RESTRICTIONS ON HILBERT COEFFICIENTS GIVE THE DEPTH OF A PRIME IDEAL INSIDE THE POLYNOMIAL RING

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ABSTRACT. In this paper, we prove that for a prime ideal P of dimension r inside a polynomial ring, if adjoining s general linear forms to the prime ideal changes the $r-s$-th Hilbert coefficient by 1 and doesn't change the 0th to $r-s-1$-th Hilbert coefficients where $s \leq r$, then the depth of S/P is $n-s-1$. This criteria also tells us about possible restrictions on the generic initial ideal of a prime ideal inside a polynomial ring.

1. SETTINGS

In this paper, we assume k is always an infinite field. Let $S(n) = k[x_1, ..., x_n]$ be the polynomial ring of n variables over k. $\mathfrak{m}(n) = (x_1, ..., x_n)$ be the graded maximal ideal of S. Let $l = c_1x_1 + c_2x_2 + ... + c_nx_n$ be a linear form in $S(n)$. There is a projection map $S(n) \to S(n)/lS(n)$. If $c_n \neq 0$, $S(n-1) \cong S(n)/lS(n)$ and this isomorphism is the composition of the embedding of $S(n-1)$ into $S(n)$ and the projection. So we identify $S(n)/lS(n)$ with $S(n-1)$ and define the new projection map to be $\pi_l : S(n) \to S(n-1)$. This makes sense when $c_n \neq 0$, so it makes sense when $l = x_n$ or when l is a general linear form. If $l = x_n$, we also denote π_{x_n} by π_n. We can also define $<_{rev}$ on $S(n-1)$ and $\pi_{n-1} : S(n-1) \to S(n-2)$. So we can compose the maps $\pi_{d+1}\pi_{d+2}...\pi_n$ and denote it by Π_{d+1}; it is the projection map $S(n) \to S(d)$. Also, $\pi_l(\mathfrak{m}(n)) = \mathfrak{m}(n-1)$. If $I \subset S(n)$, $I : \mathfrak{m}(n)^\infty$ is called the saturation of I, denoted by I^{sat}.

Now we recall the definition of the generic initial ideal. Suppose k is an infinite field. Consider the graded reverse lexicographic order $<$ on $S(n)$, then we can define the initial ideal of an ideal I in S. Let $\alpha \in GL_n(k)$ be a linear automorphism of $S(n)$. Now we can define $in_<(\alpha(I))$ for any $\alpha \in GL_n(k)$; there exist a Zariski open set U of $GL_n(k)$ such that for all $\alpha \in U$ this initial ideal $in_<(\alpha(I))$ is the same. This statement only makes sense when k is infinite because otherwise $GL_n(k)$ is discrete, so any subset of $GL_n(k)$ is open. This ideal is called the generic initial ideal with respect to $<$ and denoted by $gin_<(I)$ or $gin(I)$ if the order is clear.

We also recall the definition of Hilbert coefficients. Let M be a finitely generated graded $S(n)$-module. The function $H : \mathbb{N} \to \mathbb{N}, H(n) = dim_k(M_n)$ is called the Hilbert function. The power series $h(t) = \Sigma_{i\in\mathbb{N}} H(i)t^i$ is called the Hilbert series. It is well known that the Hilbert series is of the form $q(t)/(1-t)^d$ with $d = dim(M)$, $q(t)$ is a polynomial with integer coefficients satisfying $q(1) \neq 0$. Now we can expand $q(t)$ as linear combinations of powers of $t-1 :$ $q(t) = e_0 + e_1(t-1) + e_2(t-1)^2 + ...$, and these e_i's are called the Hilbert coefficients. All the e_i's are rational numbers.

For a monomial $u = x_1^{e_1}x_2^{e_2}...x_n^{e_n}$, we denote $\phi_i(u) = u/x_i^{e_i}$, that is, we eliminate all x_i's from the factors of u. Denote the composition $\phi_{d+1}\phi_{d+2}...\phi_n$ to be Φ_{d+1}.

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If J is a monomial ideal in $S(n)$, minimally generated by monomials $u_1, ..., u_s$. We denote $\phi_i(J) = (\phi_i(u_1), ..., \phi_i(u_s)) = J : {x_i}^{\infty}$ and $\overline{\phi_i}(J)$ = image of $\phi_i(J)$ in $S(n)/x_iS(n)$. In particular $\overline{\phi_n}(J)$ is an ideal in $S(n)/x_nS(n) = S(n-1)$. Inductively we can define $\Phi_{d+1}(J) = \phi_{d+1}\phi_{d+2}...\phi_n(J)$; it is an ideal in $S(n)$. Define $\overline{\Phi_{d+1}}(J) = \overline{\phi_{d+1}\phi_{d+2}}...\overline{\phi_n}(J)$. It is an ideal in $S(d)$.

We have the following two properties showing that the generic initial ideal in reverse lexicographic order behaves well under the projection map and saturation:

Proposition 1.1 (2, Proposition 2.14)**.** *Let I be a homogeneous ideal in $S(n)$, l be a general linear form in $S(n)$. Then $gin(\pi_l(I)) = \pi_n(gin(I))$.*

Remark 1.2. Here we view $\pi_l(I)$ as an ideal of $S(n-1)$ so that the generic initial ideal is well-defined. $\pi_n(gin(I))$ is also an ideal of $S(n-1)$, thus this equality makes sense because it compares two ideals in the same ring $S(n-1)$.

Proposition 1.3. *$gin(I^{sat}) = gin(I) : x_n^{\infty} = gin(I)^{sat}$.*

Proof. The first equality is proved in [2, Proposition 2.21]. The second statement is true because $gin(I)$ is of Borel type. □

2. GENERAL HYPERPLANE SECTION

Let I be a saturated homogeneous ideal in $S(n)$. Let l be a linear form such that the x_n-coefficient of l is nonzero. We call the ideal $\widetilde{\pi_l}(I) = (\pi_l(I))^{sat}$ the section with one hyperplane. It is an ideal in $S(n-1)$. If we have s linear forms $l_1, ..., l_s$, then inductively we can define the section with s hyperplanes: the section with one hyperplane is $I_1 = \widetilde{\pi_{l_1}}(I)$, it is an ideal in $S(n-1)$; let $\overline{l_2}$ be the image of l_2 inside $S(n-1)$, so define the section with two hyperplanes $I_2 = \pi_{\overline{l_2}}(I_1)^{sat}$, and inductively, with s hyperplanes is $I_s = \pi_{\overline{l_s}}(I_{s-1})^{sat}$. Let $d = n-s$, so I_s is an ideal in $S(d)$. In algebraic geometry, we can consider the subvariety X of $\mathbb{P}^{n-1}$ corresponding to I; the intersection of X with s hyperplanes defined by $l_1, ..., l_s$ is X_s. Then I_s will be the defining ideal of X_s inside the linear subvariety $\mathbb{P}^{d-1}$ defined by $l_1, ..., l_s$.

Proposition 2.1. *$gin(I_s) = \widetilde{\pi_{d+1}}\widetilde{\pi_{d+2}}...\widetilde{\pi_n}(I)$.*

Proof. Apply Proposition 1.1 and 1.3 inductively. □

Proposition 2.2. *Let J be a saturated monomial ideal of Borel type of $S(n)$. Then* (1)$\pi_{d+1}\widetilde{\pi_{d+2}}...\widetilde{\pi_n}(J) = \overline{\phi_{d+1}}...\overline{\phi_n}(J) = \overline{\Phi_{d+1}(J)}$ *and* (2)$\widetilde{\pi_{d+1}}\widetilde{\pi_{d+2}}...\widetilde{\pi_n}(J) = \phi_d\overline{\phi_{d+1}}...\overline{\phi_n}(J) = \phi_d\overline{\Phi_{d+1}(J)}$.

Proof. J is of Borel type and saturated, so $J = J^{sat} = J : {x_n}^{\infty} = \phi_n(J)$ and $\pi_n(J) = \overline{\phi_n}(J)$, which means (1) is true for $d = n-1$ and (2) is true for $d = n$. Now for any n, if J is a monomial ideal of Borel type in $S(n)$, $\overline{\phi_n}(J)$ is a monomial ideal in $S(n-1)$ and we claim it's also of Borel type. Suppose $J = (u_1, ..., u_s)$, then $\overline{\phi_n}(J) = (\phi_n(u_1), ..., \phi_n(u_s))$ if we view the monomials $\phi_n(u_i)$ as monomials inside $k[x_1, ..., x_{n-1}]$. So every minimial generator is of the form $\phi_n(u_k)$ for some k. Choose $1 \leq j < i \leq n-1$ such that $x_i|\phi_n(u_k)$. Since $i \neq n$, $x_i|u_k$. Since J is of Borel type, there exist t such that ${x_j}^t u_k/x_i$ is in J, so $u'_k|{x_j}^t u_k/x_i$. Since $i, j \neq n$,$\phi_n(u'_k)|{x_j}^t\phi_n(u_k)/x_i$. This means that $\overline{\phi_n(J)}$ is still of Borel type. Now inside the ring $S(n-1)$ we have $\widetilde{\pi_n}(J) = \pi_n(J) : {x_{n-1}}^{\infty} = \overline{\phi_n}(J) : {x_{n-1}}^{\infty} = \phi_{n-1}\overline{\phi_n}(J)$ and $\pi_{n-1}\widetilde{\pi_n}(J) = \overline{\phi_{n-1}\phi_n}(J)$ in $S(n-2)$. Finally we induct on $r = n-d$. □

3. THEOREM

The purpose of this section is to find any restriction on the generic initial ideal of a prime.

Theorem 3.1. *Let P be a homogeneous prime ideal in polynomial ring $S = S(n)$and $J = in(P)$ be the initial ideal of P. Assume J is of Borel type. Suppose for some $1 \leq d \leq n$ we have that $\overline{\Phi_{d+1}}(J) = \Pi_{d+1}(J) + u$ for some $u \in \Pi_{d+1}(J) : \mathfrak{m}$. Then either (1) $u = 1$ and hence J contains the ideal $(x_1, ..., x_d)$ or (2) J can be generated by all generators of J inside $k[x_1, ..., x_d]$ and u. Moreover, u is of the form vx_{d+1}^e where $v \in \pi_{d+1}\Phi_{d+2} : \mathfrak{m}$ and e is a positive ingeter, so all the minimial generator of J is inside $k[x_1, ..., x_{d+1}]$.*

Proof. Let v be a monomial which is a minimial generator of J. There exists a polynomial $f \in P$ with $in(f) = v$. We claim that f is irreducible as a polynomial in S. Otherwise $f = f_1 f_2$, f_1, f_2 are not constants, then $in(f) = in(f_1)in(f_2)$, and $in(f) \neq in(f_1)$ or $in(f_2)$. And P is a prime, which means that $f_1 \in P$ or $f_2 \in P$, then $in(f_1) \in J$ or $in(f_2) \in J$, which contradicts the minimality of $in(f) = v$.

Let $u_1, ..., u_s, v_1, ..., v_t$ be the minimial generators of J where u_i's are inside $k[x_1, ..., x_d]$ and v_j's are not. Then $\Phi_{d+1}u_i = u_i$, $\Phi_{d+1}v_j \neq v_j$. $\overline{\Phi_{d+1}(J)}$ is generated by $\Phi_{d+1}u_i = u_i, \Phi_{d+1}v_j$ and is minimially generated by u_i, u. $\overline{\Phi_{d+1}(J)}$ is also minimially generated by u_i, u, viewed as a monomial ideal in $S(d)$. If for some j, $\Phi_{d+1}v_j \neq u$, then $\Phi_{d+1}v_j = u_i$, so u_i divides v_j, which contradicts the minimality of v_j. So for all j, $\Phi_{d+1}v_j = u$, so $v_j = w_j u$ for some $w_j \in k[x_{d+1}, ..., x_n]$. There exists at least one v_j, because otherwise J is generated by monomials inside $k[x_1, ..., x_d]$ and we have $\overline{\Phi_{d+1}(J)} = \pi_{d+1}(\overline{\Phi_{d+2}}(J)) = \overline{J} \subset S/(x_{d+1}, ..., x_n)S$ but by the condition $\overline{\Phi_{d+1}(J)} \neq \pi_{d+1}(\overline{\Phi_{d+2}}(J))$. Now we claim there is only one such v_j. Otherwise, suppose $v_1 = w_1 u, v_2 = w_2 u$ are two distinct monomials inside the minimal generating set of J, where w_1, w_2 are not constants. Take f_1, f_2 be the element in the reduced Grobner basis of P satisfying $in(f_1) = v_1, in(f_2) = v_2$. By the argument above, f_1, f_2 are irreducible. Then any monomial appearing in f_1 or f_2 is not in $in(P)$ except their initial monomials. Now we write $f_1 = p_1 u + q_1, f_2 = p_2 u + q + 2$. Here $p_1 u$ is the sum of all the terms which appear in f_1 and divisible by u, and q_1 is the sum of the rest terms; and similar for p_2, q_2. It's easy to see that $in(p_1) = w_1$ and $in(p_2) = w_2$, and these initial monomials are in $k[x_{r+1}, ..., x_n]$. Besides, $u \notin (u_1, ..., u_s) : (x_1, ..., x_d)$, so any other monomial m in q_1 are not divisible by $x_1, x_2, ..., x_d$, otherwise mu divides some u_j. So they are also in the last $n - d$ variables, so q_1 is a polynomial in $k[x_{r+1}, ..., x_n]$, and similar for q_2. Now we consider the polynomial $F = p_2 f_1 - p_1 f_2 = p_2 q_1 - p_1 q_2$. Take any monomial $m_1 \in p_2, m_2 \in p_1$. Then $m_1 \in k[x_{d+1}, ..., x_n]$. m_2 is not divisible by u_i and u, so $m_2 \notin \Pi_{d+1}(J)$. So $m_1 m_2 \notin J$. This is true for any monomial in $m_1 m_2 \in p_2 q_1$ and similarly for $m_1 m_2 \in p_1 q_2$. So any possible term appearing in F is not in $J = in(P)$. But $F \in P$. This means that $F = 0$. So $p_2 f_1 = p_1 f_2$. Now f_1 is irreducible, so f_1 divides p_1 or f_2. If it divides p_1, we have $in(f_1) = v_1 = w_1 u$ is divisible by $in(p_1) = w_1$; thus it forces $f_1 = p_1$. In this case $u = 1 \in J : m$ so J contains $(x_1, ..., x_d)$. Otherwise f_1 divides f_2. Similarly if $J \neq m$ then f_2 divides f_1. So f_1 and f_2 differ by a constant multiple and their initial monomials are the same, which contradicts the assumption that $v_1 \neq v_2$ Finally $u = vx_{d+1}^e$ by stability conditions; if $u = vw$ with w not a power of x_{d+1}, then w divides a variable x_j with $j > d + 1$. Since J is of Borel type, there exist some e such that $u' = vwx_{d+1}^e / x_j \in J$, so it is divided by some minimal generator u_1 of J. Then we know $u_1 = vw_1$ with w_1 and w not dividing each other and they are both monomials in $k[x_{d+1}, x_{d+2}, ..., x_n]$, which contradicts the conclusion of the previous step. So w must be a power of x_{d+1}. □

We get the following corollary:

Corollary 3.2. *If we replace $J = in(P)$ by $J = gin(P)$ in the last theorem, then the conclusion still holds.*

Proof. We know that the generic initial ideal is strongly stable. Suppose $J = in(\alpha(P))$ for some $\alpha \in GL_n(k)$. Now $\alpha(P)$ is still a prime and we apply the theorem to $\alpha(P)$. □

Now we make this result stronger by replacing J with a larger ideal $\pi_{d+1}\overline{\Phi_{d+2}}(J)$. First we recall the Bertini Theorem:

Theorem 3.3. *Suppose k has characteristic 0. Let P be a homogeneous prime ideal in the polynomial ring S, s be a positive ingeter, $s \leq dim(S/P) - 2$. Choose s general linear forms $l_1, l_2, ..., l_s$, then $\widetilde{\pi_{l_s}}\widetilde{\pi_{l_{s-1}}}...\widetilde{\pi_{l_1}}(P)$ is still a prime ideal.*

Corollary 3.4. *Let P be a homogeneous prime ideal in polynomial ring $S = S(n)$and $J = gin(P)$. Suppose for some $1 \leq d \leq n$ we have $dim(S/P) \geq n - d + 1$, and that $\overline{\Phi_{d+1}}(J) = \pi_{d+1}\overline{\Phi_{d+2}}(J) + u$ for some $u \in \pi_{d+1}\overline{\Phi_{d+2}}(J) : \mathfrak{m}$. Then either (1) $u = 1$ and hence J contains the ideal $(x_1, ..., x_d)$ or (2) J can be generated by all generators of J inside $k[x_1, ..., x_d]$ and u. Moreover, u is of the form vx_{d+1}^e where $v \in \pi_{d+1}\Phi_{d+2} : \mathfrak{m}$ and e is a positive ingeter, so all the minimial generator of J is inside $k[x_1, ..., x_{d+1}]$.*

Proof. Let $s = n - d$. We choose s general linear forms $l_1, l_2, ..., l_s$ and consider $P_{s-1} = \widetilde{\pi_{l_{s-1}}}...\widetilde{\pi_{l_1}}(P)$. It is a prime ideal in $S(d+1)$ because $s-1 \leq dim(S/P)-2$. Now $gin(P_{s-1}) = \overline{\Phi_{d+2}}(gin(P))$ by the lemma, so $\overline{\Phi_{d+1}}(gin(P_{s-1})) = \overline{\Phi_{d+1}}(gin(P))$ and $\pi_{d+1}\overline{\Phi_{d+2}}(J) = \pi_{d+1}(gin(P_{s-1}))$. Apply the theorem to the prime ideal $P_{s-1} \subset S(d+1)$, we know $gin(P_{s-1})$ is generated in $k[x_1, ..., x_{d+1}]$. Now we claim $gin(P)$ is generated in $k[x_1, ..., x_{d+1}]$. Otherwise, we can find $u = vw$ which is a minimal generator inside $gin(P)$ where $v \in k[x_1, ..., x_{d+1}]$ and $w \in k[x_{d+2}, ..., x_n]$ with $w \neq 1$. Since $gin(P)$ is strongly stable, $vx_{d+2}^{deg(w)}$ is in $gin(P)$. Now this monomial is divisible by some minimal generator $u_1 = v_1w_1$, $v \in k[x_1, ..., x_{d+1}]$ and $w \in k[x_{d+2}, ..., x_n]$ and w_1 a pure power of x_{d+2}. w_1 cannot be 1, otherwise u_1 properly divides u, so u is not a minimal generator. Now $\Phi_{d+2}(u_1) = u$. □

By Bayer and Stillman's theorem [3, Corollary 4.3.18] we know that $S(n)/I$ and $S(n)/gin_(I)$ have the same depth as rings. So, from the above theorem we can get:

Theorem 3.5. *Let P be a prime ideal in S, $1 \leq d \leq n-1$, P_{d+1} is the $n-d$ general hyperplane section of P, Q is $\pi_{l_{r+1}}(P_{d+2})$. If $P_{d+1} = Q + (f)$ with $f \in P_{d+1} : \mathfrak{m}$, then depth $S/P = n - d - 1$.*

Proof. By the following lemma, we can take generic initial of all the ideals above. We have $gin(P_{d+1}) = \Phi_{d+1}(gin(P))$, and $gin(Q) = gin(\pi_{l_{r+1}}(P_{d+2})) = \pi_{d+1}(gin(P_{d+2})) = \pi_{d+1}\overline{\Phi_{d+2}(gin(P))}$. We have $P_{d+1} = Q + (f)$ with $f \in P_{d+1} : \mathfrak{m}$, so the Hilbert function of P_{d+1} is the same as the Hilbert function of Q except that they differ by 1 at degree f. Taking generic initial does not change the Hilbert function, so the Hilbert function of $gin(P_{d+1})$ is the same as the Hilbert function of $gin(Q)$ except that they differ by 1 at degree f. And $Q \subset P_{d+1}$, so $gin(Q) \subset gin(P_{d+1})$. So $gin(P_{d+1}) = gin(Q) + u$ for some $u \in gin(P_{d+1}) : \mathfrak{m}$. Now apply the corollary above and we know that $gin(P)$ is generated by monomials inside $k[x_1, ..., x_{d+1}]$. So depth S/P = depth $S/gin(P) = n - d - 1$. □

4. DIFFERENCE IN THE HILBERT COEFFICIENTS

The previous section talks about the condition on the generic initial ideal. However, the generic initial ideal of an ideal is implicit and requires some unknown computation. We need explicit condition on the ideal, that is, the Hilbert coefficient which only depends on the Hilbert series of the quotient ring. We want to know how the Hilbert coefficients change after going modulo s general linear forms.

Lemma 4.1. *Let M, N be two graded module over some polynomial ring S with an exact sequence $0 \to M_1 \to N \to M \to M_2 \to 0$. Let $dimM_1 = r_1, dimM_2 = r_2$, $s = max\{r_1, r_2\}$. Assume $dimM = dimN = r$. (1) $e_i(N) = e_i(M)$ for $i < r - s$. (2)If $r_1 = s > r_2$, then $e_{r-s}(N) = e_{r-s}(M) + (-1)^{r-s}e_0(M_1)$. If $r_1 < s = r_2$, then $e_{r-s}(N) = e_{r-s}(M) - (-1)^{r-s}e_0(M_2)$. If $r_1 = s = r_2$, then $e_{r-s}(N) = e_{r-s}(M) + (-1)^{r-s}e_0(M_1) - (-1)^{r-s}e_0(M_2)$.*

Proof. We know the Hilbert series is additive, so $h_M(t) = h_N(t) + h_{M_2}(t) - h_{M_1}(t)$. Let $h_M(t) = q_M(t)/(1-t)^r, h_{M_1}(t) = q_{M_1}(t)/(1-t)^{r_1}, h_{M_2}(t) = q_{M_2}(t)/(1-t)^{r_2}$. Then $h_N(t) = q_N(t)/(1-t)^r$ where $q_N(t) = q_M(t) + q_{M_2}(t)(1-t)^{r-r_2} - q_{M_1}(t)(1-t)^{r-r_1}$. Now we expand both sides in terms of powers of $t-1$ and look at the coefficients of $1, (t-1), ..., (t-1)^{r-s}$. □

Let J be a monomial ideal of Borel type, $S = S(n)$, $dim(S/J) = r$, d a positive integer satisfying $d \geq n - r$. Let J_1 be the ideal generated by minimal monomial generators of J inside $k[x_1, ..., x_d]$; $J_2 = \Phi_{d+1}(J)$. Then $J_1 \subset J \subset J_2$ and $\Pi_{d+1}J_1 = \Pi_{d+1}J$. We set $r = dim(S/J)$, then it's easy to see $r = dim(S/J_1) = dim(S/J_2)$. We have:

Lemma 4.2. *$e_i(S/J) = e_i(S/J_2)$ for $0 \leq i \leq r-n+d$, $e_i(S/J) = e_i(S/J_1)$ for $0 \leq i \leq r-n+d+1$, then $e_{r-n+d}(S/J) - e_{r-n+d}(S/J_1) = (-1)^{r-n+d}rank_{k[x_{d+1},\dots,x_n]}(J_2/J_1) = (-1)^{r-n+d}dim_k\Pi_{d+1}(J_2)/\Pi_{d+1}J_1$ which is finite.*

Proof. We know for any $d+1 \leq i \leq n$, $\Phi_{i+1}(J) \subset \Phi_i(J) = \Phi_{i+1}(J) : x_i^\infty$. For any $u \in \Phi_i(J)$, $u \notin \Phi_{i+1}(J)$, we know $x_i^{n_i}u \in \Phi_{i+1}(J)$. Now $\Phi_i(J)$ is finitely generated, so we can find n such that $x_i^n\Phi_i(J) \subset \Phi_{i+1}(J)$. Now since $\Phi_{i+1}(J)$ is of Borel type, we know that there exist n' such that $x_j^{n'}\Phi_i(J) \subset \Phi_{i+1}(J)$ for any $j \leq i$. This means that $\Phi_i(J)/\Phi_{i+1}(J)$ is a module over $S(n)/(x_1^{n'}, x_2^{n'}, ..., x_i^{n'})$ which is a ring of dimension $n - i$. So let the dimension of the module $\Phi_i(J)/\Phi_{i+1}(J)$ be r_1, then $r_1 \leq n - i \leq n - d - 1$. Now consider the exact sequence $0 \to \Phi_i(J)/\Phi_{i+1}(J) \to S(n)/\Phi_{i+1}(J) \to S(n)/\Phi_i(J) \to 0$. Apply the lemma above, we know $e_j(S/\Phi_i(J)) = e_j(S/\Phi_{i+1}(J))$ for $j < r - (n-d-1) = r - n + d + 1$. So $e_j(S/J) = e_j(S/\Phi_{d+1}(J))$ for $j \leq r - n + d$. For J_1, by the argument above it suffices to prove the equality if we replace J by J_2. We know S/J_1 and S/J_2 are both free $k[x_{d+1}, ..., x_n]$-module since J_1 and J_2 are generated by monomials in $x_1, ..., x_d$. Now for any monomial minimal generator u of J_2 satisfying $u \notin J_1$, we know $u \in k[x_1, ..., x_d]$ and $x_{d+1}^n u \in J$. So $x_i^n u \in J_1$ for all $1 \leq i \leq d$. By definition of J_1, $x_i^n u \in J_1$ for such i. This means that $J_2 \subset J_1 : (x_1, ..., x_d)^\infty$. So J_2/J_1 is a free $S(d)$-module of finite rank. It has dimension exactly $n-d$ and $e_0(J_2/J_1) = rank_{S(d)}(J_2/J_1) = dim_k(\Pi_{d+1}(J_2)/\Pi_{d+1}J_1$. So by applying the above lemma to the exact sequence $0 \to J_2/J_1 \to S/J_1 \to S/J_2 \to 0$ we know $e_i(S/J_2) = e_i(S/J_1)$ for $0 \le i \le r-(n-d)-1$ and $e_{r-n+d}(S/J_2) = e_{r-n+d}(S/J_1)+(-1)^{r-n+d}e_0(J_2/J_1)$. □

Lemma 4.3. $\overline{\Phi_{d+1}}(J) = \Pi_{d+1}(J) + u$ *for some* $u \in \Pi_{d+1}(J) : \mathfrak{m}$ *if and only if* $rank_{k[x_{d+1},...,x_n]}(J_2/J_1) = dim_k(\Pi_{d+1}(J_2)/\Pi_{d+1}J_1 = 1$.

Proof. Since $\Phi_{d+1}(J)$ and J_2 are both free $k[x_{d+1}, ..., x_n]$-modules, their quotient is a free $k[x_{d+1}, ..., x_n]$-module of rank 1 if and only if after modulo $(x_{d+1}), ..., x_n)$ the quotient is a one dimensional k-vector space, if and only if $\overline{\Phi_{d+1}}(J)/\Pi_{d+1}(J) = k$. □

Theorem 4.4. *Let* P *be a homogeneous prime ideal in* $S(n)$, $dimS(n)/P = r$. *Let* $1 \le d \le n$, $s = n-d$, *choose* s *general linear forms* $l_1, ..., l_s$. *Denote* $P_1 = \pi_{l_1}...\pi_{l_s}(P)$. *Suppose* $e_i(S/P) = e_i(S/P_1)$ *for* $1 \le i \le r-s-1$ *and* $e_{r-s}(S/P) = e_{r-s}(S/P_1) + (-1)^{r-s}$, *then* $depth(S/P) = n-d-1$.

Proof. Let $J = gin(P)$. Denote J_1, J_2 as before. Then $J_1 = gin(P_1)$ by the lemma above. Now taking generic initial ideal does not change the Hilbert series, so the Hilbert coefficients are the same, so we have $e_i(J) = e_i(J_1)$ for $1 \le i \le r-s-1$ and $e_{r-s}(J) = e_{r-s}(J_1)+(-1)^{r-n+d}$. By the above we know $\overline{\Phi_{d+1}}(J) = \Pi_{d+1}(J)+u$ for some $u \in \Pi_{d+1}(J) : \mathfrak{m}$. Now by Theorem 3.1 we know $depth(S/P) = n-d-1$. □

5. Simple cases and case where n-d=dim(S/P)

We can also talk about the generic initial ideal of a prime ideal when this prime is very simple.

Proposition 5.1. *Let* $n \ge 3$ *and* J *be a monomial ideal in* $S(n)$. *Then there exists a prime ideal* P *of* S *such that* $J = gin(P)$ *for some height 1 prime if and only if* $J = x_1^e$ *for some* $e > 0$.

Proof. Since $S(n)$ is a UFD, a height 1 prime is just a principle ideal generated by an irreducible ideal f. Now for general linear change of coordinate α, $in(\alpha f) = x_1^{deg(f)}$. Conversely for any degree d, we have a polynomial $x_1^d - x_2^{d-1}x_3$. Apply the Eisenstein criterion for the ideal (x_3) we know it is irreducible, and its generic initial monomial is just x_1^d. □

Corollary 5.2. *Let* $J = gin(P)$. *Suppose* $ht(J) = 1$, *then* $J = x_1^e$ *for some* $e > 0$.

Proof. $dim(S/P) = dim(S/J) = n-1$. So $ht(P) = 1$ because $S(n)$ is catenary and P is a prime. The rest follows by the proposition above.

If we assume $dim(S/P) = n-d$ in the previous theorems we will get some interesting results. In this case we have the following property: □

Proposition 5.3. *Let* $dim(S/P) = n-d$, $J = gin(P)$, *then: (1)* $x_1, ..., x_d \in \sqrt{J}$ *and* $x_{d+1}, ..., x_n \notin \sqrt{J}$. *(2)* $R = k[x_{d+1}, ..., x_n]$ *is a Noether normalization of* S/P *and* S/J. *(3)* $S/\Phi_{d+1}(J)$ *is free over* R *and* $\Phi_{d+1}(J)/J$ *is* R*-torsion. (4)* $deg(S/P) = deg(S/J) = rank_R(S/J) = rank_R(S/\Phi_{d+1}(J)) = dim_k(S/\Phi_{d+1}(J) + (x_{d+1}, x_{d+2}, ..., x_n))$.

Now in this case, $r-n+d=0$, so by Lemma 4.2 and 4.3 $\overline{\Phi_{d+1}}(J)=\overline{J}+u$ if and only if $deg(S/\Phi_{d+1}(J))+1=deg(\overline{S}/\overline{J})$. We can say in this case we have exactly one more degree after applying Φ_{d+1}. So by Theorem 3.1 we have:

Theorem 5.4. *Let P be a prime ideal in $S=S(n)$, $dim(S/P)=n-d$, $J=gin(P)$, and J satisfies $deg(S/\Phi_{d+1}(J))+1=deg(\overline{S}/\overline{J})$. Then J is generated by monomials in $k[x_1,...,x_{d+1}]$ and x_{d+1} appears in the minimal generator of J. Moreover, $depth(S/P)=n-d-1$ and S/P is almost Cohen-Macaulay.*

Remark 5.5. This is a generalization of a lemma in Kwak's paper [1, Theorem 5.1] where we have $\overline{S}/\overline{J}=\overline{S}/\mathfrak{m}^{r+1}$. Here r is the reduction number of S/P.

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