DECOMPOSITION OF LOCAL COHOMOLOGY TABLES IN DIMENSION 3

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ABSTRACT. Let R be a polynomial ring over a field. Motivated by the work of Smirnov and De Stefani, we describe the extremal rays of the cone of local cohomology tables of finitely generated graded R-modules of projective dimension 1 and describe some cases when the local cohomology table of a module of dimension 3 decomposes as a positive $\mathbb{Q}$-linear combination of local cohomology tables of other modules.

1. INTRODUCTION

Let $R = k[x_1, \ldots, x_n]$ be a standard graded polynomial ring over a field k. The graded Betti number and the local cohomology module are two important homological data of graded modules over R. In 2006, Boij and Söderberg [2] formulated two conjectures on the cone of graded Betti tables of finitely generated Cohen-Macaulay modules. They are proved by David Eisenbud, Gunnar Fløystad and Jerzy Weyman in characteristic 0 in [6] and by Eisenbud and Schreyer in arbitrary characteristic in [7]. These conjectures are also exteneded to the non-Cohen Macaulay case by Boij and Söderberg in [3]. We denote the Betti table of a finitely generated graded module by $\beta^\bullet(\cdot)$ and we restate the above results in the following way:

Theorem 1.1. *Let $R = k[x_1, \ldots, x_m]$ be a standard graded polynomial ring. The extremal rays of the cone generated by Betti tables of finitely generated graded R-modules are given by the modules with a pure resolution, and every Betti table in the cone decomposes, that is, there exist modules $N_1, \ldots, N_s$ with pure resolutions and $r_1, \ldots, r_s \in \mathbb{Q}$, $r_1, \ldots, r_s > 0$ such that*

$$\beta^\bullet(M) = \Sigma_{i=1}^s r_i \beta^\bullet(N_i).$$

Eisenbud and Schreyer, in [7], asked for a similar description for the cone of graded local cohomology tables. They proved a result similar to Theorem 1.1 in [8] where the decomposition is given by a converging infinite sum. Returning to the original question about writing the cohomology table as a finite sum, only the dimension 2 case has been proved recently by Smirnov and De Stefani in [5]. We denote the local cohomology table of a graded module by $H^\bullet(\cdot)$ and we summarize Smirnov and De Stefani's results below:

Theorem 1.2 ([5], Theorem 4.6)**.** *Let $R = k[x_1, \ldots, x_m]$ and $S = k[x_1, x_2]$ be two standard graded polynomial rings. Assume $m \geq 2$. Identify S-modules as R-modules via the ring map $S \cong R/(x_3, \ldots, x_m)$. Then there exists a set A of*

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finitely generated graded S-modules such that for every finitely generated graded R-module M of dimension at most 2, there exist $N_1, \dots, N_s \in A$, $r_1, \dots, r_s \in \mathbb{Q}$ and $r_1, \dots, r_s > 0$ such that

$$H^\bullet(M) = \Sigma_{i=1}^s r_i H^\bullet(N_i).$$

Moreover, the set A describes the vertex set of the cone of local cohomology tables of finitely generated graded module of dimension at most 2.

In this paper we generalize the result of Smirnov and De Stefani to higher dimensions. The main theorem is a complete description of the extremal rays in projective dimension 1 case and two partial results in dimension 3. We have:

Theorem (See Theorem 4.21)**.** *Let R be a standard graded polynomial ring of dimension n. Let e be an integer with $1 \leq e \leq n$. There exists a subset B' of finitely generated graded R-modules of depth at least $e-1$ and dimension at most e such that for every finitely generated graded module M of depth at least $e-1$ and dimension at most e, there exist $N_1, \dots, N_s \in B'$, $r_1, \dots, r_s \in \mathbb{Q}$, $r_1, \dots, r_s > 0$ such that*

$$H^\bullet(M) = \Sigma_{i=1}^s r_i H^\bullet(N_i).$$

Moreover, when $e > 2$, the extremal rays of the cone of local cohomology tables of finitely generated graded modules of depth at least $e-1$ and dimension at most e form a proper subset B of B'.

By a simple lemma the proof of Theorem 4.21 can be reduced to the case $\dim M = \dim R$ which implies $\text{projdim} M \leq 1$. In this case, the proof relies on the concept of the Auslander transpose. For a finitely generated R-module M with a presentation matrix ϕ, the Auslander transpose, denoted by $\text{Tr}(M)$, is the cokernel of ϕ^*, where $* = \text{Hom}_R(\cdot, R)$ is the dual functor. We prove that under proper assumptions on M, there is a $\mathbb{Q}$-linear map that maps $\beta^\bullet(M)$ to $H^\bullet(Tr(M))$. This proposition allows us to use the Boij-Söderberg theory of Betti tables to decompose a local cohomology table. The vertices of the cone of Betti tables are given by the pure tables which can be computed using the degree sequence of the table, so we can also compute the corresponding local cohomology table and determine when a local cohomology table decomposes.

For modules of dimension 3, we may also assume that $\dim R = 3$. We first reduce to the case where $\text{depth}(M) = 1$ and M has no dimension 1 submodule. Then we relate M to the module of global sections $\Gamma(M)$ with $\text{depth}(\Gamma(M)) \geq 2$. This means $\text{projdim}(\Gamma(M)) \leq 1$ and its cohomology table decomposes according to Theorem 4.21. The key point is whether a decomposition of $H^\bullet(\Gamma(M))$ induces a decomposition of $H^\bullet(M)$, and it does in two cases, described by the following two theorems. In the first case, there is a submodule of dimension 2 of M that induces a decomposition:

Theorem (See Theorem 5.8)**.** *Let M be a module of depth 1 and assume M has no dimension 1 submodule. Let Γ be the module of global sections of the coherent sheaf defined by M, and $Q = \Gamma/\text{Tor}\Gamma$. Then $H^\bullet(M) = H^\bullet(\text{Tor}M) + H^\bullet(M/\text{Tor}M) - e$, where e is the vector $(0, HS(H^1_\mathfrak{m}(Q)), HS(H^1_\mathfrak{m}(Q)), 0)$. In particular, if $H^1_\mathfrak{m}(Q) = 0$, then $H^\bullet(M) = H^\bullet(\text{Tor}M) + H^\bullet(M/\text{Tor}M)$.*

A description of the module $H^1_\mathfrak{m}(Q)$ is given in Proposition 5.10.

In the second case there is a submodule of dimension 3 of M that induces a decomposition:

Theorem (See Corollary 5.14)**.** *Suppose $L = \mathrm{Ext}^2_R(TrM, R) = 0$, Γ be the module of global sections, and $\Gamma^* \neq Tr(L')$ for any module L' of finite length. Then $H^\bullet(M) = H^\bullet(M \cap F) + H^\bullet(M/M \cap F)$ for some free module $F \subset \Gamma$.*

This paper consists of five sections. The first two sections are introductions and notations. In section 3, we prove that in order to decompose the local cohomology table of a module M, it suffices to study the module of global sections $\Gamma(M)$ which has depth at least 2 and describe the local cohomology table of M in terms of $\Gamma(M)$ plus a module which is a quotient module of $\Gamma(M)$ and is of finite length. In section 4, we analyze the local cohomology table of all modules of projective dimension at most 1. In dimension 3 case, this is equivalent to the condition that the depth is at least 2. Finally in section 5, we find conditions under which we can decompose the local cohomology table of $\Gamma(M)$ and its quotient module simultaneously.

2. NOTATIONS

In all the following sections, let $R = k[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field k, and let $\mathfrak{m} = (x_1, \dots, x_n)$ be its graded maximal ideal. Let M be a finitely generated graded R-module. Throughout this paper, the *Betti table* of M is the $n \cdot \mathbb{Z}$-table $\beta^\bullet(M)$ with entries $\beta^\bullet(M)_{i,j} = \dim_k \mathrm{Tor}^R_i(M, k)_j$. In this definition we do not shift the internel degrees unlike the common definition. The *local cohomology table* $H^\bullet(M)$, by definition, is the $n \cdot \mathbb{Z}$-table defined by $H^\bullet(M)_{i,j} = \dim_k H^i_{\mathfrak{m}}(M)_j$, and the *Ext-table* $E^\bullet(M)$ is the $n \cdot \mathbb{Z}$-table defined by $E^\bullet(M)_{i,j} = \dim_k \mathrm{Ext}^i_R(M, R)_j$.

To simplify the notation, we write these tables in series form. The space of Betti tables or Ext-tables is $V = \oplus_{i=0}^n \mathbb{Q}[[t]][t^{-1}]v_i$. It is a free $\mathbb{Q}[[t]][t^{-1}]$-module of rank $n+1$, and it is also a $\mathbb{Q}$-vector space. The space of local cohomology table is $V^* = \oplus_{i=0}^n \mathbb{Q}[[t^{-1}]][t]v_i$. The Betti table of M is an element $(\beta_0(M), \beta_1(M), \dots, \beta_n(M)) \in V$ defined by $\beta_i(M) = \Sigma_{j\in\mathbb{Z}}\beta_{i,j}(M)t^j$. The Ext-table is an element $(E^0(M), E^1(M), \dots, E^n(M)) \in V$ where $E^i(M) = \Sigma_{j\in\mathbb{Z}}\dim_k \mathrm{Ext}^i_R(M, R)_j t^j$. The local cohomology table is $(h^0(M), h^1(M), \dots, h^n(M)) \in V^*$ where $h^i(M) = \Sigma_{j\in\mathbb{Z}}\dim_k H^i_{\mathfrak{m}}(M)_j t^j$. These two representations of a table are equivalent. Throughout this paper, when we mention a Betti table, a local cohomology table, or an Ext-table, we mean a table in the series form .

We want to consider *convex cones* C in the vector space V or V^*, that is, subsets which are closed under multiplication by positive rational numbers and addition. We call the expression $\sum_{1\leq i\leq s} a_i c_i$ with $c_i \in c, a_i \in \mathbb{Q}, a_i > 0$ a *positive linear combination* of $c_1, c_2, \dots, c_s$. If $c \in C$ is a positive linear combination of $c_1, c_2, \dots, c_s$, we also say that c *decomposes into* $c_1, c_2, \dots, c_s$; we say the decomposition is trivial if $s = 1$ and in this case c and c_1 differ by a positive rational number. A *generating set* of the cone is a subset G of the cone C such that every element is a positive linear combination of elements in G. We also say G generates the cone C if G is a generating set. A *vertex* is an element such that it does not decompose nontrivially and the *vertex set* is the set of all vertices. We say a ray inside the cone is *extremal* if it contains a vertex; in this case every element of the ray is a vertex except for the origin. In this paper we need to consider the vertex sets of 3 kinds of cones: the cone generated by the Betti tables, the Ext-tables, and the local cohomology tables. It is easy to see that if G is a generating set and v is a vertex, then v must decompose trivially, which means that a positive multiple of v is in G. So to find the vertex set we may find a generating set G first and then find elements in G that decompose trivially.

Let $E = E_R(k)$ be the graded injective hull of $k = R/\mathfrak{m}$. Recall that by local duality, $H^i_{\mathfrak{m}}(M) = \mathrm{Hom}_R(\mathrm{Ext}^{n-i}_R(M, R(-n)), E)$. This implies $\dim_k(H^i_{\mathfrak{m}}(M)_j) = \dim_k(\mathrm{Ext}^{n-i}_R(M, R)_{-n-j})$, hence we have:

Proposition 2.1. *The $\mathbb{Q}$-linear map $L_0 : V \to V^*$, where*

$$L_0(f_0(t), f_1(t), \ldots, f_n(t)) = t^{-n}(f_n(t^{-1}), f_{n-1}(t^{-1}), \ldots, f_0(t^{-1})),$$

is invertible and $H^\bullet(M) = L_0(E^\bullet(M))$.

By the above proposition, the extremal rays of the cone of local cohomology tables and the cone of Ext-tables are in 1-1 correspondence under L_0. So to find the extremal rays of the cone generated by all local cohomology tables, it suffices to find that of all Ext-tables. The Betti numbers are nonzero for finitely many entries, and the dimension of the i-th Ext module has dimension at most i. So actually these tables sit in a smaller space.

Proposition 2.2. *The following proposition holds.*
(1) $\beta^\bullet(M) \in \oplus_{i=0}^n \mathbb{Q}[t][t^{-1}]v_i$.
(2) $E^\bullet(M) \in \oplus_{i=0}^n \mathbb{Q}[t][t^{-1}]\frac{1}{(1-t)^{n-i}}v_i$.
(3) $H^\bullet(M) \in \oplus_{i=0}^n \mathbb{Q}[t^{-1}][t]\frac{1}{(1-t^{-1})^i}v_i$.

3. Reduction to depth 2 case

Let M be a finitely generated graded R-module. We will calculate $H^\bullet(M)$ using the local cohomology table of a module of depth at least 2 plus some other information. First we consider the submodules of M of bounded dimension.

Proposition 3.1. *Let M be a finitely generated graded R-module. The following proposition holds.*
(1) There exists a maximal submodule M_i of M of dimension at most i. Every submodule of dimension at most i is a submodule of M_i.
(2) M/M_i has no submodule of dimension at most i except for the 0 module.
(3) $M_i = 0$ if and only if all associate primes $\mathfrak{p}$ of M satisfies $\dim(R/\mathfrak{p}) > i$, or equivalently, $\mathrm{ht}\mathfrak{p} < n - i$.

Proof. See Corollary 2.3 of [9]. □

Definition 3.2. The submodule M_i defined by the above proposition is called the *maximal submodule of M of dimension at most i*. In the case where $\dim M_i = i$ or $M_i = 0$ we can just call M_i the *maximal submodule of dimension i*.

Remark 3.3. If $\mathrm{depth}(M) = i$, then the submodule of dimension equal to i exists, because for any associate prime $\mathfrak{p}$ of M, $\dim R/\mathfrak{p} \geq i$, hence this is also true for any associate prime of M_i, so $\dim M_i \geq i$.

We prove that to study the decomposition of $H^\bullet(M)$, we may assume $\mathrm{depth} M \geq 1$ by removing the maximal submodule of dimension 0.

Lemma 3.4. *Let M be a finitely generated graded R-module. Then $M_0 = H^0_{\mathfrak{m}}(M)$ is the maximal submodule of dimension 0. We have an exact sequence $0 \to M_0 \to M \to M/M_0 \to 0$ which induces a decomposition of local cohomology tables $H^\bullet(M) = H^\bullet(M_0) + H^\bullet(M/M_0)$.*

The proof is trivial and we omit it.

So without loss of generality we may assume depth$M \geq 1$. In this case factoring out the maximal submodule of dimension 1 induces a decomposition:

Lemma 3.5. *Let M be a finitely generated graded R-module of depth 1, and M_1 be its maximal submodule of dimension 1. Then the exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$ induces a decomposition of local cohomology tables $H^\bullet(M) = H^\bullet(M_1) + H^\bullet(M/M_1)$.*

Proof. M_1 is a submodule of M, so if depth$M \geq 1$, then depth$M_1 \geq 1$. This means that M_1 is Cohen-Macaulay of dimension 1. Also, M/M_1 does not have submodule of dimension at most 1; hence $H^0_\mathfrak{m}(M/M_1) = 0$. Now the long exact sequence of local cohomology modules breaks up into short exact sequences:

$$0 \to H^1_\mathfrak{m}(M_1) \to H^1_\mathfrak{m}(M) \to H^1_\mathfrak{m}(M/M_1) \to 0$$

and

$$0 \to H^i_\mathfrak{m}(M) \to H^i_\mathfrak{m}(M/M_1) \to 0$$

for any $i \geq 2$, and all the connecting homomorphisms are 0, so $H^\bullet(M) = H^\bullet(M_1) + H^\bullet(M/M_1)$. □

For a module M satisfying the condition depth$M = 1$ and M has no submodule of dimension 1, we prove an important proposition.

Proposition 3.6. *Let M be a module of depth 1 and assume M has no submodule of dimension 1. Then $H^1_\mathfrak{m}(M)$ has finite length.*

Proof. The condition $l(H^1_\mathfrak{m}(M)) < \infty$ is equivalent to $l(\mathrm{Ext}^{n-1}_R(M, R)) < \infty$ by local duality. Since the module $\mathrm{Ext}^{n-1}_R(M, R)$ is finitely generated, this module has finite length if and only if $\mathrm{Ext}^{n-1}_R(M, R)_\mathfrak{p} = 0$, $\forall$ ht$\mathfrak{p} = n - 1$, which just means $\mathrm{Ext}^{n-1}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) = 0$, $\forall$ ht$\mathfrak{p} = n - 1$. Now the ring $R_\mathfrak{p}$ is a regular ring of dimension $n - 1$, so apply the local duality on $R_\mathfrak{p}$ to get the equivalent condition $H^0_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) = 0$, $\forall$ ht$\mathfrak{p} = n - 1$. Equivalently, $\mathfrak{p} \notin \mathrm{Ass}(M_\mathfrak{p})$, $\forall$ ht$\mathfrak{p} = n - 1$, or $\mathfrak{p} \notin \mathrm{Ass}M$, $\forall$ ht$\mathfrak{p} = n - 1$. This is true if and only if M has no submodule of dimension 1. □

We introduce the module of global sections here.

Proposition 3.7. *Let M be a finitely generated graded R-module. Let $\widetilde{M}$ be the coherent sheaf associated to M on $Proj(R) = \mathbb{P}^{n-1}$. Let*

$$\Gamma(M) = \oplus_{t \in \mathbb{Z}} H^0(Proj(R), \widetilde{M}(t))$$

Then we have an exact sequence of graded R-modules $0 \to H^0_\mathfrak{m}(M) \to M \to \Gamma(M) \to H^1_\mathfrak{m}(M) \to 0$. Moreover, $\mathrm{depth}(\Gamma(M)) \geq 2$ *and this exact sequence induces $H^i(M) \simeq H^i(\Gamma(M))$ for $i \geq 2$.*

Here Γ can be viewed as a functor from the category of finitely generated graded R-modules to the category of graded R-modules. The proposition is well-known and we omit the proof here. Below are some characterizations of the functor Γ.

Proposition 3.8 (universal property)**.** *Denote the natural map $M \to \Gamma(M)$ by i. Let M, N be two finitely generated graded R-modules, where* depth$M \geq 1$, depth$N \geq 2$. *Suppose $f : M \to N$ is an embedding, then there exist a unique embedding $f' : \Gamma(M) \to N$ such that $f = f'i$.*

Proof. Γ is left exact because sheafification, tensoring with $\mathcal{O}_{\mathbb{P}^{n-1}}(t)$ and H^0 are all left exact. Also, when depth$N \geq 2$, $\Gamma(N) = N$ by the exact sequence in the last proposition. So an embedding of modules $f : M \to N$ induces another embedding $\Gamma(f) : \Gamma(M) \to N$. Let $\Gamma(f) = f'$. Suppose conversely we have $f = f'i$ for some embedding $f' : \Gamma(M) \to N$, then $\Gamma(f) = \Gamma(f')\Gamma(i)$, but $\Gamma(f') = f'$ and $\Gamma(i) = \mathrm{id}_{\Gamma(M)}$, hence $f' = \Gamma(f)$ is unique. □

Proposition 3.9. *Let M, N be two finitely generated graded R-modules, M embeds into N,* depth$M \geq 1$, depth$N \geq 2$*. If $l(N/M) < \infty$, then $N = \Gamma(M)$.*

Proof. By the universal property $\Gamma(M)$ embeds into N. If it is not N, then by the depth lemma, $N/\Gamma(M)$ has depth at least 1, hence $l(N/\Gamma(M)) = \infty$, hence $l(N/M) = \infty$, which is a contradiction. □

Corollary 3.10. Let M, N be two finitely generated graded R-modules, M embeds into N, depth$M \geq 1$, depth$N \geq 2$. Let $M^{sat} = M :_N \mathfrak{m}^\infty$. Then $\Gamma(M) = M^{sat}$.

Proof. By construction, $H^0_\mathfrak{m}(N/M) = M^{sat}/M$, $(N/M)/H^0_\mathfrak{m}(N/M) = N/M^{sat}$, so depth $N/M^{sat} \geq 1$. And depth$N \geq 2$, hence we can use apply the depth lemma to get depth$M^{sat} \geq 2$, and M^{sat}/M is of finite length. By Proposition 3.9, $\Gamma(M) = M^{sat}$. □

Corollary 3.11. Let M be a finitely generated R-module of depth 1 and has no dimension 1 submodule. Then $\Gamma(M)$ is also finitely generated.

Proof. If depth$M > 0$, then $H^0_\mathfrak{m}(M) = 0$, so the long exact sequence in Proposition 3.7 becomes $0 \to M \to \Gamma(M) \to H^1_\mathfrak{m}(M) \to 0$. Now M is finitely generated, and $H^1_\mathfrak{m}(M)$ is of finite length by Proposition 3.6, hence is also finitely generated, so $\Gamma(M)$ is also finitely generated. □

In summary, we want to find a decomposition of $H^\bullet(M)$ for a general module M. By Lemma 3.4 and Lemma 3.5, without loss of generality, we may assume depth$(M) = 1$ where M has no submodule of dimension 1. In this case by Corollary 3.11 we have a finitely generated module $\Gamma(M)$ of depth at least 2 which contains M such that $H^\bullet(M)$ is equal to $H^\bullet(\Gamma(M))$ at position $i \geq 2$ and equal to the Hilbert function of $\Gamma(M)/M$ at position 1 which has finite length. So we need to study the local cohomology table of modules of depth at least 2, and their quotient of finite length.

4. Projective dimension 1 case

In this section we study the cones generated by $H^\bullet(M)$ where projdim$(M) \leq 1$, or equivalently, depth$(M) \geq n - 1$. We start with a lemma about cones which is trivial but useful in the following section.

Lemma 4.1. *Let $L : W_1 \to W_2$ be a linear map between vector spaces over $\mathbb{Q}$. Suppose $C \subset W_1$ is a cone with vertex set V_1 and a generating set G_1. Then $L(C)$ is a cone in W_2 generated by $L(G_1)$; suppose the vertex set of this cone is V_2, then $V_2 \subset L(V_1)$. V_2 is also the subset of $L(V_1)$ which is not a positive linear combination of the other elements in $L(G_1)$. If moreover L is an injection, then $V_2 = L(V_1)$.*

Next, let's recall the definition of Auslander transpose. We modify the definition in the case where the ring is local or graded-local.

Definition 4.2. Let M be a finitely generated module over a Noetherian ring R. Assume R is local or graded-local. Consider a minimal presentation $F_1 \xrightarrow{\phi} F_0 \to M \to 0$. Taking dual yields an exact sequence $0 \to M^* \to F_0^* \to F_1^* \to N \to 0$. Then the *Auslander transpose* of M is $Tr(M) = N = \mathrm{Coker}(\phi^*)$.

Remark 4.3. In general the Auslander transpose is unique up to a projective summand. Here it is unique up to isomorphism because in the definition we assume the ring is local or graded-local, and the presentation is minimal.

We recall some basic properties of the Auslander transpose.

Proposition 4.4.
(1)The Auslander transpose of a graded module is also graded.
(2)If $\mathrm{projdim}(M) \leq 1$*, then* $Tr(M) = \mathrm{Ext}^1_R(M, R)$.
(3)$Tr(M) = 0$ *if and only if* $\mathrm{projdim}(M) = 0$*, that is,* M *is free.*
(4)$Tr(M \oplus M') = Tr(M) \oplus Tr(M')$.
(5)$M = Tr(Tr(M)) \oplus F$*,* F *is free and* $Tr(Tr(M))$ *does not have a free summand.*

Let M be a finitely generated graded R-module. To study the decomposition of $H^\bullet(M)$, we may assume M is indecomposable without loss of generality. If $\mathrm{projdim} M = 0$ then M is free, and we must have $M \cong R(-i)$ for some i. In this case $H^\bullet(M)$ is clear. So we may assume that $\mathrm{projdim}(M) = 1$. Here is an important observation about properties of M and $Tr(M)$.

Proposition 4.5.
(1)Let M *be a finitely generated* R*-module with no free summand and* $\mathrm{projdim}(M) = 1$*. Let* $0 \to F_1 \xrightarrow{\phi} F_0 \to M \to 0$ *be a minimal presentation which is also a minimal resolution. Let* $N = Tr(M) \neq 0$*, then* $\mathrm{dim} N \leq n-1$*, and* $F_0^* \xrightarrow{\phi^*} F_1^* \to N \to 0$ *is a minimal presentation of* N.
(2)Let N *be a nonzero module with* $\dim(N) \leq n-1$*. Take a minimal presentation* $G_1 \xrightarrow{\psi} G_0 \to N \to 0$*. Then* $G_0^* \xrightarrow{\psi^*} G_1^*$ *is injective and has image in* $\mathfrak{m}G_1^*$*, so if* $M = Tr(N) = \mathrm{Coker}(\psi^*)$ *then* $\mathrm{projdim} M = 1$ *and* M *has a minimal resolution* $0 \to G_0^* \xrightarrow{\psi^*} G_1^* \to M \to 0$*. Also,* M *does not have a free summand.*
(3)Taking the Auslander transpose Tr *induces a 1-1 correspondence between the isomorphism classes of finitely generated graded* R*-modules* M *of projective dimension 1 without a free summand and finitely generated graded* R*-modules* N *of dimension at most* $n-1$.
(4)Under the assumption in (3), $\beta_{0,j}(M) = \beta_{1,-j}(N)$*,* $\beta_{1,j}(M) = \beta_{0,-j}(N)$*, or equivalently,* $\beta_0(M)(t) = \beta_1(N)(t^{-1}), \beta_1(M)(t) = \beta_0(N)(t^{-1})$.

Proof.
(1) Suppose $\mathrm{projdim} M = 1$ with minimal resolution $0 \to F_1 \xrightarrow{\phi} F_0 \to M \to 0$. Then $N = \mathrm{Ext}^1_R(M, R)$, so $\dim N \leq n-1$. Now consider the exact sequence $F_0^* \xrightarrow{\phi^*} F_1^* \to N \to 0$. Since ϕ has entries in $\mathfrak{m}$, so does ϕ^*, so the image of F_1^* generates N minimally. Now F_0^* surjects onto $\mathrm{Syz}_1(N)$; if the image of the basis of F_0^* is not a minimal generator, then some basis element e_i of F_0^* is mapped to 0 under ϕ^*. Then taking the dual again, Re_i^* will become a free summand of M, contradicting with our assumption. So the image of the basis of F_0^* is a minimal generating set of $\mathrm{Syz}_1(N)$. This means that $F_0^* \to F_1^* \to N \to 0$ is a minimal presentation of N.

(2) Take a minimal presentation $G_1 \xrightarrow{\psi} G_0 \to N \to 0$. Let $M = Tr(N)$, then $G_0^* \xrightarrow{\psi^*} G_1^* \to M \to 0$ is exact. By minimality ψ has entries in $\mathfrak{m}$, hence so does ψ^*. Now let $K = \mathrm{Quot}(R)$, the quotient field of R. Then $N \otimes K = 0$, hence $\psi \otimes K$ is surjective. Now it's a K-linear space where K is a field. So $(\psi \otimes K)^* = \psi^* \otimes K$ is injective. This implies that $\mathrm{Ker}(\psi^*)$ is torsion, but it is a submodule of G_0^*, so it must be 0. In other words, ψ^* is injective and this means $0 \to G_0^* \to G_1^* \to M \to 0$ is a minimal resolution of M. If M has a free summand, it must be generated by the image of some basis elements of G_1^*. Pick one of these basis elements e_i and expand to a basis of G_1^*, then we know that the e_i-coefficient of all elements in $\psi(G_0^*)$ is 0. Taking dual again, we get $\psi(e_i^*) = 0$, which means that G_1 is not map to $\mathrm{Syz}_1(N)$ minimally. This is a contradiction. Hence M has no free summands.
(3) and (4) is obvious by (1) and (2). □

For a graded module M, denote $HS(M) = \dim_k(M_i)t^i$ to be its Hilbert series. Then the Betti table of the Auslander dual describes the local cohomology table under the same assumption as above.

Proposition 4.6. *Let M be a finitely generated graded R-module without a free summand,* $\mathrm{projdim}(M) = 1$, *$N = Tr(M)$, then* $E^\bullet(M) = (1-t)^{-n}(\sum_{i=2}^n (-1)^i \beta_i(N), \sum_{i=0}^n (-1)^i \beta_i(N), 0, \ldots, 0)$.

Proof. Let $0 \to F_1 \to F_0 \to M \to 0$ be the minimal resolution of M. This induces an exact sequence $0 \to M^* \to F_0^* \to F_1^* \to N \to 0$ where $F_0^* \to F_1^* \to N \to 0$ is a minimal representation. By definition, $E^\bullet(M) = (HS(M^*), HS(N), 0, \ldots, 0)$. Now $F_0^* \to F_1^* \to N \to 0$ is a minimal presentation, so $HS(F_1^*) = \beta_0(N)HS(R) = \beta_0(N)(1-t)^{-n}, HS(F_0^*) = \beta_1(N)HS(R) = \beta_1(N)(1-t)^{-n}$. Now $HS(N) = (\sum_{i=0}^n (-1)^i \beta_i(N))(1-t)^{-n}$ and by the long exact sequence, $HS(M^*) = HS(F_0^*) + HS(N) - HS(F_1^*) = (\sum_{i=2}^n (-1)^i \beta_i(N))(1-t)^{-n}$. □

Corollary 4.7. Let V be the space of Betti tables and Ext tables, M, N be two modules satisfying the assumption in Proposition 4.6. Define $L_1 : V \to V$ to be the $\mathbb{Q}$-linear map $(\beta_0, \beta_1, \ldots, \beta_n) \to (1-t)^{-n}(\sum_{i=2}^n (-1)^i \beta_i, \sum_{i=0}^n (-1)^i \beta_i, 0, \ldots, 0)$, then $E^\bullet(M) = L_1(\beta^\bullet(N))$.

Corollary 4.8. Let C_{wf} be the cone in V generated by the Ext-tables of modules of projective dimension 1 which does not have a free summand. Then if $E^\bullet(M)$ is an extremal ray and $N = Tr(M)$, then N has a pure resolution of length at least 1, and all elements in C_{wf} is a positive linear combination of elements of the form $E^\bullet(M)$, where $Tr(M)$ has a pure resolution of length at least 1.

Proof. Let C_b be the cone generated by all Betti tables of modules of dimension at most $n-1$. Then by Proposition 4.5 (3) and Corollary 4.7, $C_{wf} = L_1(C_b)$. Applying the Boij-Söderberg theory for Betti tables we know that the extremal rays of C_b is the Betti tables of modules with pure resolutions of length s, where $1 \leq s \leq n$ and C_b is generated by these elements as a cone. Now apply Lemma 4.1. □

By the proposition above, we already know how to decompose $E^\bullet(M)$ when $N = Tr(M)$ is not pure, so to find the vertices of the cone of Ext-tables it suffices to analyze when $L_1(\beta^\bullet(N))$ is decomposable, where N has a pure resolution. First, we need two lemmas that allows us to compute $\beta^\bullet(N)$ when N is pure in terms of its degree sequence $\mathbf{d} = (d_0, \ldots, d_s)$. By Lemma 4.8 we may always assume $s \geq 1$.

Lemma 4.9. *Let $s_1 \leq s$ be two nonnegative integers, $\boldsymbol{d} = (d_0, \ldots, d_s)$ be a degree sequence, and $V_{\boldsymbol{d},s_1}$ be the vector space $\{f \in \mathbb{Q}[t, t^{-1}] | f = \sum_{i=0}^{s} \pi_{\boldsymbol{d},d_i} t^{d_i}, (1-t)^{s_1} | f\}$. Then $\dim_{\mathbb{Q}} V_{\boldsymbol{d},s_1} = s - s_1 + 1$.*

Proof. Multiplying t^{-d_0} does not affect the order of the pole at $t = 1$, so we may assume $d_0 = 0$ without loss of generality. In this case, every element in $V_{\mathbf{d},s_1}$ will be a polynomial. For a polynomial f, $(1-t)^{s_1}$ divides f if and only if $d^j f/dt^j(1) = 0, \forall j = 0, 1, \ldots, s_1 - 1$. But $f = \sum_{i=0}^{s} \pi_{\mathbf{d},d_i} t^{d_i}$, so this means that $\sum_{i=0}^{s} \pi_{\mathbf{d},d_i} \binom{d_i}{j=0, \forall j=0,1,\ldots,s_1-1}$. But the set $\{{d_i}^j, j = 0, 1, \ldots, s_1 - 1\}$ can be mapped to $\{\binom{d_i}{j}, j = 0, 1, \ldots, s_1 - 1\}$ using an invertible linear map. Thus $\sum_{i=0}^{s} \pi_{\mathbf{d},d_i}$ $\binom{d_i}{j=0, \forall j=0,1,\ldots,s_1-1}$ is equivalent to $\sum_{i=0}^{s} \pi_{\mathbf{d},d_i} {d_i}^j = 0, \forall j = 0, 1, \ldots, s_1 - 1$. Note that the matrix $({d_i}^j)_{(i,j)}, 0 \leq i \leq s, 1 \leq j \leq s_1$ has full rank s_1 because it has a Vandermonde submatrix of rank s_1 and we have $s + 1$ variables, so the dimension of the solution space is $s - s_1 + 1$. □

In the case where $s = s_1$, $\dim_{\mathbb{Q}} V_{\mathbf{d},s} = 1$, so there is a unique vector up to a scalar. Denote the sign function by *sgn*, then this vector has the alternating sign property, described as below.

Lemma 4.10.
(1)For each degree sequence $\boldsymbol{d} = (d_0, \ldots, d_s)$, $\dim_{\mathbb{Q}} V_{\boldsymbol{d},s} = 1$, hence there exist a unique polynomial $\pi_{\boldsymbol{d}}(t) \in \mathbb{Q}[t, t^{-1}]$ up to multiplying a nonzero rational number inside $V_{\boldsymbol{d},s}$, denoted by $\pi_{\boldsymbol{d}}(t) = \sum_{i=0}^{s} \pi_{\boldsymbol{d},d_i} t^{d_i}$.
(2)If we rescale these coefficients so that $\pi_{\boldsymbol{d},d_0} = 1$, then $\pi_{\boldsymbol{d},d_i} = \frac{\Pi_{j \neq 0}(d_j - d_0)}{\Pi_{j \neq i}(d_j - d_i)}$, and $sgn(\pi_{\boldsymbol{d},d_i}) = (-1)^i$, that is, the coefficients are nonzero and have alternating signs.
(3)Under the assumption in (2), $\pi_{\boldsymbol{d}}(t)/(1-t)^s|_{t=1} > 0$.
(4)Up to multiplying a scalar, $\beta^\bullet(N) = (\pi_{\boldsymbol{d},d_i} t^{d_i})$ if N is pure of type $\boldsymbol{d}$.

The proof is shown in [2, Section 2.1]. For example, we have $\pi_{0,1,2,3}(t) = 1 - 3t + 3t^2 - t^3$, and $\pi_{0,1,3,4}(t) = 1 - 2t + 2t^3 - t^4$.

To analyze the positive relation we need more notions. We define another invertible linear map $L_2 : V \to V$, $L_2(f_0, f_1, f_2, \ldots, f_n) = (1-t)^n (f_1 - f_0, f_0, f_2, \ldots, f_n)$. Since L_2 is an isomorphism of $\mathbb{Q}$-vector spaces, it induces a bijection between the vertex set of a cone with the vertex set of the image of the cone by Lemma 4.1. Under this notation, $L_2 L_1(\beta^\bullet(N)) = (\beta_0(N) - \beta_1(N), \sum_{i=2}^{n} (-1)^i \beta_i(N), 0, \ldots, 0)$. So this element separates the polynomial $\pi_{\mathbf{d}}(t)$ into two parts, the first part is the sum of the first two terms and the second part is the sum of the rest. Hence it is natural to introduce the following notation for a degree sequence $\mathbf{d}$: let $\alpha_{\mathbf{d}}(t) = \pi_{\mathbf{d},d_0} t^{d_0} - \pi_{\mathbf{d},d_1} t^{d_1}$ and $\alpha'_{\mathbf{d}}(t) = \sum_{i=2}^{s} (-1)^i \pi_{\mathbf{d},d_i} t^{d_i} = \pi_{\mathbf{d}}(t) - \alpha_{\mathbf{d}}(t)$. More generally, for an integer d, define $\tau_{<d}$ which maps a Laurent polynomial $f \in \mathbb{Q}[t, t^{-1}]$ to the sum of terms of f of degree less than d and $\tau_{\geq d}$ which maps a Laurent polynomial $f \in \mathbb{Q}[t, t^{-1}]$ to the sum of terms of f of degree at least d. It is easy to see that for a degree sequence $\mathbf{d} = (d_0 < d_1 < \ldots < d_s)$, $\alpha_{\mathbf{d}}(t) = \tau_{\leq d_1} \pi_{\mathbf{d}}(t)$ and $\alpha'_{\mathbf{d}}(t) = \tau_{\geq d_2} \pi_{\mathbf{d}}(t)$. We have:

Proposition 4.11. *$L_2 L_1(\beta^\bullet(N)) = (\alpha_{\boldsymbol{d}}(t), \alpha'_{\boldsymbol{d}}(t))$.*

We need to check whether $L_2 L_1(\beta^\bullet(N))$ can be decomposed for various $\mathbf{d}$'s. The next proposition shows that if there is a space between d_i and d_{i+1} for $i \neq 1$, then we can decompose $L_2 L_1(\beta^\bullet(N))$.

Proposition 4.12. *Let $s \geq 1$ be a positive integer. Let $\boldsymbol{d} = (d_0 < d_1 < \ldots < d_s)$ be a degree sequence. Assume $d_i < d_{i+1} - 1$ for some i and pick an integer a such that $d_i < a < d_{i+1}$. Define two degree sequences $\boldsymbol{d'} = (d_0 < d_1 < \ldots < d_i < a < d_{i+2} < d_s)$ and $\boldsymbol{d''} = (d_0 < d_1 < \ldots < d_{i-1} < a < d_{i+1} < d_s)$. Then $\pi_{\boldsymbol{d}}(t) = c_1\pi_{\boldsymbol{d'}}(t) + c_2\pi_{\boldsymbol{d''}}(t)$ where $c_1, c_2 > 0$. Moreover, if $i \neq 1$, then we also have $\alpha_{\boldsymbol{d}}(t) = c_1\alpha_{\boldsymbol{d'}}(t) + c_2\alpha_{\boldsymbol{d''}}(t)$ and $\alpha'_{\boldsymbol{d}}(t) = c_1\alpha'_{\boldsymbol{d'}}(t) + c_2\alpha'_{\boldsymbol{d''}}(t)$, so in particular, let N be a pure module of type $\boldsymbol{d}$, then $L_2L_1(\beta^\bullet(N)) = c'_1L_2L_1(\beta^\bullet(N')) + c'_2L_2L_1(\beta^\bullet(N''))$ where N' is pure of type $\boldsymbol{d'}$, N'' is pure of type $\boldsymbol{d''}$, and $c'_1, c'_2 > 0$ are elements in $\mathbb{Q}$.*

Proof. If $s = 1$, then $\pi_{\mathbf{d}}(t) = t^{d_0} - t^{d_1}$, $\pi_{\mathbf{d'}}(t) = t^{d_0} - t^a$, $\pi_{\mathbf{d''}}(t) = t^a - t^{d_1}$, so $\pi_{\mathbf{d}}(t) = \pi_{\mathbf{d'}}(t) + \pi_{\mathbf{d''}}(t)$. Now assume $s \geq 2$, then by 4.10.(2) we know $\pi_{\mathbf{d'}}(t)$ has nonzero coefficients at degree $d_j, j \neq i+1$ and degree a, and $\pi_{\mathbf{d''}}(t)$ has nonzero coefficients at degree $d_j, j \neq i$ and degree a. So cancelling the coefficients in degree a, there is a linear combination $c_1\pi_{\mathbf{d'}}(t) + c_2\pi_{\mathbf{d''}}(t)$ which is a polynomial with possible nonzero coefficients at degree $d_i, 0 \leq i \leq s$. Now $sgn(\pi_{\mathbf{d'},a}) = (-1)^{i+1} \neq sgn(\pi_{\mathbf{d''},a}) = (-1)^i$. Hence we have $sgn(c_1) = sgn(c_2)$. This polynomial is still divisible by $(1-t)^s$, so by 4.10.(1), it's a multiple of $\pi_{\mathbf{d}}(t)$, and after rescaling we may assume $\pi_{\mathbf{d}}(t) = c_1\pi_{\mathbf{d'}}(t) + c_2\pi_{\mathbf{d''}}(t)$ and $sgn(c_1) = sgn(c_2)$. Now since $s \geq 2$, we have $i \geq 1$ or $i + 1 \leq s - 1$. In the first case, $sgn(\pi_{\mathbf{d},d_0}) = sgn(\pi_{\mathbf{d'},d_0}) = sgn(\pi_{\mathbf{d''},d_0}) = 1$ and in the second case $sgn(\pi_{\mathbf{d},d_s}) = sgn(\pi_{\mathbf{d'},d_s}) = sgn(\pi_{\mathbf{d''},d_s}) = (-1)^s$, so $c_1, c_2 > 0$. If $i \neq 1$, then either $i = 0, i + 1 = 1$ or $i \geq 2$. We can apply $\tau_{\leq d_1}$ to the equation $\pi_{\mathbf{d}}(t) = c_1\pi_{\mathbf{d'}}(t) + c_2\pi_{\mathbf{d''}}(t)$ to get $\alpha_{\mathbf{d}}(t) = c_1\alpha_{\mathbf{d'}}(t) + c_2\alpha_{\mathbf{d''}}(t)$ and apply $\tau_{\geq d_2}$ to get $\alpha'_{\mathbf{d}}(t) = c_1\alpha'_{\mathbf{d'}}(t) + c_2\alpha'_{\mathbf{d''}}(t)$. The last statement is true for $c'_1 = c_1$ and $c'_2 = c_2$ by Proposition 4.11. □

Corollary 4.13. Let $\mathbf{d}_0 = (d_{0,0} < d_{0,1} < \ldots < d_{0,s})$ be a degree sequence. For a degree sequence $\mathbf{d}_i = (d_{i,0} < d_{i,1} < \ldots < d_{i,s}), 1 \leq i \leq d_1 - d_0$ we say it satisfies condition $\mathcal{P}$ if $d_{i,1} - d_{i,0} = 1, d_{i,j} = d_{i,2} + j - 2\ \forall 2 \leq i \leq s$. Let N be pure of type $\mathbf{d}_0$, then there exist N_i which is pure of type $\mathbf{d}_i$ that satisfies $\mathcal{P}$ such that $L_2L_1(\beta^\bullet(N))$ decompose into $L_2L_1(\beta^\bullet(N_i))$.

Proof. We fix the degree sequence $\mathbf{d}_0 = (d_{0,0} < d_{0,1} < \ldots < d_{0,s})$. Let A be the set of degree sequences $\{\mathbf{d} = (d_0 < d_1 < \ldots < d_s) | d_{0,0} \leq d_0 \leq d_1 \leq d_{0,1}, d_{0,2} \leq d_2 \leq d_s \leq d_{0,s}\}$. Then A is a finite set since $d_{0,0}, d_{0,1}, d_{0,2}, d_{0,s}$ are fixed. If N is pure of type $\mathbf{d}$ that does not satisfy $\mathcal{P}$, then $\mathbf{d}$ satisfies the hypothesis of Proposition 4.12 so $L_2L_1(\beta^\bullet(N))$ decomposes, and moreover, using the notations in Proposition, the two degree sequences $\mathbf{d}', \mathbf{d}''$ is still in A. Let C' be the cone generated by $L_2L_1(\beta^\bullet(N))$, where N is pure of type $\mathbf{d} \in A$. Consider the set $B = \{L_2L_1(\beta^\bullet(N)) | N$ pure of type $\mathbf{d}$, $\mathbf{d} \in A$ satisfies $\mathcal{P}\}$. Then B contains the vertex set because every element in $C' \backslash B$ can be decomposed into elements in C'. We know C' is finitely generated as a cone, so a vertex set of C' also generates C' by [4, Theorem 1.26], hence B generates C', and $L_2L_1(\beta^\bullet(N)) \in C$. This means that $L_2L_1(\beta^\bullet(N))$ decomposes into elements in B, which proves the corollary. □

The next proposition shows that $L_2L_1(\beta^\bullet(N))$ is decomposable if the length of the degree sequence $s \neq 1, n$. Note that we always assume $1 \leq s \leq n$.

Proposition 4.14. *Let $2 \leq s \leq n-1$. Let Let $\boldsymbol{d} = (d_0 < d_1 < \ldots < d_s)$ be a degree sequence. Construct two degree sequences $\boldsymbol{d'} = (d_0 < d_1 < \ldots < d_{s-1} < d_s + 1)$ and $\boldsymbol{d''} = (d_0 < d_1 < \ldots < d_{s-1} < d_s < d_s + 1)$. The first degree sequence has length s*

and the second degree sequence has length $s+1$. Then $c_1\pi_{\boldsymbol{d}}(t)+c_2\pi_{\boldsymbol{d}'}(t)=\pi_{\boldsymbol{d}''}(t)$ where $c_1>0, c_2<0$. Moreover we also have $\alpha_{\boldsymbol{d}''}(t)=c_1\alpha_{\boldsymbol{d}}(t)+c_2\alpha_{\boldsymbol{d}'}(t)$ and $\alpha'_{\boldsymbol{d}''}(t)=c_1\alpha'_{\boldsymbol{d}}(t)+c_2\alpha'_{\boldsymbol{d}'}(t)$. In particular, let N be a pure module of type $\boldsymbol{d}$, then $L_2L_1(\beta^\bullet(N))=c'_1L_2L_1(\beta^\bullet(N'))+c'_2L_2L_1(\beta^\bullet(N''))$ where N' is pure of type $\boldsymbol{d}'$ and N'' is pure of type $\boldsymbol{d}''$, where $c'_1,c'_2>0$ are elements in $\mathbb{Q}$.

Proof. Consider the $\mathbb{Q}$-vector space spanned by $\pi_{\mathbf{d}}(t)$ and $\pi_{\mathbf{d}'}(t)$. The two polynomials are linear independent because $\pi_{\mathbf{d},d_s}\neq 0, \pi_{\mathbf{d}',d_s}=0, \pi_{\mathbf{d},d_s+1}=0, \pi_{\mathbf{d},d_s}\neq 0$. So they span $V_{\mathbf{d}'',s}$. Also $(1-t)^{s+1}|\pi_{\mathbf{d}''}(t)$, hence there exist $c_1,c_2\in\mathbb{Q}$ such that $c_1\pi_{\mathbf{d}}(t)+c_2\pi_{\mathbf{d}'}(t)=\pi_{\mathbf{d}''}(t)$. Now $sgn(c_1)=sgn(\pi_{\mathbf{d},d_s})/sgn(\pi_{\mathbf{d}'',d_s})=(-1)^s/(-1)^s=1$ and $sgn(c_2)=sgn(\pi_{\mathbf{d}',d_s+1})/sgn(\pi_{\mathbf{d}'',d_s+1})=(-1)^s/(-1)^{s+1}=-1$. Since $s\neq 1$, $s\geq 2$, so apply $\tau_{\leq d_1}$ and τ_{d_2} to this equation we get $\alpha_{\mathbf{d}''}(t)=c_1\alpha_{\mathbf{d}}(t)+c_2\alpha_{\mathbf{d}'}(t)$ and $\alpha'_{\mathbf{d}''}(t)=c_1\alpha'_{\mathbf{d}}(t)+c_2\alpha'_{\mathbf{d}'}(t)$. By Proposition 4.11 this just means $L_2L_1(\beta^\bullet(N''))=c_1L_2L_1(\beta^\bullet(N))+c_2L_2L_1(\beta^\bullet(N''))$, therefore

$$L_2L_1(\beta^\bullet(N))=-\frac{c_2}{c_1}L_2L_1(\beta^\bullet(N'))+\frac{1}{c_1}L_2L_1(\beta^\bullet(N'')).$$

Let $c'_1=-\frac{c_2}{c_1}$ and $c'_2=\frac{1}{c_1}$. Since $c_2<0<c_1$, $c_1,c_2\in\mathbb{Q}$, we know $c'_1,c'_2>0$ and $c'_1,c'_2\in\mathbb{Q}$. □

Let C_t be the cone generated by the Ext-tables of modules of projective dimension at most 1; let C_{wf} be the same as Corollary 4.8, that is, the cone generated by the Ext-tables of modules of projective dimension 1 without a free summand; let C_f be the cone generated by the Ext-tables of free modules. Then $C_t=C_{wf}+C_f$. We want to know a generating set and the vertex set of C_t. To simplify we define some more notations here:

Let $L_1(\beta^\bullet(N))=a_{\mathbf{d}}$ for a module N of type $\mathbf{d}=d_0<d_1<\ldots<d_s$. Let $L_2(a_{\mathbf{d}})=b_{\mathbf{d}}$. If N is pure of type $\mathbf{d}=d_0<d_1<\ldots<d_s$ that does not satisfy the assumption in Proposition 4.12 or Proposition 4.13, then $\mathbf{d}$ must satisfy the following condition: either $s=1, \mathbf{d}=d_0<d_0+1$, or $s=n, \mathbf{d}=d_0,d_0+1,d_2,d_2+1,\ldots,d_2+n-2$. Consider the following 4 kinds of tables that are Ext-tables of some modules:

(1) $A_1=\{L_1(\beta^\bullet(N))=a_{\mathbf{d}}=(1-t)^{-n}(0,t^{d_0}-t^{d_0+1},0,\ldots,0)$, where N is pure of type $\mathbf{d}$, $\mathbf{d}=(d_0,d_0+1)\}$.

(2) $A_2=\{L_1(\beta^\bullet(N))=a_{\mathbf{d}}$, where N is pure of type $\mathbf{d}$, $\mathbf{d}=(d_0,d_0+1,d_2,d_2+1,\ldots,d_2+n-2)\}$. In this case we have

$$a_{\mathbf{d}}=(1-t)^{-n}(\sum_{i=2}^{n}(-1)^i\pi_{\mathbf{d},d_i}t^{d_i},\sum_{i=0}^{n}(-1)^i\pi_{\mathbf{d},d_i}t^{d_i},0,\ldots,0).$$

(3) $A_3=\{L_1(\beta^\bullet(N))=a_{\mathbf{d}}$, where N is pure of type $\mathbf{d}$, $\mathbf{d}=(d_0,d_0+1,d_2,d_2+1,\ldots,d_2+s-2), 2\leq s\leq n-1\}$. In this case we have

$$a_{\mathbf{d}}=(1-t)^{-n}(\sum_{i=2}^{s}(-1)^i\pi_{\mathbf{d},d_i}t^{d_i},\sum_{i=0}^{s}(-1)^i\pi_{\mathbf{d},d_i}t^{d_i},0,\ldots,0).$$

(4) $A_4=\{E^\bullet(R(d))=(1-t)^{-n}(t^d,0,\ldots,0), d\in\mathbb{Z}\}$.

As a summary of the propositions above, we know:

Proposition 4.15.
(1) C_{wf} is generated by $A_1\cup A_2\cup A_3$.
(2) C_f is generated by A_4.

(3) C_t is generated by $A_1 \cup A_2 \cup A_3 \cup A_4$.
(4) An element in A_3 decomposes into elements in $A_3 \cup A_4$, so it cannot be a vertex.

Proof. (1) This is proved by Corollary 4.8 and Corollary 4.13. (2) is trivial because every free module is a direct sum of free modules of rank 1. (3) is trivial by (1) (2). (4) is proved by Proposition 4.14. □

Now to find the vertex set of C_t it suffices to determine whether elements in $A_1 \cup A_2 \cup A_4$ decompose into elements in $A_1 \cup A_2 \cup A_3 \cup A_4$ nontrivially.

Proposition 4.16. *The vertex set of C_t is $A_1 \cup A_2 \cup A_4$.*

Proof. Observe that only elements in A_1 have a 0 entry in the first component and the elements in A_2, A_3 and A_4 have positive entries. So if an element in A_1 decomposes, it can only decompose into elements in A_1, but elements in A_1 are linearly independent, therefore, the decomposition is trivial. Similarly checking the second component we know elements in A_4 only has trivial decompositions. So it remains to check elements in A_2. We apply L_2 again to the elements in A_1, A_2, A_3 and A_4. We have:

$$L_2((1-t)^{-n}(0, t^{d_0} - t^{d_0+1}, 0, \dots, 0)) = (t^{d_0} - t^{d_0+1}, 0, 0, \dots, 0),$$

$$L_2((1-t)^{-n}(\sum_{i=2}^{n}(-1)^i \pi_{\mathbf{d},d_i} t^{d_i}, \sum_{i=0}^{n}(-1)^i \pi_{\mathbf{d},d_i} t^{d_i}, 0, \dots, 0)) =$$

$$(\sum_{i=0}^{1}(-1)^i \pi_{\mathbf{d},d_i} t^{d_i}, \sum_{i=2}^{n}(-1)^i \pi_{\mathbf{d},d_i} t^{d_i}, 0, \dots, 0),$$

$$L_2((1-t)^{-n}(\sum_{i=2}^{s}(-1)^i \pi_{\mathbf{d},d_i} t^{d_i}, \sum_{i=0}^{n}(-1)^i \pi_{\mathbf{d},d_i} t^{d_i}, 0, \dots, 0)) =$$

$$(\sum_{i=0}^{1}(-1)^i \pi_{\mathbf{d},d_i} t^{d_i}, \sum_{i=2}^{s}(-1)^i \pi_{\mathbf{d},d_i} t^{d_i}, 0, \dots, 0),$$

$$L_2((1-t)^{-n}(t^d, 0, \dots, 0)) = (-t^d, t^d, 0, \dots, 0).$$

The first 3 kinds of elements is also equal to $L_2(a_{\mathbf{d}}) = b_{\mathbf{d}}$. Now assume we have an equation

$$(*) \qquad b_{\mathbf{d}_0} = \sum_{j \in J} q_j b_{\mathbf{d}_j} + \sum_{k \in K} q_k b_{\mathbf{d}_k} + \sum_{l \in L} q_l b_{\mathbf{d}_l} + \sum_{m \in M} q_m (-t^m, t^m, 0, \dots, 0)$$

with $b_{\mathbf{d}_j}, b_{\mathbf{d}_k}, b_{\mathbf{d}_l}$ belongs to $L_2(A_1), L_2(A_2), L_2(A_3)$ respectively and q_j, q_k, q_l, q_m being positive rational numbers. We prove that this decomposition is trivial in the following steps.

(1) Observe the following fact: for each $b_{\mathbf{d}_j}, b_{\mathbf{d}_k}, b_{\mathbf{d}_l}, (-t^m, t^m, 0, \dots, 0)$, the lowest term of the second component has a positive coefficient. So let $b_{\min 2} = \min\{\mathbf{d}_{j,2}, \mathbf{d}_{k,2}, \mathbf{d}_{l,2}, m\}$, then $b_{\min 2} = \mathbf{d}_{0,2}$. In fact, if $b_{\min 2} < \mathbf{d}_{0,2}$ then on the right side of (*) the coefficient of $t^{b_{\min 2}}$ in the second component is positive while on the left side it is 0. If $b_{\min 2} > \mathbf{d}_{0,2}$ then on the left side of (*) the coefficient of $t^{\mathbf{d}_{0,2}}$ in the second component is positive while on the right side it is 0.

(2) Observe another fact: for each $b_{\mathbf{d}_j}, b_{\mathbf{d}_k}, b_{\mathbf{d}_l}, (-t^m, t^m, 0, \dots, 0)$, the highest term of the first component has a positive coefficient. So use the same method as in (1) we can prove that $b_{\max 1} = \max\{\mathbf{d}_{j,1}, \mathbf{d}_{k,1}, \mathbf{d}_{l,1}, m\}$, then $b_{\max 1} = \mathbf{d}_{0,1}$.

(3) For an integer m, $m \leq \mathbf{d}_{0,1}$ and $m \geq \mathbf{d}_{0,2}$ are contradictory to each other when $\mathbf{d}_{0,2} > \mathbf{d}_{0,1}$. So in (*) the term $(-t^m, t^m, 0, \dots, 0)$ cannot appear.

(4) For each $b_{\mathbf{d}_j}, b_{\mathbf{d}_k}, b_{\mathbf{d}_l}$, the lowest term of the first component has a positive coefficient. So $b_{\min 1} = \min\{\mathbf{d}_{j,0}, \mathbf{d}_{k,0}, \mathbf{d}_{l,0}\}$, then $b_{\min 1} = \mathbf{d}_{0,0}$.

(5) Observe the fact that $\mathbf{d}_{0,1} = \mathbf{d}_{0,0} + 1$, $\mathbf{d}_{j,1} = \mathbf{d}_{j,0} + 1$, $\mathbf{d}_{k,1} = \mathbf{d}_{k,0} + 1$, $\mathbf{d}_{l,1} = \mathbf{d}_{l,0} + 1$ for any j, k, l. This, together with (2) and (4) implies that $\mathbf{d}_{0,0} = \mathbf{d}_{j,0} = \mathbf{d}_{k,0} = \mathbf{d}_{l,0}$ for any j, k, l.

(6) Apply L_2^{-1} to (*) to get

$$(**) \qquad a_{\mathbf{d}_0} = \sum_{j \in J} q_j a_{\mathbf{d}_j} + \sum_{k \in K} q_k a_{\mathbf{d}_k} + \sum_{l \in L} q_l a_{\mathbf{d}_l}$$

The second entry of $a_{\mathbf{d}}$ is $(1-t)^{-n}\pi_{\mathbf{d}}(t)$. If the length of $\mathbf{d}$ is s, then the order of zero at $t = 1$ of $\pi_{\mathbf{d}}(t)$ is s, so the order of pole at $t = 1$ of $(1-t)^{-n}\pi_{\mathbf{d}}(t)$ is $n - s$; as $1 \leq s \leq n$, $(1-t)^{-n}\pi_{\mathbf{d}}(t)$ is a Laurent polynomial if and only if $n = s$, and if $n \neq s$, $(1-t)^{-n}\pi_{\mathbf{d}}(t)$ is the second entry of a multiple of an Ext-table of a module, hence all the coefficients are positive. So in (**) the term $q_k a_{\mathbf{d}_k}$ and $q_l a_{\mathbf{d}_l}$ does not appear, otherwise the second exponent of the right side is a power series with infinitely many terms with positive coefficients, while the second exponent of the left side is a Laurent polynomial, which is a contradiction.

(7) We get that in (*),

$$b_{\mathbf{d}_0} = \sum_{j \in J} q_j b_{\mathbf{d}_j}.$$

All the elements are in A_2, so they are of the form $b_{\mathbf{d}_j} = d_{j,0} < d_{j,0} + 1 < d_{2,j} < \dots < d_{2,j} + n - 2$. Also by (5) all the $d_{j,0}$ are equal to $d_{0,0}$. But in this case the second entry of $b_{\mathbf{d}_j}$, which is $\alpha'_{\mathbf{d}_j}(t)$, only has nonzero entries in $d_{j,2}, \dots, d_{j,2} + n - 2$, so all these $\alpha'_{\mathbf{d}_j}(t)$'s are linearly independent, which implies that all the $b_{\mathbf{d}_j}$'s are linearly independent. Therefore, the decomposition is trivial. □

Smirnov and De Stefani, in [5], express each local cohomology table as a finite positive linear combination of the vertex set. However this is not the case here in projective dimension 1. When $n > 2$, $A_3 \neq \emptyset$, and we have:

Proposition 4.17. *Any element in A_3 is not a positive linear combination of elements in $A_1 \cup A_2 \cup A_4$.*

Proof. For elements in A_4 the second component is 0. For elements in A_1 the second component is $(t^d - t^{d+1})/(1-t)^n$. It has a pole at $t = 1$ of order $n - 1$, and $\lim_{t \to 1}(1-t)^{n-1}(t^d - t^{d+1})/(1-t)^n = 1 > 0$. For elements in A_4 the second component is $\pi_{\mathbf{d}}(t)/(1-t)^n$ which is regular at $t = 1$. So for every linear combination of elements in $A_1 \cup A_4$ the second component is regular at $t = 1$; for every linear combination of elements in $A_1 \cup A_2 \cup A_4$ where an element in A_2 appears, the second component of this sum is a series $f(t)$ which has a pole of order $n - 1$ such that $\lim_{t \to 1}(1-t)^{n-1}f(t) > 0$. But for an element in A_3 the second component has a pole at $t = 1$ of order $n - s$ where $2 \leq s \leq n - 1$, so it cannot be a positive linear combination of elements in $A_1 \cup A_2 \cup A_4$. □

Proposition 4.15, 4.16 and 4.17 describe the cone of Ext-tables. By the local duality, the following theorem holds for local cohomology tables of modules of projective dimension at most 1.

Theorem 4.18. *Let B_d,B'_d be the following two sets of degree sequences: B_d consists of all degree sequences of the form (d_0, d_0+1), and $(d_0, d_0+1, d_2, d_2+1, \ldots, d_2+n-2)$; B'_d consists of all degree sequences of the form $(d_0, d_0+1, d_2, d_2+1, \ldots, d_2+s-2)$ for some $2 \le s \le n$. Let B_M be the set of modules $\{M|$ Either $\mathrm{projdim} M = 1$, M does not have a free summand, $Tr(M)$ is pure of type $\boldsymbol{d}$ for $\boldsymbol{d} \in B_d$ or M is free $\}$, and $B'_M = \{M|$ Either $\mathrm{projdim} M = 1$, M does not have a free summand, $Tr(M)$ is pure of type $\boldsymbol{d}$ for $\boldsymbol{d} \in B'_d$, or M is free$\}$. Then:*
(1) For every module M, $H^\bullet(M)$ is a linear combination of $H^\bullet(M_i)$ for some $M_i \in B'_M$ with positive rational coefficients.
(2) $H^\bullet(M)$ is an extremal ray if and only if $M \in B_M$.
(3) If $n > 2$, not every element in the cone of local cohomology table is a positive linear combination of the extremal rays.

We've already get the set $\{H^\bullet(M), M \in B_M\}$ of extremal rays, proved that its convex hull is not the cone, and find a larger set $\{H^\bullet(M), M \in B'_M\}$ such that its convex hull is the cone.

Finally, we may change the dimension of the base ring R via the following lemma:

Lemma 4.19. *Let $R = k[x_1, \ldots, x_n]$ and $S = k[y_1, \ldots, y_d]$ be two standard graded polynomial rings. For a finitely generated graded R-module M with $\dim(M) \le d$, M can be given an S-module structure such that M is a finitely generated S-module and the local cohomology tables of the R-module M and the S-module M are the same. Moreover if $m \le d$, then the set of local cohomology tables of R-modules of dimension at most d is equal to the set of local cohomology tables of S-modules.*

Proof. First we prove the first claim where $\dim(M) = d$. In this case, $n \ge d$ and $\dim R/ann_R M = d$. We choose d general linear forms $y_1, \ldots, y_d$ in $R/ann_R M$, then they form a homogeneous system of parameters. Let $S = k[y_1, \ldots, y_d]$ be the subring of $R/ann_R M$ generated by $y_1, \ldots, y_d$. Then $R/ann_R M$ is a finite S-module, and S be a polynomial ring. Since M is a finitely generated graded $R/ann_R M$-module, it is a finitely generated graded S-module. Let $\mathfrak{m}$ be the homogeneous maximal ideal of R and $\mathfrak{n}$ be the homogeneous maximal ideal of S. Then $\mathfrak{n} \cdot R/ann_R M$ is $\mathfrak{m} \cdot R/ann_R M$-primary because $R/ann_R M/\mathfrak{n} \cdot R/ann_R M$ is a positively graded Artinian ring, so its graded maximal ideal is nilpotent. This means that for any i, $H^i_{\mathfrak{m}}(M) = H^i_{\mathfrak{m}R/ann_R M}(M) = H^i_{\mathfrak{n}}(M)$ as graded modules. Hence we have $H^\bullet_{\mathfrak{m}}(M) = H^\bullet_{\mathfrak{n}}(M)$. In general, assume $\dim(M) = d' \le d$. Let $S_0 = k[y_1, \ldots, y_{d'}]$ be a polynomial ring and $\mathfrak{n}_0$ be its graded maximal ideal; then M can be given an S_0-module structure such that $H^i_{\mathfrak{m}}(M) = H^i_{\mathfrak{n}_0}(M)$. Now consider the projection map $\pi : S \to S_0 \cong S/(y_{d'} + 1, \ldots, y_d)$ and view M as an S-module via π, then $\mathfrak{n}_0 = \pi(\mathfrak{n})$, so $H^i_{\mathfrak{n}_0}(M) = H^i_{\mathfrak{n}}(M)$. This means that $H^\bullet_{\mathfrak{m}}(M) = H^\bullet_{\mathfrak{n}}(M)$ and we are done. To prove the second claim, one inclusion is already proved by the first claim. For the other direction, let M be an S-module, then $\dim(M) \le \dim S = d \le n$. Now switch n and d and apply the first claim again. □

Remark 4.20. The above proof is a variant of [5, Lemma 2.2] and the proof is similar except for two points: first, we do not change the base field, so we require that the local cohomology tables are the same while in [5] they may differ by a scalar; second, we do not require that the base field is infinite.

Theorem 4.21. *Let e be an integer with $e \le n$. Let C_e be the cone generated by local cohomology tables of modules M with $\dim M \le e$, $\mathrm{depth} M \ge e - 1$. Let*

$S = k[x_1, \dots, x_e]$, $\pi : R \to S \cong R/(x_{e+1}, \dots, x_n)$. View S-modules as R-modules via π. Define $B_{S,d}$, $B'_{S,d}$, $B_{S,M}$ and $B'_{S,M}$ as in Theorem 4.18 where we replace R-modules by S-modules. Then:
(1) For every module M, $H^\bullet(M)$ is a linear combination of $H^\bullet(M_i)$ for some $M_i \in B'_{S,M}$ with positive rational coefficients.
(2) $H^\bullet(M)$ is an extremal ray if and only if $M \in B_{S,M}$.
(3) If $n > 2$, not every element in the cone of local cohomology table is a positive linear combination of the extremal rays.

Proof. Use Theorem 4.18 and Lemma 4.19. □

5. Facts in dimension 3

In this section we analyze whether the decomposition of $H^\bullet\Gamma(M)$ induces that of $H^\bullet(M)$. In dimension 2, this is the case, but things get complicated in dimension 3. In this section, we assume $n = 3$, that is, R is a polynomial ring over 3 variables. Let M, Γ be two finitely generated graded R-modules such that $M \subset \Gamma$. Take another submodule F of Γ. Then we have a 3*3 exact diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \cap F & \longrightarrow & M & \longrightarrow & M/M \cap F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & \Gamma & \longrightarrow & \Gamma/F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F/M \cap F & \longrightarrow & \Gamma/M & \longrightarrow & \Gamma/M + F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This diagram induces 3 horizontal long exact sequences, 3 vertical long exact sequences, and 4 morphisms between these 6 exact sequences. Denote the 3 horizontal long exact sequences from top to bottom by C_1, C_2, C_3 and 3 vertical ones from left to right D_1, D_2, D_3. The four morphism are $f : C_1 \to C_2$, $f' : C_2 \to C_3$, $g : D_1 \to D_2$ and $g' : D_2 \to D_3$. These morphisms of complexes consists of 12 R-linear maps, denoted by $f_i, f'_i, g_i, g'_i, 1 \le i \le 12$. For example, f is given by:

$$\begin{array}{cccccccc}
0 \longrightarrow & H^0_{\mathfrak{m}}(M \cap F) & \longrightarrow & H^0_{\mathfrak{m}}(M) & \longrightarrow & H^0_{\mathfrak{m}}(M/M \cap F) & \longrightarrow & H^1_{\mathfrak{m}}(M \cap F) \\
 & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\
0 \longrightarrow & H^0_{\mathfrak{m}}(F) & \longrightarrow & H^0_{\mathfrak{m}}(\Gamma) & \longrightarrow & H^0_{\mathfrak{m}}(\Gamma/F) & \longrightarrow & H^1_{\mathfrak{m}}(F)
\end{array}$$

$$\begin{array}{cccccccc}
\longrightarrow & H^1_{\mathfrak{m}}(M) & \longrightarrow & H^1_{\mathfrak{m}}(M/M \cap F) & \longrightarrow & H^2_{\mathfrak{m}}(M \cap F) & \longrightarrow & H^2_{\mathfrak{m}}(M) \\
 & \downarrow f_5 & & \downarrow f_6 & & \downarrow f_7 & & \downarrow f_8 \\
\longrightarrow & H^1_{\mathfrak{m}}(\Gamma) & \longrightarrow & H^1_{\mathfrak{m}}(\Gamma/F) & \longrightarrow & H^2_{\mathfrak{m}}(F) & \longrightarrow & H^2_{\mathfrak{m}}(\Gamma)
\end{array}$$

$$\begin{array}{ccccccccc}
\longrightarrow & H^2_{\mathfrak{m}}(M/M \cap F) & \longrightarrow & H^3_{\mathfrak{m}}(M \cap F) & \longrightarrow & H^3_{\mathfrak{m}}(M) & \longrightarrow & H^3_{\mathfrak{m}}(M/M \cap F) & \longrightarrow 0 \\
 & \downarrow f_9 & & \downarrow f_{10} & & \downarrow f_{11} & & \downarrow f_{12} & \\
\longrightarrow & H^2_{\mathfrak{m}}(\Gamma/F) & \longrightarrow & H^3_{\mathfrak{m}}(F) & \longrightarrow & H^3_{\mathfrak{m}}(\Gamma) & \longrightarrow & H^3_{\mathfrak{m}}(\Gamma/F) & \longrightarrow 0
\end{array}$$

We prove a general decomposition principle, which allows us to decompose $H^\bullet(M)$ as the sum of local cohomology tables of a submodule and a quotient module of M.

Proposition 5.1 (General decomposition principle)**.** *Given a 3*3 diagram as above. Suppose this diagram satisfies the following condition:*
(1) $\mathrm{depth} F \geq 2$, $\mathrm{depth}\Gamma/F \geq 2$, $\mathrm{depth}\Gamma \geq 2$.
(2) $l(\Gamma/M) < \infty$, *or equivalently,* $\Gamma = \Gamma(M)$.
(3) The connecting homomorphism $H^2_{\mathfrak{m}}(\Gamma/F) \to H^3_{\mathfrak{m}}(F)$ *is 0.*
Then the following holds.
(4) $H^\bullet(\Gamma) = H^\bullet(F) + H^\bullet(\Gamma/F)$
(5) $F = \Gamma(M \cap F)$ *and* $\Gamma/F = \Gamma(M/M \cap F)$
(6) $H^\bullet(M) = H^\bullet(M \cap F) + H^\bullet(M/M \cap F)$

Proof. (4)Consider the long exact sequence C_2. By condition (3), it decomposes into 2 short exact sequences, hence the equality holds.
(5)By the short exact sequence at the bottom, Γ/M is of finite length implies that $F/M \cap F$ and $\Gamma/M + F$ are of finite length. So by condition (1), (5) holds.
(6)By (4) it suffices to consider $H^1_{\mathfrak{m}}$. But by (5), $H^1_{\mathfrak{m}}(M) = \Gamma/M$, $H^1_{\mathfrak{m}}(M \cap F) = F/M \cap F$, $H^1_{\mathfrak{m}}(M/M \cap F) = \Gamma/M + F$. So by the short exact sequence at the bottom, $H^1_{\mathfrak{m}}$ also decomposes. □

The general decomposition principle tells us that if we can find a submodule F satisfying (1)-(3), then $H^\bullet(M)$ decomposes.

Corollary 5.2. Using the notations above and assume (2). If Γ is decomposable, then $H^\bullet(M)$ decomposes.

Proof. Suppose F is a direct summand, then $\Gamma = F \oplus \Gamma/F$. In this case the condition (1)(3) is satisfied. □

Corollary 5.3. Using the notations above and assume (2). If $H^2_{\mathfrak{m}}(M) = 0$ or $H^3_{\mathfrak{m}}(M) = 0$, then $H^\bullet(M)$ decomposes as a sum of $H^\bullet(M_i)$ such that $\Gamma(M_i)$ is cyclic.

Proof. If $H^3_{\mathfrak{m}}(M) = 0$, then $\Gamma(M)$ is Cohen-Macaulay of dimension 2, then it reduces to the case in [5]. If $H^2_{\mathfrak{m}}(M) = 0$, then $H^i_{\mathfrak{m}}(\Gamma)$ is zero except for $i = 3$, so Γ is free, hence it's a direct sum of free cyclic modules. Now use (4)-(6). □

Remark 5.4. The above two lemmas explain why things are different in dimension 2 and dimension 3. In dimension 2, Γ is free because $\mathrm{depth}\Gamma \geq 2$, so it reduces to the cyclic case. In dimension 3 this is not always true.

Pick a module M of depth 1 without a dimension 1 submodule and let $\Gamma = \Gamma(M)$. In general it's hard to find a submodule which satisfies (1)(3). There are two ways to approach this. The first way is to go modulo a submodule of dimension 2; and the second way is to go modulo a free submodule. In the first way, (3) is satisfied but the quotient Γ/F may violate (1) because we may have $\mathrm{depth}\Gamma/F = 1$.

Lemma 5.5. *Let* Γ *be a module of depth 2. Then the maximal submodule of dimension at most 2 is the torsion submodule* $\mathrm{Tor}(\Gamma)$*. If* $\mathrm{depth}\Gamma \geq 2$*, then* $\mathrm{Tor}(\Gamma)$ *is Cohen-Macaulay of dimension 2.*

Proof. The maximal submodule of dimension at most 2 is generated by all $m \in \Gamma$ such that $\mathrm{ann}_R(m) \neq 0$, so it is $\mathrm{Tor}(\Gamma)$. We have an exact sequence $0 \to \mathrm{Tor}(\Gamma) \to \Gamma \to Q \to 0$ where Q is the quotient module. It induces a long exact sequence of local cohomology modules:

$$0 \longrightarrow H^0_{\mathfrak{m}}\mathrm{Tor}(\Gamma) \longrightarrow H^0_{\mathfrak{m}}(\Gamma) \longrightarrow H^0_{\mathfrak{m}}(Q)$$

$$\longrightarrow H^1_{\mathfrak{m}}\mathrm{Tor}(\Gamma) \longrightarrow H^1_{\mathfrak{m}}(\Gamma) \longrightarrow H^1_{\mathfrak{m}}(Q)$$

$$\longrightarrow H^2_{\mathfrak{m}}\mathrm{Tor}(\Gamma) \longrightarrow H^2_{\mathfrak{m}}(\Gamma) \longrightarrow H^2_{\mathfrak{m}}(Q)$$

$$\longrightarrow H^3_{\mathfrak{m}}\mathrm{Tor}(\Gamma) \longrightarrow H^3_{\mathfrak{m}}(\Gamma) \longrightarrow H^3_{\mathfrak{m}}(Q) \longrightarrow 0$$

Since $\mathrm{depth}(\Gamma) \geq 2$, $H^0_{\mathfrak{m}}(\Gamma) = H^1_{\mathfrak{m}}(\Gamma) = 0$. Also, $H^0_{\mathfrak{m}}(Q) = 0$ because Q does not have submodule of dimension less than 2, so by the long exact sequence $H^1_{\mathfrak{m}}\mathrm{Tor}(\Gamma) = 0$. Also $H^0_{\mathfrak{m}}\mathrm{Tor}(\Gamma) = 0$ because $H^0_{\mathfrak{m}}(\Gamma) = 0$. Therefore we have $\mathrm{depth}(\mathrm{Tor}\Gamma) \geq 2$, but $\dim(\mathrm{Tor}\Gamma) \leq 2$, so it's Cohen-Macaulay of dimension 2. □

The measurement of Γ/F violating (1) is given by the module $H^1_{\mathfrak{m}}(Q)$. It suffices to give a description of this module.

Lemma 5.6. *Let Γ be a module of depth 2, then Γ^* and Γ^{**} are modules of depth at least 2.*

Proof. If Γ has a free summand F, then F^* is a free summand of Γ^*, so it suffices to prove in the case where Γ has no free summands. In this case Γ^* is the second syzygy of $Tr(\Gamma)$. Hence $\mathrm{projdim}(\Gamma^*) \leq 3 - 2 = 1$. So $\mathrm{depth}(\Gamma^*) \geq 2$. Replace Γ by Γ^*, we get $\mathrm{depth}(\Gamma^{**}) \geq 2$. □

Proposition 5.7. *There is an exact sequence $0 \to \mathrm{Tor}(\Gamma) \to \Gamma \to \Gamma^{**} \to L \to 0$. Let $N = Tr(\Gamma)$, then $L = \mathrm{Ext}^2_R(N, R)$. Let $Q = \Gamma/\mathrm{Tor}\Gamma$, then Q has depth at least 1, $H^1_{\mathfrak{m}}(Q) = H^0_{\mathfrak{m}}(L)$ is of finite length.*

Proof. The exact sequence is well-known, so we omit the proof of the exactness. Now let $Q = \Gamma/\mathrm{Tor}\Gamma$. The composition map $\mathrm{Tor}(\Gamma) \to \Gamma \to \Gamma^{**}$ is 0, hence we have a map $Q \to \Gamma^{**}$. This map is injective because it's a map between torsion-free modules over R and it's injective after tensoring K. So there are two short exact sequences $0 \to \mathrm{Tor}(\Gamma) \to \Gamma \to Q \to 0$ and $0 \to Q \to \Gamma^{**} \to L \to 0$. This leads to two long exact sequences:

$$0 \longrightarrow H^1_{\mathfrak{m}}\mathrm{Tor}(\Gamma) = 0 \longrightarrow H^1_{\mathfrak{m}}(\Gamma) = 0 \longrightarrow H^1_{\mathfrak{m}}(Q)$$

$$\longrightarrow H^2_{\mathfrak{m}}\mathrm{Tor}(\Gamma) \longrightarrow H^2_{\mathfrak{m}}(\Gamma) \longrightarrow H^2_{\mathfrak{m}}(Q)$$

$$\longrightarrow H^3_{\mathfrak{m}}\mathrm{Tor}(\Gamma) \longrightarrow H^3_{\mathfrak{m}}(\Gamma) \longrightarrow H^3_{\mathfrak{m}}(Q) \longrightarrow 0$$

$$
\begin{array}{l}
0 \longrightarrow H^0_{\mathfrak{m}}Q = 0 \longrightarrow H^0_{\mathfrak{m}}(\Gamma^{**}) = 0 \longrightarrow H^0_{\mathfrak{m}}(L) \\
\longrightarrow H^1_{\mathfrak{m}}Q \longrightarrow H^1_{\mathfrak{m}}(\Gamma^{**}) = 0 \longrightarrow H^1_{\mathfrak{m}}(L) \\
\longrightarrow H^2_{\mathfrak{m}}Q \longrightarrow H^2_{\mathfrak{m}}(\Gamma^{**}) \longrightarrow H^2_{\mathfrak{m}}(L) = 0 \\
\longrightarrow H^3_{\mathfrak{m}}Q \longrightarrow H^3_{\mathfrak{m}}(\Gamma^{**}) \longrightarrow H^3_{\mathfrak{m}}(L) = 0 \longrightarrow 0
\end{array}
$$

Now $H^1_{\mathfrak{m}}\mathrm{Tor}(\Gamma) = H^0_{\mathfrak{m}}(\Gamma) = 0$, so $H^0_{\mathfrak{m}}(Q) = 0$, hence depth$Q \geq 1$. Also depth$\Gamma^{**} \geq 2$, so $H^0_{\mathfrak{m}}(\Gamma^{**}) = H^1_{\mathfrak{m}}(\Gamma^{**}) = 0$, hence $H^1_{\mathfrak{m}}Q \cong H^0_{\mathfrak{m}}(L)$. L is a finitely generated module over R, so $H^0_{\mathfrak{m}}(L)$ is of finite length. □

Theorem 5.8. *Let M be a module of depth 1 and assume M has no dimension 1 submodule. Let $\Gamma = \Gamma(M)$. Then $H^\bullet(M)$=$H^\bullet(\mathrm{Tor}M) + H^\bullet(M/\mathrm{Tor}M) - e$, where e is the vector $(0, HS(H^1_{\mathfrak{m}}(Q)), HS(H^1_{\mathfrak{m}}(Q)), 0)$. In particular, if $H^1_{\mathfrak{m}}(Q) = 0$, then $H^\bullet(M)$=$H^\bullet(\mathrm{Tor}M) + H^\bullet(M/\mathrm{Tor}M)$.*

Proof. Take $F = \mathrm{Tor}\Gamma$ in the 3*3 diagram given at the beginning of this section, then $M \cap F = \mathrm{Tor}(M)$, $\Gamma/F = Q$. Let $f : C_1 \to C_2$ be the corresponding morphism. Then $f_i, 7 \leq i \leq 12$ are isomorphisms, and f_6 is surjective. Note that in this case $H^3_{\mathfrak{m}}(F) = H^3_{\mathfrak{m}}(M \cap F) = 0$. Now eliminate all the 0's and get a diagram:

$$
\begin{array}{ccccccc}
0 \longrightarrow & H^1_{\mathfrak{m}}(\mathrm{Tor}M) & \longrightarrow & H^1_{\mathfrak{m}}(M) & \longrightarrow & H^1_{\mathfrak{m}}(M/\mathrm{Tor}M) & \\
& \downarrow f_4 & & \downarrow f_5 & & \downarrow f_6 & \\
0 \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^1_{\mathfrak{m}}(Q) &
\end{array}
$$

$$
\begin{array}{ccccccc}
\longrightarrow & H^2_{\mathfrak{m}}(\mathrm{Tor}M) & \longrightarrow & H^2_{\mathfrak{m}}(M) & \longrightarrow & H^2_{\mathfrak{m}}(M/\mathrm{Tor}M) & \longrightarrow 0 \\
& \downarrow f_7 & & \downarrow f_8 & & \downarrow f_9 & \\
\longrightarrow & H^2_{\mathfrak{m}}(\mathrm{Tor}\Gamma) & \longrightarrow & H^2_{\mathfrak{m}}(\Gamma) & \longrightarrow & H^2_{\mathfrak{m}}(Q) & \longrightarrow 0
\end{array}
$$

plus isomorphisms $H^3_{\mathfrak{m}}(M) = H^3_{\mathfrak{m}}(\Gamma) = H^3_{\mathfrak{m}}(M/\mathrm{Tor}M) = H^3_{\mathfrak{m}}(Q)$. So the kernel of the map $H^2_{\mathfrak{m}}(\mathrm{Tor}M) \to H^2_{\mathfrak{m}}(M)$ is isomorphic to $H^1_{\mathfrak{m}}(Q)$. This leads to 3 equations:

$$HS(H^3_{\mathfrak{m}}(M)) = HS(H^3_{\mathfrak{m}}(\mathrm{Tor}M)),$$

$$HS(H^2_{\mathfrak{m}}(M)) = HS(H^2_{\mathfrak{m}}(M/\mathrm{Tor}M)) + HS(H^2_{\mathfrak{m}}(\mathrm{Tor}M)) - HS(H^1_{\mathfrak{m}}(Q)),$$

$$HS(H^1_{\mathfrak{m}}(M)) = HS(H^1_{\mathfrak{m}}(M/\mathrm{Tor}M)) + HS(H^1_{\mathfrak{m}}(\mathrm{Tor}M)) - HS(H^1_{\mathfrak{m}}(Q)).$$

Equivalently, we have

$$H^\bullet(M) = H^\bullet(\mathrm{Tor}M) + H^\bullet(M/\mathrm{Tor}M) - e,$$

where $e = (0, HS(H^1_{\mathfrak{m}}(Q)), HS(H^1_{\mathfrak{m}}(Q)), 0)$. □

The above theorem shows that $H^1_{\mathfrak{m}}(Q)$ is the error term we want to get rid of in the sense that if it vanishes then $H^\bullet(M)$ decomposes as a sum of two local cohomology tables $H^\bullet(\mathrm{Tor}M)$ and $H^\bullet(M/\mathrm{Tor}M)$. Here $\mathrm{Tor}M$ is a submodule of M of dimension 2. In general $H^1_{\mathfrak{m}}(Q)$ does not vanish. By Proposition 5.7 we see $H^1_{\mathfrak{m}}(Q) \cong H^0_{\mathfrak{m}}(L)$, and the next two proposition shows when it vanishes and how to calculate it in terms of M.

Proposition 5.9. *Using the same notation as Theorem 5.8, let* $L = \mathrm{Ext}^2_R(Tr(\Gamma), R)$. *Then* $H^1_{\mathfrak{m}}(Q) = 0$ *if and only* $L = 0$ *or* L *is Cohen-Macaulay of dimension* 1.

Proof. For a finitely generated module Γ, $\dim L \leq 3 - 2 = 1$. So if $L \neq 0$, then $H^0_{\mathfrak{m}}(L) = 0$ if and only if $\mathrm{depth} L \geq 1$, if and only if L is a Cohen-Macaulay module of dimension 1. □

Proposition 5.10. *Let* Γ *be a finitely generated module over* R *without a free summand. Let* $N = Tr(\Gamma)$, N_1 *be the maximal submodule of dimension at most 1,* $Q' = N/N_1$, *and* $L = \mathrm{Ext}^2_R(N, R)$. *Let* $\Gamma' = \oplus_{t\in\mathbb{Z}} H^0(Proj(R), \widetilde{Q'}(t))$. *Then* $H^0_{\mathfrak{m}}(L) = \mathrm{Hom}_R(\Gamma'/Q', E)$.

Proof. We may assume $\mathrm{depth} N \geq 1$ because $\mathrm{Ext}^2_R(N, R) = \mathrm{Ext}^2_R(N/H^0(N), R)$. In this case N_1 is Cohen-Macaulay of dimension 1. The short exact sequence $0 \to N_1 \to N \to Q' \to 0$ induces a long exact sequence:

$$0 \longrightarrow H^0_{\mathfrak{m}} N_1 \longrightarrow H^0_{\mathfrak{m}}(N) \longrightarrow H^0_{\mathfrak{m}}(Q') = 0$$

$$\longrightarrow H^1_{\mathfrak{m}} N_1 \longrightarrow H^1_{\mathfrak{m}}(N) \longrightarrow H^1_{\mathfrak{m}}(Q')$$

$$\longrightarrow H^2_{\mathfrak{m}} N_1 = 0 \longrightarrow H^2_{\mathfrak{m}}(N) \longrightarrow H^2_{\mathfrak{m}}(Q').$$

So we have an exact sequence $0 \to H^1_{\mathfrak{m}} N_1 \to H^1_{\mathfrak{m}}(N) \to H^1_{\mathfrak{m}}(Q') \to 0$. By local duality, $0 \to \mathrm{Ext}^2_R(Q', R) \to L \to \mathrm{Ext}^2_R(N_1, R) \to 0$ is exact. Now $\dim L \leq 1$, $\mathrm{Ext}^2_R(N_1, R)$ is Cohen-Macaulay of dimension 1, and $H^1_{\mathfrak{m}}(Q')$ is of finite length, hence $\mathrm{Ext}^2_R(Q', R)$ is of finite length. This means that $\mathrm{Ext}^2_R(Q', R) = H^0_{\mathfrak{m}}(L)$. Finally $\mathrm{Ext}^2_R(Q', R) = \mathrm{Hom}_R(H^1_{\mathfrak{m}}(Q'), E)$ and $H^1_{\mathfrak{m}}(Q') = \Gamma'/Q'$, so we are done. □

Theorem 5.8 describes a way to decompose a local cohomology table of a module M of dimension 3 using a submodule $\mathrm{Tor}M$ of dimension 2. There is another way to decompose $H^\bullet(M)$, which is induced by a free submodule $F \subset \Gamma$. Note that if such F does not exist, then $\dim \Gamma \leq 2$, $\dim M \leq 2$, and the decomposition of the local cohomology table of M is known by [5]. So without loss of generality we may always assume that F exists. In this case (1) is automatically satisfied because when $\mathrm{depth}\Gamma \leq 2$ and $\mathrm{depth} F \leq 3$, we always have $\mathrm{depth}\Gamma/F \geq 2$ by standard facts on the depth. hence F may only violate (3) in the general decomposition principle. We observe that $H^2_{\mathfrak{m}}(\Gamma/F) \to H^3_{\mathfrak{m}}(F)$ is 0 if and only if $H^3_{\mathfrak{m}}(F) \to H^3_{\mathfrak{m}}(\Gamma)$ is injective, if and only if $\Gamma^* \to F^*$ is surjective.

Lemma 5.11. *In the 3·3 diagram at the beginning of this section, let* $F = Re, e \in \Gamma$ *be a free submodule. Then (3) of Proposition 5.1 is equivalent to the following condition: there exists* $h \in \Gamma^*$ *such that* $h(e) = 1$.

Proof. Since F is a free cyclic module generated by e, F^* is free cyclic, and generated by e^*. So the map $\Gamma^* \to F^*$ is surjective if and only if e^* is in the image. And $h(e) = 1$ if and only if the image of h is e^*. □

Proposition 5.12. *Let Γ be a module of dimension 3 and depth at least 2. Suppose Γ^* has a free summand $G = Re'$ and $L = \mathrm{Ext}^2_R(Tr(\Gamma), R) = 0$. Then $F = G^* \subset \Gamma$ is a free summand, so F, Γ satisfies (3) of Proposition 5.1.*

Proof. Since F is a free summand of Γ^{**}, Γ^{**} surjects onto F. Now since $L = 0$, Γ surjects onto Γ^{**}, so Γ surjects onto F, but F is projective, hence F is a free summand. The map $\Gamma^* \to F^*$ induced by the inclusion $F \to \Gamma$ is just the projection onto the summand G which it is surjective. This means that F, Γ satisfies (3) of Proposition 5.1. □

Proposition 5.13. *Let Γ be a module of dimension 3 and depth at least 2. Then Γ^* does not have a free summand if and only if $\Gamma^* = Tr(L')$ for a module L' of finite length.*

Proof. By the previous proposition $\mathrm{projdim}(\Gamma^*) \leq 1$. So Γ^* does not have a free summand if and only if $\Gamma^* = Tr(L')$ for $L' = Tr(\Gamma^*)$. But Γ^* is the second syzygy of $N = Tr(\Gamma)$, hence $L' = \mathrm{Ext}^3_R(N, R)$, and this module has finite length. □

By Proposition 5.12 and 5.13 we immediately have:

Corollary 5.14. We use the notations of Proposition 5.12 and 5.13. Suppose $L = 0$, and $\Gamma^* \neq Tr(L')$ for any module L' of finite length. Then there exists a free submodule $F \subset \Gamma$ such that $H^\bullet(M) = H^\bullet(M \cap F) + H^\bullet(M/M \cap F)$.

Since $l(F/M \cap F) < \infty$, we know $\dim(M \cap F) = \dim F = 3$, so $H^\bullet(M)$ is the sum of $H^\bullet(M \cap F)$ and $H^\bullet(M/M \cap F)$ where $M \cap F$ is a submodule of M of dimension 3.

In conclusion, we decompose $H^\bullet(M)$ in two cases using submodules of M; in Theorem 5.8 we use a submodule of dimension 2 and in Corollary 5.14 we use a submodule of dimension 3. So in these cases, $H^\bullet(M)$ is not extremal.

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