2) \pi_{d+1} \pi_{d+2} \ldots \pi_n(J) = \phi_d \phi_{d+1} \ldots \phi_n(J)$.

Proof. $J$ is of Borel type and saturated, so $J = J^{sat} = J : x_n^{\infty} = \phi_n(J)$ and $\pi_n(J) = \phi_n(J)$, which means $(1)$ is true for $d = n - 1$ and $(2)$ is true for $d = n$. Now for any $n$, if $J$ is a monomial ideal of Borel type in $S(n)$, $\phi_n(J)$ is a monomial ideal in $S(n-1)$ and we claim it’s also of Borel type. Suppose $J = (u_1, \ldots, u_k)$, then $\phi_n(J) = (\phi_n(u_1), \ldots, \phi_n(u_k))$ if we view the monomials $\phi_n(u_k)$ as monomials inside $k[x_1, \ldots, x_{n-1}]$. So every minimal generator is of the form $\phi_n(u_k)$ for some $k$. Choose $1 \leq j < i \leq n - 1$ such that $x_i | \phi_n(u_k)$. Since $i \neq n$, $x_i | u_k$. Since $J$ is of Borel type, there exist $t$ such that $x_j^t u_k / x_i$ is in $J$, so $u_k^t | x_j^t u_k / x_i$. Since $i, j \neq n, \phi_n(u_k) x_j^t \phi_n(u_k) / x_i$. This means that $\Phi_n(J)$ is still of Borel type. Now inside the ring $S(n-1)$ we have $\pi_n(\phi_n(J)) = \pi_n(\phi_n(J)) : x_n^{\infty} = \phi_n(\phi_n(J))$ and hence $\pi_n(\phi_n(J)) = \phi_n(\phi_n(J))$ in $S(n-2)$. Finally we induct on $r = n - d$.

4. Theorem

The main purpose is to find any restriction on the generic initial ideal of a prime.
Theorem 4.1. Let $P$ be a homogeneous prime ideal in polynomial ring $S = S(n)$ and $J = \text{in}(P)$ be the initial ideal of $P$. Assume $J$ is strongly stable. Suppose for some $1 \leq d \leq n$ we have that $\Phi_{d-1}(J) = \Pi_{d+1}(J) + u$ for some $u \in \Pi_{d+1}(J) : m$. Then either (1) $u = 1$ and hence $J$ contains the ideal $\langle x_1, \ldots, x_d \rangle$ or (2) $J$ can be generated by all generators of $J$ inside $k[x_1, \ldots, x_d]$ and $u$. Moreover, $u$ is of the form $ex_{d+1}$ where $v \in \pi_{d+1}(J) : m$ and $e$ is a positive integer, so all the minimal generator of $J$ is inside $k[x_1, \ldots, x_{d+1}]$.

Proof. Let $v$ be a monomial ideal which is a minimal generator of $J$. There exists a polynomial $f \in P$ with $\text{in}(f) = v$. We claim that $f$ is irreducible as a polynomial in $S$. Otherwise $f = f_1f_2$, $f_1$, $f_2$ are not constants, then $\text{in}(f) = \text{in}(f_1)\text{in}(f_2)$, and $\text{in}(f) \neq \text{in}(f_1)$ or $\text{in}(f_2)$. And $P$ is a prime, which means that $f_1 \in P$ or $f_2 \in P$, then $\text{in}(f_1) \in J$ or $\text{in}(f_2) \in J$, which contradicts the minimality of $\text{in}(f) = v$.

Let $u_1, \ldots, u_s, v_1, \ldots, v_t$ be the minimal generators of $J$ where $u_i$'s are inside $k[x_1, \ldots, x_d]$ and $v_j$'s are not. Then $\Phi_{d+1}u_i = u_i$, $\Phi_{d+1}v_j \neq v_j$, $\Phi_{d+1}(J)$ is generated by $\Phi_{d+1}u_i = u_i$, $\Phi_{d+1}v_j$ and is minimally generated by $u_i, u$. If for some $j$, $\Phi_{d+1}v_j \neq u_j$, then $\Phi_{d+1}v_j = u_j$, so $u_j$ divides $v_j$, which contradicts the minimality of $v_j$. So for all $j$, $\Phi_{d+1}v_j = u_j$, so $v_j = w_ju_j$ for some $w_j \in k[x_{d+1}, \ldots, x_n]$. There exists at least one $v_j$, because otherwise $J$ is generated by monomials inside $k[x_1, \ldots, x_d]$ and we have $\Phi_{d+1}(J) = \pi_{d+1}(\Phi_{d+2}(J)) = J \subset S/(x_{d+1}, \ldots, x_n)S$ but by the condition $\Phi_{d+1}(J) \neq \pi_{d+1}(\Phi_{d+2}(J))$. Now we claim there is only one such $v_j$. Otherwise, suppose $v_1 = w_1u_1$, $v_2 = w_2u_2$ are two distinct monomials inside the minimal generating set of $J$, where $w_1, w_2$ are not constants. Take $f_1, f_2$ be the element in the reduced Grobner basis of $P$ satisfying $\text{in}(f_1) = v_1, \text{in}(f_2) = v_2$. By the argument above, $f_1, f_2$ are irreducible. Then any monomial appearing in $f_1$ or $f_2$ is not in $\text{in}(P)$ except their initial monomials. Now we write $f_1 = p_1u + q_1f_2 = p_2u + q + 2$. Here $p_1u$ is the sum of all the terms which appear in $f_1$ and divisible by $u$, and $q_1$ is the sum of the rest terms; and similar for $p_2, q_2$. It’s easy to see that $\text{in}(p_1) = w_1$ and $\text{in}(p_2) = w_2$, and these initial monomial is in $k[x_{d+1}, \ldots, x_n]$. Besides, because $u \in J : m$, any other monomial $m$ in $q_1$ are not divisible by $x_1, x_2, \ldots, x_r$, otherwise $mu$ divides some $u_j$. So they are also in the last $n-r$ variables, so $q_1$ is a polynomial in $k[x_{r+1}, \ldots, x_n]$. We have the same thing for $q_2$. Now we consider the polynomial $S_{yz} = p_2f_1 - p_1f_2 = p_2q_1 - p_1q_2$. Take any monomial $m_1 \in p_2, m_2 \in p_1$. Then $m_1 \in k[x_{r+1}, \ldots, x_n]$, $m_2$ is not divisible by $u_i$ and $u$, so $m_2 \notin J : \langle x_{r+1}, x_{r+2}, \ldots, x_n \rangle$. So $m_1m_2 \notin J$. This is true for any monomial in $m_1m_2 \in p_2q_1$ and similarly for $m_1m_2 \in p_1q_2$. So any possible term appearing in $S_{yz}$ is not in $J = \text{in}(P)$. But $S_{yz} \in P$. This means that $S_{yz} = 0$. So $p_2f_1 = p_1f_2$. Now $f_1$ is irreducible, so $f_1$ divides $p_1$ or $f_2$. If it divides $p_1$, we have $\text{in}(f_1) = v_1 = w_1u_1$ is divisible by $\text{in}(p_1) = w_1$; thus it forces $f_1 = p_1$. In this case $u = 1 \in J : m$ so $J$ contains $(x_1, \ldots, x_d)$. Otherwise $f_1$ divides $f_2$. Similarly if $J \neq m$ then $f_2$ divides $f_1$. So $f_1$ and $f_2$ differ by a constant multiple and their initial monomials are the same, which contradicts the assumption that $v_1 \neq v_2$. Finally $u = vx_{d+1}$ by stability conditions; if $u = vw$ with $w$ not a power of $x_{d+1}$, then $w$ divides a variable $x_j$ with $j \geq d + 1$. Then consider $u' = vwx_{d+1}/x_j$, it is divided by some minimal generator $u_1$. Then we know $u_1 = vw_1$ with $w_1$ and $w$ not dividing each other and they are both monomials in $k[x_{d+1}, x_{d+2}, \ldots, x_n]$, which contradicts the conclusion of the previous step.

We get the following corollary:
Corollary 4.2. Suppose the characteristic of \( k \) is 0. If we replace \( J = \text{in}(P) \) by \( J = \text{gin}(P) \) in the last theorem, then the conclusion still holds.

Proof. Suppose \( J = \text{in}(\alpha(P)) \) for some \( \alpha \in GL_n(k) \). Now \( \alpha(P) \) is still a prime and we apply the theorem to \( \alpha(P) \).

Now we make this result stronger by replacing \( J \) with a larger ideal \( \pi_{d+1}\Phi_{d+2}(J) \).

First we have the following lemma:

Proposition 4.3. Let \( P \) be a homogeneous prime ideal in polynomial ring \( S = S(n) \) and \( J = \text{gin}(P) \). Suppose for some \( 1 \leq d \leq n \) we have \( \dim(S/P) \geq n - d + 1 \), and that \( \Phi_{d+1}(J) = \pi_{d+1}\Phi_{d+2}(J) + u \) for some \( u \in \pi_{d+1}\Phi_{d+2}(J) : m \). Then either (1) \( u = 1 \) and hence \( J \) contains the ideal \( (x_1, \ldots, x_d) \) or (2) \( J \) can be generated by all generators of \( J \) inside \( k[x_1, \ldots, x_d] \) and \( u \). Moreover, \( u \) is of the form \( vx_{d+1}^\alpha \) where \( v \in \pi_{d+1}\Phi_{d+2} : m \) and \( \alpha \) is a positive integer, so all the minimal generator of \( J \) is inside \( k[x_1, \ldots, x_{d+1}] \).

Proof. Still looking for ref...

Theorem 4.4. Let \( P \) be a homogeneous prime ideal in polynomial ring \( S = S(n) \) and \( J = \text{gin}(P) \). Suppose for some \( 1 \leq d \leq n \) we have \( \dim(S/P) \geq n - d + 1 \), and that \( \Phi_{d+1}(J) = \pi_{d+1}\Phi_{d+2}(J) + u \) for some \( u \in \pi_{d+1}\Phi_{d+2}(J) : m \). Then either (1) \( u = 1 \) and hence \( J \) contains the ideal \( (x_1, \ldots, x_d) \) or (2) \( J \) can be generated by all generators of \( J \) inside \( k[x_1, \ldots, x_d] \) and \( u \). Moreover, \( u \) is of the form \( vx_{d+1}^\alpha \) where \( v \in \pi_{d+1}\Phi_{d+2} : m \) and \( \alpha \) is a positive integer, so all the minimal generator of \( J \) is inside \( k[x_1, \ldots, x_{d+1}] \).

Proof. Let \( s = n - d \). We choose \( s \) general linear forms \( l_1, l_2, \ldots, l_s \) and consider \( P_s = \pi_{(d+1)} \cdots \pi_s(P) \). It is a prime ideal in \( S(d+1) \) because \( s - 1 \leq \dim(S/P) \) - 2. Now \( \text{gin}(P_{s-1}) = \Phi_{d+2}(\text{gin}(P)) \) by the lemma, so \( \Phi_{d+1}(\text{gin}(P_{s-1})) = \Phi_{d+1}(\text{gin}(P)) \) and \( \pi_{d+1}\Phi_{d+2}(J) = \pi_{d+1}(\text{gin}(P_{s-1})) \). Apply the theorem to the prime ideal \( P_{s-1} \subseteq S(d+1) \), we know \( \text{gin}(P_{s-1}) \) is generated in \( k[x_1, \ldots, x_{d+1}] \). Now we claim \( \text{gin}(P) \) is generated in \( k[x_1, \ldots, x_{d+1}] \). Otherwise, we can find \( u = vu \) which is a minimal generator inside \( \text{gin}(P) \) where \( v \in k[x_1, \ldots, x_{d+1}] \) and \( w \in k[x_{d+2}, \ldots, x_n] \) with \( w \neq 1 \). Since \( \text{gin}(P) \) is strongly stable, \( v * x_{d+2}^\alpha \) is in \( \text{gin}(P) \). Now this monomial is divisible by some minimal generator \( u_1 = v_1 * w_1, v \in k[x_1, \ldots, x_{d+1}] \) and \( w \in k[x_{d+2}, \ldots, x_n] \) and \( w_1 \) a pure power of \( x_{d+2} \). \( w_1 \) cannot be 1, otherwise \( u_1 \) properly divides \( u \), so \( u \) is not a minimal generator. Now \( \Phi_{d+2}(u_1) = u \).

From the above theorem we can get:

Theorem 4.5. Let \( P \) be a prime ideal in \( S \), \( 1 \leq d \leq n - 1 \), \( P_{d+1} \) is the \( n-d \) general hyperplane section of \( P \), \( Q \) is \( \pi_{r+1}(P_{d+2}) \). If \( P_{d+1} = Q + (f) \) with \( f \in P_{d+1} : m \), then depth \( S/P = n - d - 1 \).

Proof. By the following lemma, we can take generic initial of all the ideals above. We have \( \text{gin}(P_{d+1}) = \Phi_{d+1}(\text{gin}(P)) \) and \( \text{gin}(Q) = \text{gin}(\pi_{r+1}(P_{d+2})) \) where \( \pi_{d+1}\Phi_{d+2}(\text{gin}(P)) \). We have \( P_{d+1} = Q + (f) \) with \( f \in P_{d+1} : m \), so the Hilbert function of \( P_{d+1} \) is the same as the Hilbert function of \( Q \) except that they differ by 1 at degree \( f \). Taking generic initial does not change the Hilbert function, so the Hilbert function of \( \text{gin}(P_{d+1}) \) is the same as the Hilbert function of \( \text{gin}(Q) \) except that they differ by 1 at degree \( f \). And \( Q \subseteq P_{d+1} \), so \( \text{gin}(Q) \subseteq \text{gin}(P_{d+1}) \). So \( \text{gin}(P_{d+1}) = \text{gin}(Q) + u \) for some \( u \in \text{gin}(P_{d+1}) : m \). Now apply the corollary above and we know that \( \text{gin}(Q) \) is generated by monomials inside \( k[x_1, \ldots, x_{d+1}] \). So by [HH], depth \( S/P = \text{depth} S/\text{gin}(P) = n - d - 1 \).
5. DIFFERENCE IN THE HILBERT COEFFICIENTS

We want to know the difference between the Hilbert coefficients after modulo $s$ general linear forms.

**Lemma 5.1.** Let $M, N$ be two graded module over some polynomial ring $S$ with an exact sequence $0 \rightarrow M_1 \rightarrow N \rightarrow M \rightarrow M_2 \rightarrow 0$. Let $\dim M = r, \dim M_1 = r_1, \dim M_2 = r_2, s = \max\{r_1, r_2\}$. (1) If $r_1 < r, r_2 < r$, then $e_i(N) = e_i(M)$ for $i < r - s$. (2) If $r_1 = r, r_2 < r$ and $\dim N = r$ then $e_0(N) = e_0(M) - e_0(M_1)$. (3) If $r_1 < r, r_2 = r$ and $\dim N = r$ then $e_0(N) = e_0(M) + e_0(M_2)$.

**Proof.** We know the Hilbert series is additive, so $h_M(t) = h_N(t) + h_{M_2}(t) - h_{M_1}(t)$. Let $h_M(t) = q_M(t)/(1 - t)^r, h_{M_1}(t) = q_{M_1}(t)/(1 - t)^{r_1}, h_{M_2}(t) = q_{M_2}(t)/(1 - t)^{r_2}$. Then $h_N(t) = q_N(t)/(1 - t)^{r_2}$ where $q_N(t) = q_M(t) + q_{M_1}(t)(1 - t)^{r_2} - q_{M_2}(t)(1 - t)^{r_2}$. In case (1) we know $q_N(1) = 0$. Now we know $q_M(t) = e_0(M) + e_1(M)(t - 1) + e_2(M)(t - 1)^2 + \ldots$ and $q_N(t) = e_0(N) + e_1(N)(t - 1) + e_2(N)(t - 1)^2 + \ldots$ and their difference is $q_{M_1}(t)(1 - t)^{r_2} - q_{M_2}(t)(1 - t)^{r_2}$ which can be divided by $(1 - t)^{r_2}$. So this means $e_i(N) = e_i(M)$ for $i < r - s$. In case (2) or (3) we still get $q_M(t) - q_N(t) = q_{M_1}(t)(1 - t)^{r_2} - q_{M_2}(t)(1 - t)^{r_1}$. Now in case (2) or (3) we substitute $t = 1$ and use the condition, then we get the conclusion.

Let $J$ be a strongly stable monomial ideal, $S = S(n), \dim(S/J) = r, d$ a positive integer satisfying $d \geq n - r$. Let $J_1$ be the ideal generated by minimal monomial generators of $J$ inside $\{x_1, \ldots, x_d\}; J_2 = \Phi_{d+1}(J)$. Then $J_1 \subset J \subset J_2$ and $\Pi_{d+1}J_1 = \Pi_{d+1}J_2$. We set $r = \dim(S/J)$, then it’s easy to see $r = \dim(S/J_1) = \dim(S/J_2)$.

We have:

**Lemma 5.2.** $e_i(S/J_2) = e_i(S/J_2)$ for $0 \leq i \leq r - n + d$, $e_i(S/J_2) = e_i(S/J_1)$ for $0 \leq i \leq r - n + d + 1$, then $e_{r-n+d}(S/J_2) - e_{r-n+d}(S/J_1) = \text{rank}_{S/J_2}(J_2/J_1)$.

**Proof.** We know for any $d + 1 \leq i \leq n$, $\Phi_{i+1}(J) \subset \Phi_i(J) = \Phi_{i+1}(J) : x_i^n$. For any $u \in \Phi_i(J), u \notin \Phi_{i+1}(J)$, we know $x_i^n u \in \Phi_{i+1}(J)$. Now $\Phi_i(J)$ is finitely generated, so we can find $n$ such that $x_i^n \Phi_i(J) \subset \Phi_{i+1}(J)$. Now since $\Phi_{i+1}(J)$ is strongly stable, we know $x_i^n \Phi_i(J) \subset \Phi_{i+1}(J)$ for any $j \leq i$. This means that $\Phi_i(J)/\Phi_{i+1}(J)$ is a module over $S(n)/(x_1^n, x_2^n, \ldots, x_i^n)$ which is a ring of dimension $n - i$. So let the dimension of the module $\Phi_i(J)/\Phi_{i+1}(J)$ be $r_1$, then $r_1 \leq n - i \leq n - d - 1$. Now consider the exact sequence $0 \rightarrow \Phi_i(J)/\Phi_{i+1}(J) \rightarrow S(n)/\Phi_{i+1}(J) \rightarrow S(n)/\Phi_i(J) \rightarrow 0$. Apply the lemma above, we know $e_j(S(\Phi_i(J))) = e_j(S(\Phi_{i+1}(J)))$ for $j \leq r - n - d$. For $J_1, J_2$, by the argument above it suffices to prove the equality if we replace $J$ by $J_2$. We know $S/J_1$ and $S/J_2$ are both free $k[x_{d+1}, \ldots, x_n]$-module since $J_1$ and $J_2$ are generated by monomials in $x_1, \ldots, x_d$. Now for any monomial minimal generator $u$ of $J_2$ satisfying $u \notin J_1$, we know $u \in k[x_1, \ldots, x_d]$ and $x_i^n u \in J_1$ for all $1 \leq i \leq d$. By definition of $J_1, x_i^n u \in J_1$ for such $i$. This means that $J_2 \subset J_1 : (x_1, \ldots, x_d)\infty$. So $J_2/J_1$ is a free $S(d)$-module of finite rank. It has dimension exactly $n - d$ and $e_0(J_2/J_1) = \text{rank}_{S(d)}(J_2/J_1) = \dim_k(\Pi_{d+1}(J_2)/\Pi_{d+1}J_1)$. So by applying the above lemma to the exact sequence $0 \rightarrow J_2/J_1 \rightarrow S/J_1 \rightarrow S/J_2 \rightarrow 0$ we know $e_i(S/J_2) = e_i(S/J_1)$ for $0 \leq i \leq r - (n - d) - 1$ and $e_{r-n+d}(S/J_2) = e_{r-n+d}(S/J_1) + e_0(J_2/J_1)$.
Remark 5.3. \( \Phi_{d+1}(J) = \Pi_{d+1}(J) + u \) for some \( u \in \Pi_{d+1}(J) : \text{m} \) if and only if 
\[ \text{rank}_{k[x_{d+1}, \ldots, x_n]}(J_2/J_1) = \text{dim}_k(\Pi_{d+1}(J_2)/\Pi_{d+1}(J_1) = 1. \]

Proof. Since \( \Phi_{d+1}(J) \) and \( J_2 \) are both free \( k[x_{d+1}, \ldots, x_n] \)-modules, their quotient is a free \( k[x_{d+1}, \ldots, x_n] \)-module of rank 1 if and only if after modulo \( (x_{d+1}), \ldots, x_n) \) the quotient is a one dimensional \( k \)-vector space, if and only if \( \Phi_{d+1}(J)/\Pi_{d+1}(J) = k. \)

\[ \Box \]

Theorem 5.4. Let \( P \) be a homogeneous prime ideal in \( S(n), \text{dim}(S(n))/P = r. \)

Let \( 1 \leq d \leq n, s = n - d, \) choose \( s \) general linear forms \( l_1, \ldots, l_s. \) Denote \( P_1 = \pi_1 \ldots \pi_s(P). \) Suppose \( e_i(P) = e_i(P_1) \) for \( 1 \leq i \leq r - s - 1 \) and \( e_{r-s}(P) = e_{r-s}(P_1) + 1, \) then \( \text{depth}(S/P) = n - d - 1. \)

Proof. Let \( J = \text{gin}(P). \) Denote \( J_1, J_2 \) as before. Then \( J_1 = \text{gin}(P_1) \) by the lemma above. Now taking generic initial ideal does not change the Hilbert series, so the Hilbert coefficients are the same, so we have \( e_i(J) = e_i(J_1) \) for \( 1 \leq i \leq r - s - 1 \) and \( e_{r-s}(P) = e_{r-s}(P_1) + 1. \) By the above we know \( \Phi_{d+1}(J) = \Pi_{d+1}(J) + u \) for some \( u \in \Pi_{d+1}(J) : \text{m}. \) Now by the main theorem we know \( \text{depth}(S/P) = n - d - 1. \) 

\[ \Box \]

6. Simple cases and case where \( n-d=\text{dim}(S/P) \)

We can also talk about the generic initial ideal of a prime ideal when this prime is very simple.

Proposition 6.1. Let \( n \geq 3. \) Then a strongly stable ideal \( J = \text{gin}(P) \) for some height 1 prime if and only if \( J = x_1^e \) for some \( e > 0. \)

Proof. Since \( S(n) \) is a UFD, a height 1 prime is just a principle ideal generated by an irreducible ideal \( f. \) Now for general linear change of coordinate \( \alpha, \text{inv}(\alpha f) = x_1^{\deg(f)}. \) Conversely for any degree \( d, \) we have a polynomial \( x_1^d - x_2^{d-1}x_3. \) Apply the Eisenstein criterion for the ideal \( (x_3) \) we know it is irreducible, and its initial monomial is just \( x_1^d. \)

\[ \Box \]

Corollary 6.2. Let \( J = \text{gin}(P). \) Suppose \( \text{ht}(J) = 1, \) then \( J = x_1^e \) for some \( e > 0. \)

Proof. \( \text{dim}(S/P) = \text{dim}(S/J) = n - 1. \) So \( \text{ht}(P) = 1 \) because \( S(n) \) is catenary and \( P \) is a prime. The rest follows by the proposition above.

If we assume \( \text{dim}(S/P) = n - d \) in the previous theorems we will get some interesting results. In this case we have the following property:

Proposition 6.3. Let \( \text{dim}(S/P) = n - d, J = \text{gin}(P), \) then: (1) \( x_1, \ldots, x_d \in \text{rad}(J) \) and \( x_{d+1}, \ldots, x_n \notin \text{rad}(J). \) (2) \( R = k[x_{d+1}, \ldots, x_n] \) is a Noether normalization of \( S/P \) and \( S/J. \) (3) \( S/\Phi_{d+1}(J) \) is free over \( R \) and \( \Phi_{d+1}(J)/J \) is \( R \)-torsion. (4) \( \text{deg}(S/P) = \text{deg}(S/J) = \text{rank}_R(S/J) = \text{rank}_R(S/\Phi_{d+1}(J)) = \text{dim}_R(S/\Phi_{d+1}(J) + (x_{d+1}, x_{d+2}, \ldots, x_n)). \)

Now in this case, \( \Phi_{d+1}(J) \) is \( J + u \) if and only if \( \text{deg}(S/\Phi_{d+1}(J)) + 1 = \text{deg}(S/J). \) We can say in this case we have exactly one more degree after applying \( \Phi_{d+1}. \) So by the previous result we have:

Theorem 6.4. Let \( P \) be a prime ideal in \( S = S(n), \text{dim}(S/P) = n - d, J = \text{gin}(P), \) and \( J \) satisfies \( \text{deg}(S/\Phi_{d+1}(J)) + 1 = \text{deg}(S/J). \) Then \( J \) is generated by monomials in \( k[x_1, \ldots, x_{d+1}] \) and \( x_{d+1} \) appears in the minimal generator of \( J. \) Moreover, \( \text{depth}(S/P) = n - d - 1 \) and \( S/P \) is almost Cohen-Macaulay.
REFERENCES


