3.5 Derivatives of Trig Functions

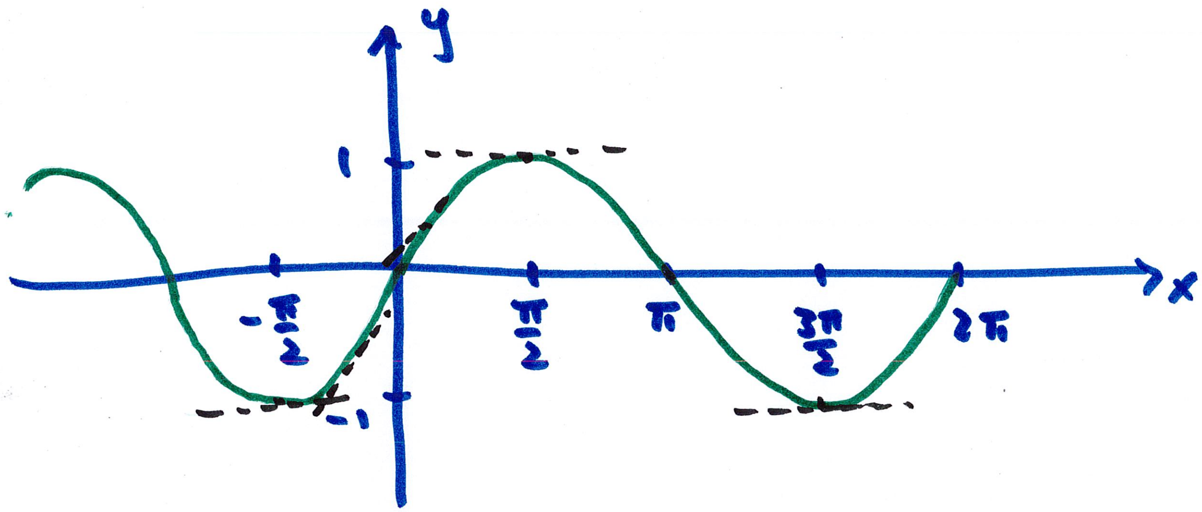
$f(x) = \sin x$ $\qquad f'(x) = ?$

tangent lines are horizontal $(f' = 0)$ at $\pi/2, \frac{3\pi}{2}, \ldots$ and $-\pi/2, -\frac{3\pi}{2}, \ldots$

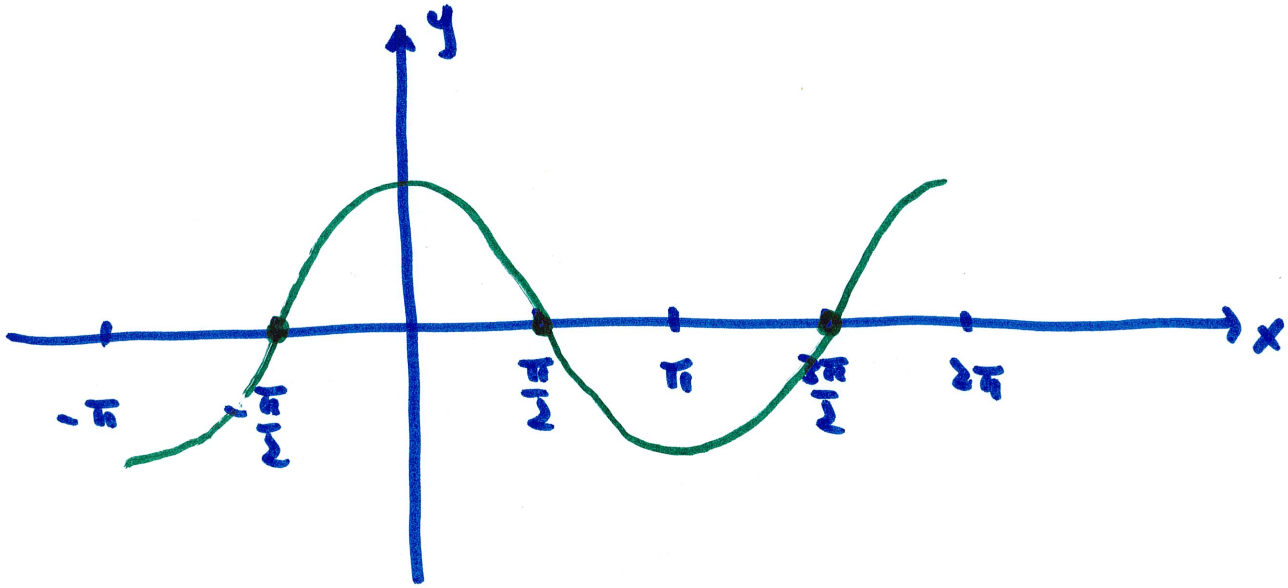
$f' > 0$ on $(-\pi/2, \pi/2)$

$f' < 0$ on $(\pi/2, 3\pi/2)$

continue to see rest of f'



sketch $f'(x)$ using slopes of tangent lines



this looks just like the graph of $\cos x$

so, $\boxed{\frac{d}{dx} \sin x = \cos x}$

following the same process, we can find $\boxed{\frac{d}{dx}\cos x = -\sin x}$

knowing the derivatives of $\sin x$ and $\cos x$, we can use them and the rules of differentiation to find the deriv. of the others.

<u>example</u> $\frac{d}{dx}(\tan x)$

we know $\tan x = \frac{\sin x}{\cos x}$

$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$ now use quotient rule

$$= \frac{(\cos x)\frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \left(\frac{1}{\cos x}\right)^2 = (\sec x)^2 = \boxed{\sec^2 x}$$

following the same process, we find the derivs of the other basic trig functions

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

observations: derivative of co-function is negative

they are related to their co-function, too

e.g. $\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = -\sin x$

all x here are in <u>RADIANS</u>. Do <u>NOT</u> use degrees.

example $y = \dfrac{\tan x - 1}{\sec x}$

we can differentiate it directly, using quotient rule

$$y' = \frac{(\sec x)\frac{d}{dx}(\tan x - 1) - (\tan x - 1)\frac{d}{dx}(\sec x)}{(\sec x)^2}$$

$$= \frac{(\sec x)(\sec^2 x) - (\tan x - 1)(\sec x \cdot \tan x)}{(\sec x)^2}$$

factor out $\sec x$

$$= \frac{\cancel{(\sec x)}\left[(\sec^2 x) - (\tan x - 1)(\tan x)\right]}{(\sec x)^{\cancel{2}}}$$

$$= \frac{\sec^2 x - \tan^2 x + \tan x}{\sec x}$$

identities: $\sin^2 x + \cos^2 x = 1$

divide by $\cos^2 x$: $\tan^2 x + 1 = \sec^2 x$

$\sec^2 x - \tan^2 x = 1$

$$= \boxed{\frac{1 + \tan x}{\sec x}}$$

another option: rewrite y in terms of $\sin x$ and $\cos x$ then differentiate

$$y = \frac{\tan x - 1}{\sec x} = \frac{\frac{\sin x}{\cos x} - 1}{\frac{1}{\cos x}} \cdot \frac{\cos x}{\cos x} = \frac{\sin x - \cos x}{1}$$

same

$$y = \sin x - \cos x$$

$$y' = \cos x - (-\sin x) = \boxed{\cos x + \sin x}$$ much easier!

verify that $\cos x + \sin x = \frac{1 + \tan x}{\sec x} = \frac{1 + \frac{\sin x}{\cos x}}{\frac{1}{\cos x}} \cdot \frac{\cos x}{\cos x}$

$$= \frac{\cos x + \sin x}{1}$$

so the derivs. match

Always consider rewriting in terms of $\sin x$ and $\cos x$ first

Special Trig Limits

$$\lim_{x \to 0} \frac{\sin x}{x}$$ if $x = 0$, $\frac{\sin x}{x} \to \frac{0}{0}$ indeterminate, limit = ?

there are many ways to find the limit, table of values is easiest

x : in radians

x	-0.01	-0.001	0	0.001	0.01
$\frac{\sin x}{x}$	0.99998	0.9999998		0.9999998	0.99998

clearly, $$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

similarly, we can find $$\boxed{\lim_{x \to 0} \frac{\cos x - 1}{x} = 0}$$

$\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ we can replace "x" with identical thing w/o changing the limit

as long as that thing $\to 0$

for example, $\lim\limits_{x \to 0} \dfrac{\sin(x^2)}{(x^2)} = 1$

$\lim\limits_{x \to 0} \dfrac{\sin(\sqrt{x})}{(\sqrt{x})} = 1$ $\lim\limits_{x \to 0} \dfrac{\sin(x^2 - x)}{(x^2 - x)} = 1$

but $\lim\limits_{x \to 0} \dfrac{\sin(5x)}{x} \neq 1$ since they are not identical

to find $\lim\limits_{x \to 0} \dfrac{\sin(5x)}{x}$ we do this

$$\lim_{x \to 0} \frac{\sin(5x)}{(5x)} \cdot \frac{5}{1} = \underbrace{\left(\lim_{x \to 0} \frac{\sin(5x)}{(5x)}\right)}_{1} \cdot \underbrace{\left(\lim_{x \to 0} \frac{5}{1}\right)}_{5} = 5$$

example

$$\lim_{x\to 0} \frac{\sin 19x}{\tan x} \qquad x \to 0 \quad \frac{\sin 19x}{\tan x} \to \frac{0}{0} = ?$$

try to bring $\lim_{x\to 0} \frac{\sin x}{x} = 1$ into this

$$\lim_{x\to 0} \frac{\sin 19x}{1} \cdot \frac{1}{\tan x} = \lim_{x\to 0} \frac{\sin 19x}{1} \cdot \frac{1}{\frac{\sin x}{\cos x}}$$

$$\lim_{x\to 0} \frac{\sin 19x}{1} \cdot \frac{\cos x}{\sin x} = \lim_{x\to 0} \underbrace{\frac{\sin 19x}{1}}_{\text{if } 19x \text{ in denom then limit is } 1} \cdot \underbrace{\frac{1}{\sin x}}_{\text{if } x \text{ in numerator then limit is } 1} \cdot \cos x$$

$\cos x$ goes to 1 as $x \to 0$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{x}{\sin x} = 1$$

$$\lim_{x\to 0} \underbrace{\frac{\sin 19x}{19x}}_{1} \cdot \underbrace{\frac{x}{\sin x}}_{1} \cdot \underbrace{\cos x}_{1} \cdot \underbrace{\frac{19x}{x}}_{19} = \boxed{19}$$