5.4 Euler Equations and Regular Singular Points

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

$P(x_0) \neq 0 \rightarrow x_0$ is an ordinary point

$P(x_0) = 0 \rightarrow x_0$ is a singular point

$$y'' + \frac{Q}{P}\,y' + \frac{R}{P}\,y = 0$$

Regular Singular Point

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$$

are BOTH defined.

then x_0 is a regular singular pt.

else irregular s.p.

<u>example</u>: $x^2(1-x)^2 y'' + 2x y' + 4y = 0$

classify all singular points

singular pts: $x=0, \quad x=1$

$x_0=0$: $\displaystyle\lim_{x\to 0} (x-0)\frac{2x}{x^2(1-x)^2} = \lim_{x\to 0} \frac{2\cancel{x^2}}{\cancel{x^2}(1-x)^2} = 2$

$\displaystyle\lim_{x\to 0} \cancel{x^2}\cdot\frac{4}{\cancel{x^2}(1-x)^2} = 4$

<u>BOTH</u> defined so $x_0=0$ is a regular s.p.

$x_0=1$: $\displaystyle\lim_{x\to 1} (x-1)\frac{2x}{x^2(1-x)^2}$ undefined

$\rightarrow$ STOP $x_0=1$ is an irregular s.p.

Solutions near r.s.p. are more complicated than those near ordinary pts.

Simplest case $\rightarrow$ Euler Equation

$$x^2 y'' + \alpha x y' + \beta y = 0 \qquad \alpha, \beta \text{ constants}$$

note $x_0 = 0$ is a r.s.p.

assume solution is of form $y = x^r$

$$y' = r x^{r-1} \qquad y'' = r(r-1) x^{r-2}$$

$$\underbrace{x^2 (r)(r-1) x^{r-2}}_{x^r} + \underbrace{\alpha x (r) x^{r-1}}_{x^r} + \beta x^r = 0$$

$$x^r\left[r(r-1)+\alpha r+\beta\right]=0$$

$$\boxed{r(r-1)+\alpha r+\beta=0}$$ solve for r

$$r^2+(\alpha-1)r+\beta=0$$

a) real and distinct

$r_1 \neq r_2$ $\quad \boxed{y=c_1|x|^{r_1}+c_2|x|^{r_2}}$

$|\cdot|$ often omitted

b) repeated

$r_1=r_2=r$ $\quad \boxed{y=c_1|x|^{r}+c_2|x|^{r}\ln|x|}$

c) complex pairs

$r_1=\lambda+i\mu$

$r_2=\lambda-i\mu$

$$\boxed{\begin{aligned}y=c_1|x|^{\lambda}&\cos(\mu\ln|x|)\\&+c_2|x|^{\lambda}\sin(\mu\ln|x|)\end{aligned}}$$

<u>example</u>: $x^2 y'' + 3xy' + \frac{3}{4} y = 0$

sub in $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

typo on assignment sheet

$r(r-1) + 3r + \frac{3}{4} = 0$

$4(r)(r-1) + 12r + 3 = 0$

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$4r^2 + 8r + 3 = 0$

$|x|^{-1/2} = \frac{1}{|x|^{1/2}}$

↳ skip graph

$(2r+1)(2r+3) = 0$

$r = -1/2 \qquad r = -3/2$

$$\boxed{y = c_1 |x|^{-1/2} + c_2 |x|^{-3/2}}$$

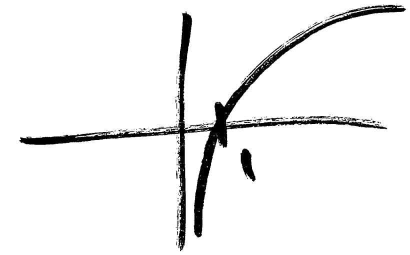
$x_0 = 0$ is an r.s.p.

near 0 as $x \to 0$, $y \to$ ● ?

y becomes unbounded

<u>example</u>: $4x^2y'' + 8xy' + 17y = 0$

ln x



sub $y = x^r$

$$4r(r-1) + 8r + 17 = 0$$

$$4r^2 + 4r + 17 = 0$$

$$r = \frac{-4 \pm \sqrt{4^2 - 4(4)(17)}}{2(4)}$$

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$$= \frac{-4 \pm \sqrt{-256}}{8} = \frac{-4 \pm \sqrt{-1}\sqrt{256}}{8}$$

$$= \cancel{-\frac{1}{2} \pm i\,16} \quad -\frac{1}{2} \pm i\,2$$

$$\boxed{y = c_1 |x|^{-1/2} \cos(2\ln|x|) + c_2 |x|^{-1/2} \sin(2\ln|x|)}$$

$x \to 0$

as $x \to 0$ $\quad \ln|x| \to -\infty$

$\cos(2\ln|x|)$ and $\sin(2\ln|x|)$

are bounded between 1 and -1

$|x|^{-1/2}$ is unbounded