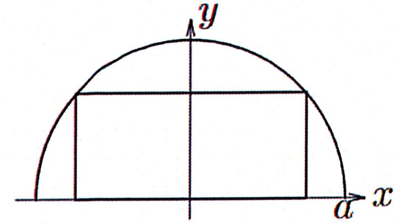
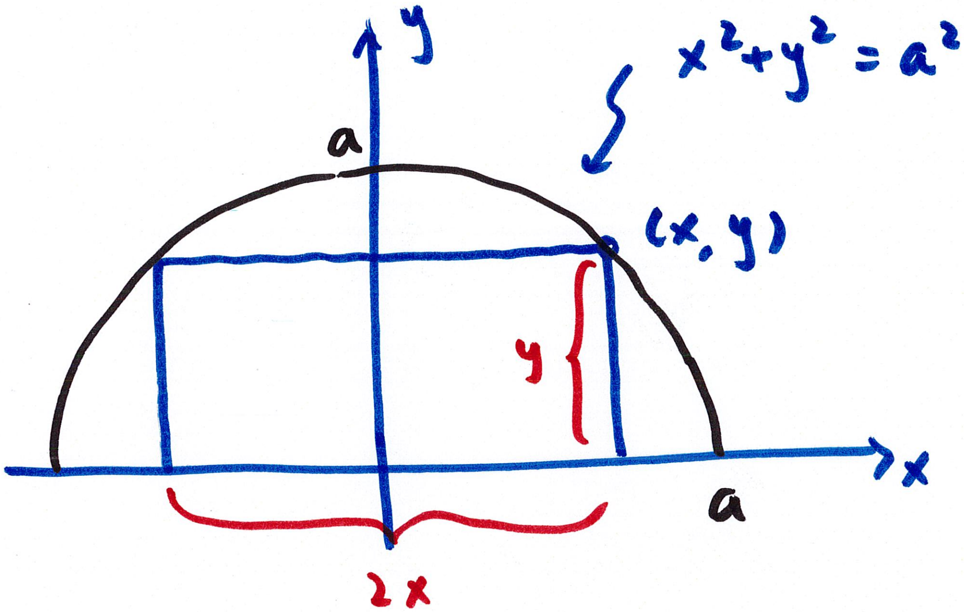
A rectangle is inscribed in the upper half of the circle $x^2 + y^2 = a^2$ as shown at right. Calculate the area of the largest such rectangle. A. $\frac{a^2}{2}$ B. $3a\sqrt{2}$ C. $2a^2$ D. $4a^2$ E. a^2.





height : y

length : $2x$

area: $A = 2xy$

need to eliminate one variable

circle eq: $x^2+y^2=a^2$

$y^2 = a^2 - x^2$

$y = \sqrt{a^2-x^2} = (a^2-x^2)^{1/2}$

area: $A(x) = 2x(a^2-x^2)^{1/2}$

$0 \le x \le a$

find critical numbers, then compare $A(x)$ at critical numbers and at end points

$$A'(x) = \underbrace{(2x)\left(\tfrac{1}{2}\right)(a^2-x^2)^{-1/2}(-2x)}_{\text{Chain Rule}} + (a^2-x^2)^{1/2}(2)$$

$$= (-2x^2)(a^2-x^2)^{-1/2} + 2(a^2-x^2)^{1/2}$$

$$= \frac{-2x^2}{(a^2-x^2)^{1/2}} + 2(a^2-x^2)^{1/2}$$

$$A' = 0 \rightarrow \frac{-2x^2}{(a^2-x^2)^{1/2}} + 2(a^2-x^2)^{1/2} = 0$$

$$2(a^2-x^2)^{1/2} = \frac{2x^2}{(a^2-x^2)^{1/2}}$$

$$a^2 - x^2 = x^2$$

$$2x^2 = a^2$$

$$x^2 = \frac{1}{2}a^2$$

$$x = \frac{1}{\sqrt{2}}a, \quad \cancel{-\frac{1}{\sqrt{2}}}a$$

not inside interval of x $0 \le x \le a$

$$A(x) = 2x\,(a^2 - x^2)^{1/2}$$

$$A(0) = 0$$

$$A\left(\frac{1}{\sqrt{2}}a\right) = \frac{2}{\sqrt{2}}a\left(a^2 - \frac{1}{2}a^2\right)^{1/2} = \sqrt{2}\,a\left(\frac{1}{2}a^2\right)^{1/2} = \sqrt{2}\,a \cdot \frac{1}{\sqrt{2}}a = a^2$$

$$A(a) = 0$$

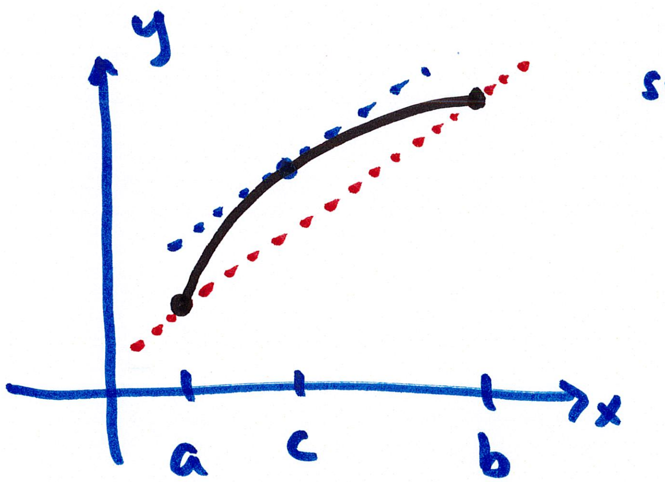
Given that $f(x)$ is differentiable for all x, $f(2) = 4$, and $f(7) = 10$, then the Mean Value Theorem states that there is a number c such that A. $2 < c < 7$ and $f'(c) = \frac{6}{5}$ B. $2 < c < 7$ and $f'(c) = \frac{5}{6}$ C. $4 < c < 10$ and $f'(c) = \frac{6}{5}$ D. $2 < c < 7$ and $f'(c) = 0$ E. $4 < c < 10$ and $f'(c) = 0$.

Mean Value Theorem: $f(x)$ continuous on $[a, b]$

$f(x)$ differentiable on (a, b)

then there is a "c" s.t., $a \le c \le b$

such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$



somewhere between a and b

there is at least one place c

where the tangent line slope =

secant line slope through endpoints

know: $f(2) = 4$

$f(7) = 10$

$$\underset{a}{2} \le x \le \underset{b}{7}$$

somewhere between $x = 2$ and $x = 7$ there is a place

where $f' = \dfrac{f(7) - f(2)}{7-2} = \dfrac{10-4}{7-2} = \dfrac{6}{5}$

Which of the following is/are true about the function $g(x) = 4x^3 - 3x^4$? (1) g is decreasing for $x > 1$. (2) g has a relative extreme value at $(0, 0)$. (3) the graph of g is concave up for all $x < 0$.

A. (1), (2) and (3) B. only (2) C. only (1) D. (1) and (2) E. (1) and (3).

$g(x) = 4x^3 - 3x^4$

$g'(x) = 12x^2 - 12x^3 = 0$

$12x^2(1-x) = 0 \rightarrow x = 0,\ x = 1$

sign of $g'(x)$: $+ +$ 0 $+ +$ 0 $- -$

$\nearrow$ $\nearrow$ $\searrow$

$x=0$ $x=1$

inc: $(-\infty, 0), (0, 1)$

dec: $(1, \infty)$ $\rightarrow$ statement (1) is correct

(2) is FALSE because for an extreme value to exist at $x=0$ there has to be a sign change in g' across $x=0$

$g''(x) = 24x - 36x^2 = 0$

$12x(2-3x) = 0 \rightarrow x = 0,\ x = 2/3$

sign of g'': $- -$ 0 $+ +$ 0 $- -$

$\cap$ $\cup$ $\cap$

$x=0$ $x=2/3$

(3) is ~~TRUE~~ FALSE

$\lim_{x\to 0}\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x} =$ A. 1/2 B. 2 C. 1/3 D. 1 E. 0.

$$\lim_{x\to 0}\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x} \xrightarrow{x=0} \frac{0}{0}$$

l'Hospital's Rule can be used (the other form is $\frac{\infty}{\infty}$)

l'Hospital's Rule: $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ if limit $\to \frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\overset{L}{=} \lim_{x\to 0}\frac{2-\frac{1}{\sqrt{1-x^2}}}{2+\frac{1}{1+x^2}} \xrightarrow{x=0} \frac{2-1}{2+1} = \frac{1}{3}$$

Do <u>NOT</u> use l'Hospital's Rule if not $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Find $\frac{d}{dx}\int_1^{2x}\sqrt{t^2+1}dt$ at $x=\sqrt{2}$. A. 6 B. 3 C. $\sqrt{2}$ D. $\sqrt{4x^2+1}$ E. $\frac{1}{2\sqrt{3}}$.

deriv. of an integral → Fundamental Theorem of Calculus (part 1)

FTC 1: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

here, the upper limit is not just x so we need to use Chain Rule

let $u = 2x$

then $\frac{d}{dx}\int_1^{2x}\sqrt{t^2+1}\,dt = \frac{d}{dx}\underbrace{\int_1^{u}\sqrt{t^2+1}\,dt}_{y(u)}$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$= \underbrace{\left(\frac{d}{du}\int_1^{u}\sqrt{t^2+1}\,dt\right)}_{\text{FTC 1}}\left(\frac{d}{dx}2x\right)$$

$$= \left(\sqrt{u^2+1}\right)(2) \qquad u = 2x$$

$$= 2\sqrt{(2x)^2+1} = 2\sqrt{4x^2+1}$$

at $x = \sqrt{2}$ we get $2\sqrt{4(\sqrt{2})^2+1} = 6$

$\int_3^4 x\sqrt{25-x^2}dx =$ A. 0 B. -37 C. $\frac{37}{3}$ D. $-\frac{74}{3}$ E. $\frac{7}{12}$

$$\int_3^4 x\sqrt{25-x^2}\,dx$$ need to use a substitution

$$= \int_3^4 (\boxed{x})(\boxed{25-x^2})^{1/2}\,dx$$

compare x and $25-x^2$

deriv. of $25-x^2$ is $-2x$ which is a constant multiple of x, so we let $\boxed{u=25-x^2}$

$$\frac{du}{dx} = -2x$$

$$\boxed{du = -2x\,dx}$$

now we adjust the integration limits

old upper limit: $x=4 \to u=25-x^2=25-16=9$

old lower limit: $x=3 \to u=25-x^2=25-9=16$

$$\int_{3 \to 16}^{4 \to 9} \underbrace{(25-x^2)^{1/2}}_{u^{1/2}} \overbrace{\underbrace{(x)\,dx}_{du=-2x\,dx}}^{-\frac{1}{2}du}$$

so $x\,dx = -\frac{1}{2}du$

$$= \int_{16}^{9} -\frac{1}{2}u^{1/2}\,du = -\frac{1}{2}\int_{16}^{9} u^{1/2}\,du = -\frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right)\Bigg|_{16}^{9}$$

DO NOT go back to x

$$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Bigg|_{16}^{9} = -\frac{1}{2}\left(\frac{2}{3}\cdot 9^{3/2}\right) - -\frac{1}{2}\left(\frac{2}{3}\cdot 16^{3/2}\right)$$

$$= -\frac{1}{2}\left(\frac{2}{3}\cdot 27\right) + \frac{1}{2}\left(\frac{2}{3}\cdot \overset{64}{\cancel{8}}\right)$$

$$= -\frac{1}{3}\cdot 27 + \frac{1}{3}\cdot \overset{64}{\cancel{8}} = \cancel{-9+18=9} \quad = -9 + \frac{64}{3} = \frac{37}{3}$$

Suppose that the mass of a radioactive substance decays from 18 gms to 2 gms in 2 days. How long will it take for 12 gms of this substance to decay to 4 gms? A. $\frac{\ln 3}{\ln 2}$ days B. 1 day C. $\frac{\ln 2}{\ln 3}$ days D. 2 days E. $(\ln 3)^2$ days

exponential growth/decay: $y(t) = y_0 e^{kt}$

$y_0 = 18$ $\quad$ t: time in days

$y(2) = 2$

now find k

$y(t) = 18e^{kt}$

$y(2) = 18e^{k \cdot 2} = 2 \rightarrow e^{2k} = \frac{1}{9} \rightarrow \ln e^{2k} = \ln \frac{1}{9}$

$2k = \ln \frac{1}{9} \rightarrow \frac{1}{2} \ln \frac{1}{9} = k$

$12 \rightarrow 4$ $\quad$ $y_0 = 12$, $y(t) = 4$

$y(t) = y_0 e^{kt}$

$4 = 12 e^{(\frac{1}{2} \ln \frac{1}{9})t}$

$\frac{1}{3} = e^{\ln\left(\frac{1}{9}\right)^{1/2} t}$

$\cancel{e^{\ln x} = x}$

$\cancel{\frac{1}{3} = \left(\frac{1}{9}\right)^{1/2} t}$

$\ln \frac{1}{3} = \ln e^{\ln\left(\frac{1}{9}\right)^{1/2} t}$

$= \ln\left(\frac{1}{9}\right)^{1/2} t$

$\ln\left(\frac{1}{3}\right) = \ln\left(\frac{1}{3}\right) t$

so $t = 1$