Find the area of the region between the graph of $y = \frac{1}{1+x^2}$ and the x–axis, from $x = -\sqrt{3}$ to $x = 1$.

A. $\frac{\pi}{2}$ B. $\frac{3\pi}{4}$ C. $\frac{15\pi}{12}$ D. $\frac{\pi}{3}$ E. $\frac{7\pi}{12}$

$\int_a^b f(x)\,dx \rightarrow$ area of region bounded by $f(x)$ and x-axis from $x=a$ to $x=b$

FTC 2: $\int_a^b f(x)\,dx = F(b) - F(a)$

where $F'(x) = f(x)$

$F(x)$ is antiderivative of $f(x)$

here, $\int_{-\sqrt{3}}^{1} \underbrace{\frac{1}{1+x^2}}_{f(x) \rightarrow F(x) = ?} dx$

$= \tan^{-1}(x) \Big|_{-\sqrt{3}}^{1}$

$= \tan^{-1}(1) - \tan^{-1}(-\sqrt{3})$

$\theta = \tan^{-1}(-\sqrt{3})$

$\tan\theta = -\sqrt{3}$

$= \frac{-\sqrt{3}/2}{1/2}$

$\theta = -\frac{\pi}{3}$

$= \frac{\pi}{4} - -\frac{\pi}{3}$

$= \frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$

$\int_0^1 \frac{e^x}{1+e^x}dx =$ A. $\ln\frac{1+e}{2}$ B. $\ln(1+e)$ C. $\frac{1}{2}$ D. $1-\ln 2$ E. e

$$\int_0^1 \frac{e^x}{1+e^x}\,dx$$

e^x vs. $1+e^x$

find one part whose derivative is a constant multiple of the other part

rule of thumb: the more complicated part

here, $\frac{d}{dx}(1+e^x) = e^x \rightarrow$ exactly the other part

so, let $\boxed{u = 1+e^x}$

$\frac{du}{dx} = e^x$ $\boxed{du = e^x\,dx}$

old upper limit: $x = 1 \rightarrow u = 1+e^x \rightarrow u = 1+e$

old lower limit: $x = 0 \rightarrow u = 1+e^x \rightarrow u = 1+e^0 = 1+1 = 2$

$$\int_{0 \to 2}^{1 \to 1+e} \underbrace{\frac{1}{1+e^x}}_{\frac{1}{u}} \underbrace{e^x\,dx}_{du} = \int_2^{1+e} \frac{1}{u}\,du = \ln|u| \Big|_2^{1+e} = \ln|1+e| - \ln|2|$$

$$= \ln\frac{1+e}{2}$$

do NOT go back to x

Compute $\int_{\ln\frac{\pi}{4}}^{\ln\frac{\pi}{2}} e^x \cos e^x \, dx$.

A. 1

B. $\frac{\sqrt{3}+1}{2}$

C. $1-\sqrt{2}$

D. $\frac{\sqrt{3}-1}{2}$

E. $\frac{2-\sqrt{2}}{2}$

$$\int_{\ln\frac{\pi}{4}}^{\ln\frac{\pi}{2}} e^x \cos e^x \, dx$$

look at parts: e^x, $\cos e^x$

neither has a derivative that is a constant multiple of the other

but if we focus on the two e^x, notice

if $u = e^x$ then we can substitute cleanly

$\boxed{u = e^x}$

$\frac{du}{dx} = e^x$

$\boxed{du = e^x dx}$

$\ln\frac{\pi}{2} \to \frac{\pi}{2}$

$\ln\frac{\pi}{4} \to \frac{\pi}{4}$

$$\underbrace{\cos(e^x)}_{\cos(u)} \cdot \underbrace{e^x dx}_{du}$$

adjust limits: $x = \ln\frac{\pi}{2} \to u = e^x = e^{\ln\frac{\pi}{2}} = \frac{\pi}{2}$

$x = \ln\frac{\pi}{4} \to u = e^x = e^{\ln\frac{\pi}{4}} = \frac{\pi}{4}$

new integral: $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(u)\,du = \sin(u)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \sin(\frac{\pi}{2}) - \sin(\frac{\pi}{4}) = 1 - \frac{1}{\sqrt{2}} = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$

Compute $\int_0^{\frac{\pi}{2}} \boxed{\frac{d}{dx}(x^4 \sin(x))} dx$.

$f(x)$

A. $\frac{\pi}{2}$

B. 0

C. 1

D. $\frac{\pi^3}{8}$

E. $\frac{\pi^4}{16}$

FTC 2: $\int_a^b f(x)dx = F(b) - F(a)$

where $F' = f(x)$

or F is antideriv of $f(x)$

$f(x) = \frac{d}{dx}(x^4 \sin(x))$

$F(x)$ is antideriv. of $\frac{d}{dx}(x^4 \sin(x))$

so $F(x) = x^4 \sin(x)$

$$\int_0^{\frac{\pi}{2}} \frac{d}{dx}(x^4 \sin(x))dx = x^4 \sin(x) \Big|_0^{\pi/2}$$

$$= \left(\frac{\pi}{2}\right)^4 \sin\left(\frac{\pi}{2}\right) - 0$$

$$= \frac{\pi^4}{16}$$

what if deriv. is on the outside ?

$$\frac{d}{dx}\boxed{\int_0^{\pi/2} x^4 \sin(x)\, dx}$$

FTC I: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

one has to be a non-constant

↳ area between $x^4\sin(x)$ and x-axis from $x=0$ to $x=\pi/2$ which is always a number (constant)

$$\frac{d}{dx}(\text{constant}) = 0$$

S2018 #4

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{3x} \xrightarrow{x\to\infty} 1^{\infty} \quad \text{indeterminate form}$$

need to turn it into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then use l'Hospital's Rule

$$\lim_{x\to\infty}\underbrace{\left(1+\frac{1}{2x}\right)^{3x}}_{y} \qquad \text{so we want } \lim_{x\to\infty} y$$

$$y=\left(1+\frac{1}{2x}\right)^{3x}$$

$$\ln y=\ln\left(1+\frac{1}{2x}\right)^{3x}$$

$$=3x\cdot\ln\left(1+\frac{1}{2x}\right)=\frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{3x}} \xrightarrow{x\to\infty} \frac{\ln(1)=0}{0}\to\frac{0}{0}$$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{3x}}$$

$$\overset{L}{=} \lim_{x\to\infty} \frac{\dfrac{1}{1+\frac{1}{2x}} \dfrac{d}{dx}\left(1+\frac{1}{2x}\right)}{\dfrac{d}{dx}\left(\frac{1}{3x}\right)} = \lim_{x\to\infty} \frac{\dfrac{1}{1+\frac{1}{2x}} \cdot \frac{1}{2} \cdot \cancel{-\frac{1}{x^2}}}{\frac{1}{3} \cdot \cancel{-\frac{1}{x^2}}}$$

$$\frac{d}{dx}\left(\frac{1}{2x}\right) = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{2} \cdot \frac{-1}{x^2}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2} \cdot \dfrac{1}{1+\underbrace{\frac{1}{2x}}_{\to 0 \text{ as } x\to\infty}}}{\frac{1}{3}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}$$

$\lim_{x\to\infty} \ln y = \frac{3}{2}$ but we want $\lim_{x\to\infty} y$

$$y = e^{\ln y}$$

so $\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{3/2}$

S19 # 8

find $f'(1)$ if $f(x) = \ln\left(\frac{x}{x^2+2}\right)$

one way to do this: $f'(x) = \frac{1}{\frac{x}{x^2+2}} \cdot \underbrace{\frac{d}{dx}\left(\frac{x}{x^2+2}\right)}_{\text{quotient rule}}$

it's ok but might get messy

another way: $f(x) = \ln\left(\frac{x}{x^2+2}\right) = \underbrace{\ln(x)} - \underbrace{\ln(x^2+2)}$

each has simple deriv.

$$f'(x) = \frac{1}{x} - \frac{1}{x^2+2} \cdot \frac{d}{dx}(x^2+2)$$

$$= \frac{1}{x} - \frac{2x}{x^2+2}$$

$$f'(1) = \frac{1}{1} - \frac{2}{1+2} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$f(x) = \int_3^{\tan x} \sqrt{\sqrt{t} + 6t}\, dt$$

find $f'(\frac{\pi}{4})$

deriv. of area function:

FTC 1 : $\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$

if x is not just x, use Chain Rule

$$f(x) = \int_3^{\tan x} \sqrt{\sqrt{t} + 6t}\, dt$$

let $u = \tan x$ then

$$f(u) = \int_3^{u} \sqrt{\sqrt{t} + 6t}\, dt$$

Chain Rule: $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$

$$\frac{df}{dx} = \underbrace{\left(\frac{d}{du}\int_3^u \sqrt{\sqrt{t}+6t}\, dt\right)}_{\text{FTC 1}}\left(\frac{d}{dx}\tan x\right)$$

$$= \left(\sqrt{\sqrt{u} + 6u}\right)\left(\sec^2 x\right) = \left(\sqrt{\sqrt{\tan x} + 6\tan x}\right)\left(\sec^2 x\right)$$

at $x = \frac{\pi}{4}$ $\tan x = 1$

$$\sec x = \frac{1}{\cos x} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

so $f'\left(\frac{\pi}{4}\right) = \sqrt{1+6} \cdot (\sqrt{2})^2 = 2\sqrt{7}$