3.4 The Product and the Quotient Rules

we know $\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$ sum/difference rules

but $\frac{d}{dx}\left[f(x)\, g(x)\right] \neq f'(x)\, g'(x)$

the <u>Product Rule</u> takes care of the $\frac{d}{dx}\left[f(x)\, g(x)\right]$

$$\frac{d}{dx}\left[f(x)\, g(x)\right] = f(x) g'(x) + f'(x) g(x)$$

the order is irrelevant

$$\frac{d}{dx}\left[g(x)\, f(x)\right] = g(x) f'(x) + g'(x) f(x)$$

another notation: $(fg)'(x) = f'(x)\, g(x) + f(x)\, g'(x)$

<u>example</u> $y = (1+x^2)(2-x^2)$

let's try this in two ways

1) Product Rule

$$y = \underbrace{(1+x^2)}_{f}\underbrace{(2-x^2)}_{g}$$

$$y' = \underbrace{(1+x^2)}_{f}\underbrace{\frac{d}{dx}(2-x^2)}_{g'} + \underbrace{(2-x^2)}_{g}\underbrace{\frac{d}{dx}(1+x^2)}_{f'}$$

$$= (1+x^2)(-2x) + (2-x^2)(2x)$$

$$= -2x - 2x^3 + 4x - 2x^3 = -4x^3 + 2x$$

2) multiply y out first

$$y = 2 - x^2 + 2x^2 - x^4 = 2 + x^2 - x^4$$

deriv is simple: $y' = +2x - 4x^3$

example $y = x^2 e^x$

find y''

$y = x^2 e^x$

product rule: $y' = (x^2)\frac{d}{dx}(e^x) + (e^x)\frac{d}{dx}(x^2)$

$= (x^2)(e^x) + (e^x)(2x)$

$= \underbrace{(e^x)}_{f}\underbrace{(x^2+2x)}_{g}$

$y'' = (e^x)\frac{d}{dx}(x^2+2x) + (x^2+2x)\frac{d}{dx}(e^x)$

$= (e^x)(2x+2) + (x^2+2x)(e^x)$

$= 2xe^x + 2e^x + x^2e^x + 2xe^x = x^2e^x + 4xe^x + 2e^x$

now the Quotient Rule

note the sign : order matters!

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x) - f(x)\, g'(x)}{[g(x)]^2} \qquad g(x) \neq 0$$

$$\frac{d}{dx}\left[\frac{\text{high}}{\text{low}}\right] = \frac{(\text{low})\cdot(d\cdot\text{high}) - (\text{high})\cdot(d\cdot\text{low})}{(\text{low})(\text{low})}$$

example

$$y = \frac{6x-1}{2x-4}$$

$$y' = \frac{(2x-4)\frac{d}{dx}(6x-1) - (6x-1)\frac{d}{dx}(2x-4)}{(2x-4)(2x-4)}$$

$$= \frac{(2x-4)(6) - (6x-1)(2)}{(2x-4)^2} = \frac{12x-24-12x+2}{(2x-4)^2}$$

$$= \frac{-22}{(2x-4)^2}$$

example

$$y = \frac{xe^x}{x+2}$$

Start w/ quotient rule

xe^x is a product of two functions of x so needs product rule

$$y' = \frac{(x+2)\boxed{\frac{d}{dx}(xe^x)} - (xe^x)\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$= \frac{(x+2)\boxed{\left[(x)\cdot\frac{d}{dx}(e^x) + (e^x)\frac{d}{dx}(x)\right]} - (xe^x)(1)}{(x+2)^2}$$

$$= \frac{(x+2)(xe^x + e^x) - xe^x}{(x+2)^2}$$

$$= \frac{x^2e^x + xe^x + 2xe^x + 2e^x - xe^x}{(x+2)^2} = \frac{x^2e^x + 2xe^x + 2e^x}{(x+2)^2}$$

Just like w/ product rule, sometimes using the quotient rule directly is not the easiest way

for example, $y = \frac{8}{x} \xrightarrow{\text{quotient}} y' = \frac{(x)\frac{d}{dx}(8) - (8)\frac{d}{dx}(x)}{(x)^2}$

$$= \frac{(x)(0) - (8)}{(x)^2} = -\frac{8}{x^2}$$

$$y = 8x^{-1}$$

$$y' = 8(-1 \cdot x^{-2}) = -8x^{-2} = -\frac{8}{x^2}$$

another example, $y = \frac{2 - x^3}{x^4}$

$$= \frac{2}{x^4} - \frac{x^3}{x^4} = 2x^{-4} - x^{-1}$$

quotient rule not needed

$$y' = -8x^{-5} + x^{-2} = -\frac{8}{x^5} + \frac{1}{x^2}$$

but we have no choice if $y = \frac{x^4}{2-x^3} \neq \frac{x^4}{2} - \frac{x^4}{x^3}$

quotient rule is the only way

$$y' = \frac{(2-x^3)\frac{d}{dx}(x^4) - (x^4)\frac{d}{dx}(2-x^3)}{(2-x^3)^2}$$

$$= \frac{(2-x^3)(4x^3) - (x^4)(-3x^2)}{(2-x^3)^2}$$

$$= \frac{8x^3 - 4x^6 + 3x^6}{(2-x^3)^2} = \frac{8x^3 - x^6}{(2-x^3)^2}$$

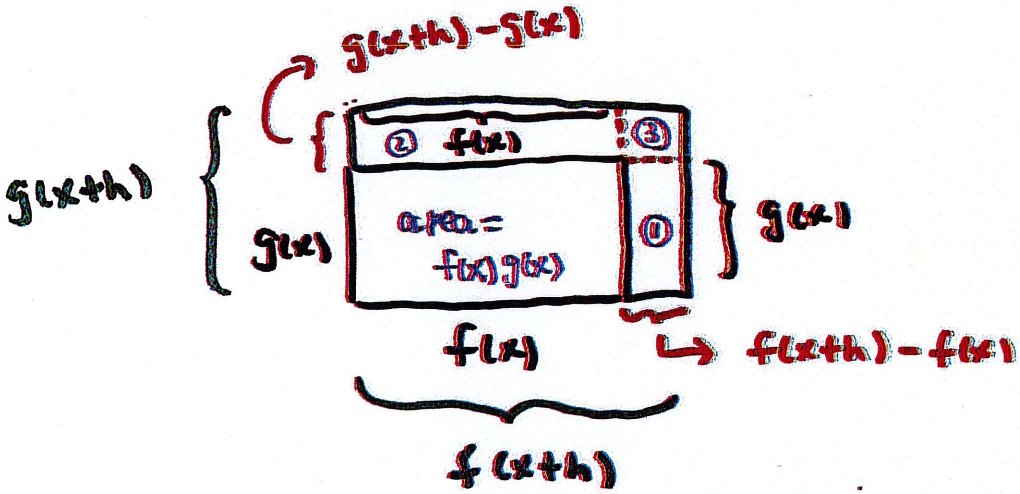
Optional

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

why?

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

let's think of $f(x)g(x)$ as area of a rectangle with length $f(x)$ and width $g(x)$, $f(x+h)g(x+h)$ as a bigger rectangle w/ sides $f(x+h)$ and $g(x+h)$



the difference in area between the two rectangles is $f(x+h)g(x+h) - f(x)g(x)$, which is the numerator of the difference quotient we see from the picture that is also the L-shaped area, which can be expressed as

$$\underbrace{[f(x+h)-f(x)]\,g(x)}_{\text{area of rectangle mark } ①} + \underbrace{[g(x+h)-g(x)]\,f(x)}_{②} + \underbrace{[f(x+h)-f(x)][g(x+h)-g(x)]}_{③}$$

replace numerator of limit w/ the above

$$\frac{d}{dx}\left[f(x)g(x)\right] = \lim_{h\to 0} \frac{[f(x+h)-f(x)]g(x) + [g(x+h)-g(x)]f(x) + [f(x+h)-f(x)][g(x+h)-g(x)]}{h}$$

rewrite:

$$= \lim_{h\to 0} \left\{ \frac{[f(x+h)-f(x)]g(x)}{h} + \frac{[g(x+h)-g(x)]f(x)}{h} + \frac{[f(x+h)-f(x)][g(x+h)-g(x)]}{h} \right]$$

$$= \underbrace{\lim_{h\to 0} \frac{[f(x+h)-f(x)]}{h}}_{f'(x)} \cdot g(x) + \underbrace{\lim_{h\to 0} \frac{[g(x+h)-g(x)]}{h}}_{g'(x)} \cdot f(x) + \lim_{h\to 0} \frac{[f(x+h)-f(x)]}{h} \cdot \underbrace{[g(x+h)-g(x)]}_{\text{goes to } 0 \text{ as } h\to 0}$$

$$= f'(x)\, g(x) + g'(x)\, f(x) \qquad \text{product rule.}$$

Quotient rule is much more complicated, but basic idea is similar