The length of a rectangle is increasing at a constant rate of 8 cm/s and its width is decreasing at a constant rate of 3 cm/s. How fast, in cm^2/s, is the area of the rectangle increasing at the moment when the length is 20 cm and the width is 10 cm?

↳ do NOT plug in until taking derivative

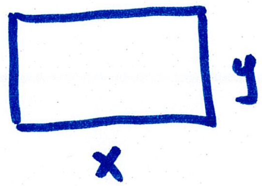
A. 140

B. 130

C. 40

D. 24

E. 20



x: length

y: width

given: $\frac{dx}{dt} = 8$

$\frac{dy}{dt} = -3$

find: $\frac{dA}{dt}$ when $x = 20$, $y = 10$

$A = xy$

both x and y are functions of t (time)

$$\frac{d}{dt} A = \frac{d}{dt}(xy) = x\frac{dy}{dt} + y\frac{dx}{dt}$$

now we can plug in numbers

$$= x(-3) + y(8)$$

$$= -3x + 8y$$

at the moment when $x = 20$, $y = 10$

$$\frac{dA}{dt} = -60 + 80 = \boxed{20}$$

increasing at 20 cm²/s

Use a linear approximation to estimate the value of $e^{-0.01}$

A. 1.001

B. 1.01

C. 0.9

D. 0.99

E. 0.999

linear approx: $L = f(a) + f'(a)(x-a)$

identify $f(x)$: e^x

a: a convenient number where we know $f(a)$ easily and close to where we want to be

here, we use $a=0$ since we know $e^0 = 1$ and 0 is close to -0.01

$L = f(a) + f'(a)(x-a)$

$f(a) = e^0 = 1$

$f'(x) = e^x$

$f'(a) = e^0 = 1$

$L = 1 + (1)(x-0) = 1 + x \approx e^x$

$e^{-0.01} \approx 1 + (-0.01) = 0.99$

always work with a given closed interval

Find the absolute maximum and minimum values of the function $f(x) = \cos x + \sin x$ on the interval $[0, \pi]$.

procedure: find critical pts → $f' = 0$ or f' DNE

then compare $f(x)$ at critical pts and at the end points of interval

we do NOT use first or second derivative test with absolute max/min

A. max: $\sqrt{2}$; min: -1

B. max: $\frac{\pi}{4}$; min: π

C. max: 2; min: -2

D. max: 1; min: -1

E. max: π; min: 0

$f'(x) = -\sin x + \cos x = 0$

Solve $\cos x = \sin x$ on $[0, \pi]$

or $\tan x = 1$ on $[0, \pi]$

$x = \pi/4$ only crit. pt.

compare $f(x) = \cos x + \sin x$ at $x = 0, \pi/4, \pi$

$f(0) = 1$

$f(\pi/4) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \approx 1.4$ max

$f(\pi) = -1$ min

$$\lim_{x\to\pi}\frac{\cos x+\sin 2x+1}{x^2-\pi^2} = \xrightarrow{\text{if } x=\pi} \frac{-1+0+1}{\pi^2-\pi^2} \to \frac{0}{0}$$

A. $\frac{1}{2\pi}$

B. $\frac{1}{\pi}$

C. 1

D. $-\frac{1}{\pi}$

E. $-\frac{1}{2\pi}$

l'Hospital's Rule

↳ when limit seems to go to $\frac{0}{0}$ or $\frac{\infty}{\infty}$

take the deriv. of top and bottom and then re-evaluate limit

repeat until no longer $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\overset{L}{=} \lim_{x\to\pi} \frac{-\sin x + 2\cos 2x}{2x}$$

now try $x=\pi$

$$= \frac{-\sin(\pi)+2\cos(2\pi)}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi}$$

do NOT apply l'Hospital's Rule because not $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\lim_{x\to 0}(1+2x)^{\cot x} = \qquad \cot x = \frac{1}{\tan x}$$

as $x\to 0$ $\rightarrow$ $1^{\infty} = ?$ indeterminate form

turn into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then use l'Hospital's Rule

A. 0

B. 1

C. 2

D. e

E. e^2

$$\lim_{x\to 0} \underbrace{(1+2x)^{\cot x}}_{y}$$

let $y = (1+2x)^{\cot x}$

so we want $\lim\limits_{x\to 0} y$

$$\ln y = \ln (1+2x)^{\cot x}$$

$$= (\cot x)\ln(1+2x)$$

$$= \frac{\ln(1+2x)}{\tan x}$$

note as $x\to 0$ $\frac{\ln(1+2x)}{\tan x} \to \frac{0}{0}$

so can use l'Hospital to find $\lim\limits_{x\to 0} \ln y$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1+2x)}{\tan x}$$

$$\stackrel{L}{=} \lim_{x\to 0} \frac{\frac{2}{1+2x}}{\sec^2 x} \qquad \sec x = \frac{1}{\cos x} \text{ and } \cos(0) = 1$$

if $x = 0$

$$= \frac{\frac{2}{1}}{1} = 2$$

not done : $\lim_{x\to 0} \ln y = 2$ but we want $\lim_{x\to 0} y$

$$y = e^{\ln y} = e^2$$

Find the locations (x values) of the inflection points of $f(x) = e^{-x^2}$. Note that its first and second derivatives are: $f'(x) = -2xe^{-x^2}$, $f''(x) = (4x^2 - 2)e^{-x^2}$.

inflection pts: $f'' = 0$ or f'' DNE

and f'' changes sign

A. $x = 1$ and $x = -1$

B. $x = 2$ and $x = -2$

C. $x = 0$

D. $x = \sqrt{2}$ and $x = -\sqrt{2}$

E. $x = \frac{1}{\sqrt{2}}$ and $x = \frac{-1}{\sqrt{2}}$

$$f'' = (4x^2-2)e^{-x^2} = 0$$

$$4x^2-2=0 \quad \text{or} \quad \underbrace{e^{-x^2} = 0}_{\text{never}}$$

$$\hookrightarrow x^2 = \tfrac{1}{2}$$

$$x = \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}$$

Sign of f'': + + 0 − − 0 + +

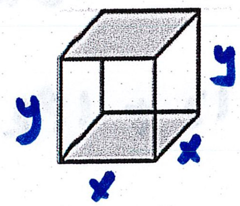
∪ ∩ ∪

$-\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$

$f''(-2) \geq 0$ $f''(0) =$ $f''(2) \geq 0$

sign change across both, so both are where inflections pts are.

A six-sided box is to have four clear plastic sides, a wooden square top, and a wooden square bottom. The volume of the box must be 24 ft^3. Plastic costs \$1 per ft^2 and wood costs \$3 per ft^2. Find the dimensions of the box which minimize cost.



A. 2 ft $\times$ 2 ft $\times$ 6 ft

B. $\sqrt{6}$ ft $\times$ $\sqrt{6}$ ft $\times$ 4 ft

C. $\sqrt[3]{4}$ ft $\times$ $\sqrt[3]{4}$ ft $\times$ $6\sqrt[3]{4}$ ft

D. $\sqrt[3]{3}$ ft $\times$ $\sqrt[3]{3}$ ft $\times$ $8\sqrt[3]{3}$ ft

E. $2\sqrt[3]{2}$ ft $\times$ $2\sqrt[3]{2}$ ft $\times$ $3\sqrt[3]{2}$ ft

volume: $x^2 y = 24$ constraint

four sides

$$\text{cost} = \underbrace{(x^2)(3)}_{\text{base}} + \underbrace{(x^2)(3)}_{\text{top}} + \underbrace{(4)(xy)(1)}_{\text{four sides}}$$

$= 6x^2 + 4xy$ eliminate y ~~from~~ using constraint

$x^2y = 24 \rightarrow y = \frac{24}{x^2}$

$C(x) = 6x^2 + 4x\left(\frac{24}{x^2}\right) = 6x^2 + \frac{96}{x}$

$C'(x) = 12x - \frac{96}{x^2} = 0 \rightarrow 12x = \frac{96}{x^2} \rightarrow x^3 = \frac{96}{12} = 8 \rightarrow x = 2$

check $x=2$ minimizes cost

use 2nd deriv. test

$y = \frac{24}{x^2} = 6$

$C'' = 12 + \frac{192}{x^3}$ $\quad C''(2) > 0$ so $x=2$ min C

F17 E3 #4 $\quad\quad \frac{1}{2}, \frac{1}{9}$

$f(x) = \frac{\ln x}{x^2}$ on $[\frac{1}{e}, e]$

find abs. max/min

find critical numbers: $f' = 0$ or f' DNE

$$f'(x) = \frac{(x^2)(\frac{1}{x}) - (\ln x)(2x)}{(x^2)^2} = \frac{x - 2x\ln x}{x^4}$$

$f' = 0 \rightarrow$ $x - 2x\ln x = 0$

$x(1 - 2\ln x) = 0 \rightarrow$ $\cancel{x = 0}$ or $1 - 2\ln x = 0$

discard because not in $[\frac{1}{e}, e]$

$\ln x = \frac{1}{2}$

$x = e^{1/2}$ keep

f' DNE $\rightarrow$ $x^4 = 0 \rightarrow \cancel{x = 0}$

$f(\frac{1}{e}) = \frac{\ln \frac{1}{e}}{(\frac{1}{e})^2} = \frac{-1}{\frac{1}{e^2}} = -e^2$ min

$f(e^{1/2}) = \frac{\ln e^{1/2}}{(e^{1/2})^2} = \frac{1/2}{e} = \frac{1}{2e}$ max

$f(e) = \frac{\ln e}{e^2} = \frac{1}{e^2}$

F18E3 #4

$\sqrt{9.1}$

identify $f(x)$: $f(x) = \sqrt{x} = x^{1/2}$

identify a: close to 9.1 and $\sqrt{a}$ is known

so, $a = 9$

$L = f(a) + f'(a)(x-a)$

$f(a) = \sqrt{9} = 3$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$f'(a) = f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

$L = 3 + \frac{1}{6}(x-9) \approx \sqrt{x}$

$\sqrt{9.1} \approx 3 + \frac{1}{6}(0.1) = 3 + \frac{1}{60} = 3\frac{1}{60}$