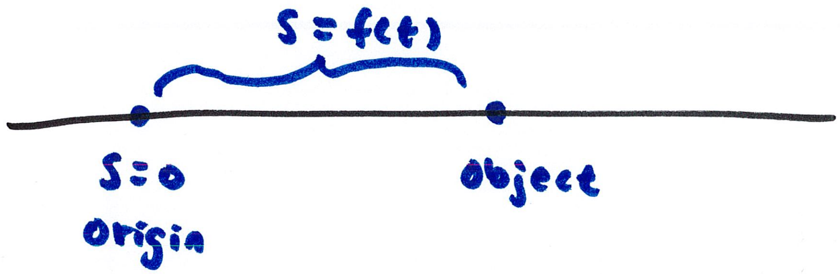
3.6 Derivatives as Rates of Change

example The position of an object moving horizontally on a straight line is $S = f(t) = t^3 - 12t^2 + 45t \qquad 0 \le t \le 7$

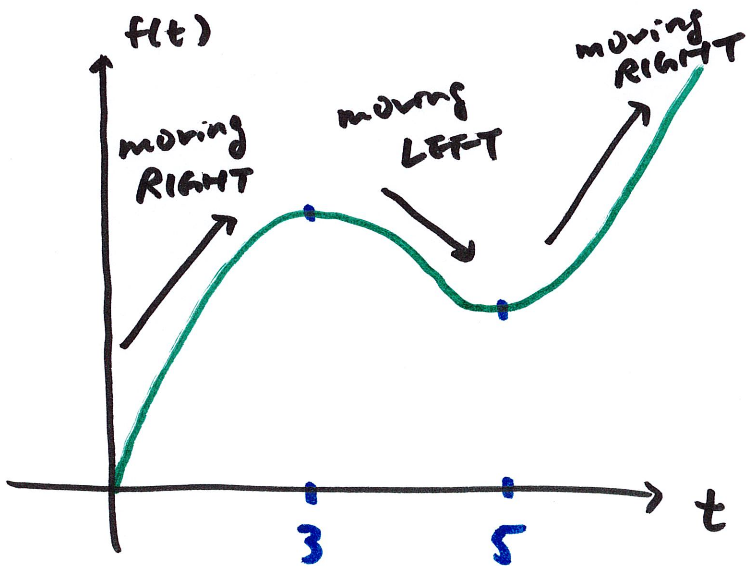
$S > 0$ corresponds to the right of the origin.



$S > 0$: right of origin

$S < 0$: left of origin

graph of $S = f(t) = t^3 - 12t^2 + 45t$



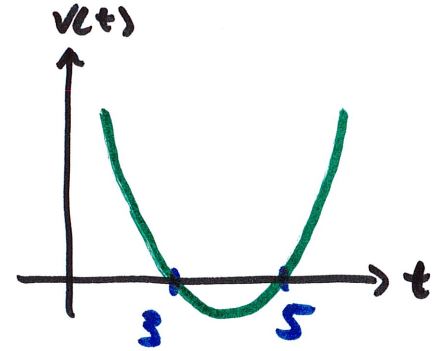
$0 < t < 3$: $f(t)$ increasing so getting farther away from origin moving RIGHT

$3 < t < 5$: $f(t)$ decreasing moving LEFT the velocity is the rate of change of position

↳ derivative

$v(t) = s' = f'(t) = \cancel{3t^2 - 24t + 45}$

$3t^2 - 24t + 45$ ↗



at t-intercepts can be found by setting $v(t) = 0$

zero velocity ("stationary")

$3t^2 - 24t + 45 = 0$

$t^3 - 8t + 15 = 0$ $\quad (t-3)(t-5) = 0 \quad t = 3,\ t = 5$

we can also see the motion of the object from the velocity graph

$0 < t < 3$: $v(t) > 0$ (above t-axis) means $s = f(t)$ is increasing so move RIGHT

$3 < t < 5$: $v(t) < 0$ means $s = f(t)$ is decreasing so move LEFT

$t > 5$: move RIGHT again

at $t = 3$, $t = 5$ object stops ($v = 0$) when changing direction

velocity can be negative because it contains direction information but speed cannot be negative → speed = absolute value of velocity

velocity is rate of change (derivative) of position

acceleration is the rate of change (derivative) of velocity

3.7 The Chain Rule (part 1)

$y = (3x+5)^2 \qquad y' = ?$

one option: $y = (3x+5)(3x+5) = 9x^2 + 30x + 25$

then $y' = 18x + 30$

but notice we can't really do that with $y = (3x+5)^{17}$

or $y = (3x+5)^{3/4}$

or $y = (3x+5)^{-9}$

we can use the Chain Rule

basic idea: think of the function as a function of another function
then the derivative is the product of how each is affected

$$y = (3x+5)^2$$

think of it as $\underbrace{y = u^2}_{\text{y as function of u}}$ where $\underbrace{u = 3x+5}_{\text{u as function of x}}$

the derivative is product of how y is affected by u and how u is affected by x

$$\boxed{\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{\text{how u affects y}} \underbrace{\frac{du}{dx}}_{\text{how x affects u}}}$$

another form: if $y = f(g(x))$

then $y' = f'(g(x))\, g'(x)$

$$\frac{dy}{dx} = (2u)\cdot(3) = 6u = 6(3x+5) = 18x+30$$ same as before

$$y = (3x+5)^{17}$$

$$y = u^{17} \quad \text{where} \quad u = 3x+5$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(17u^{16}\right)(3)$$

$$= \underbrace{17(3x+5)^{16}}_{\substack{\text{looks like} \\ \text{the basic} \\ \text{power rule} \\ \text{if we pretend} \\ 3x+5 \text{ is just} \\ \text{"}x\text{"}}} \cdot \underbrace{(3)}_{\substack{\text{derivative} \\ \text{of } 3x+5 \\ \text{to account} \\ \text{for the fact} \\ \text{that } 3x+5 \\ \text{is not just } x}} = 51(3x+5)^{16}$$

$$\boxed{\text{another way to summarize: } \frac{d}{dx}(\square)^n = n(\square)^{n-1} \cdot \frac{d}{dx}(\square)}$$

example

$y = (3x^2+5)^{-3/4}$

$y = u^{-3/4} \qquad u = 3x^2+5$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(-\frac{3}{4}u^{-7/4}\right)(6x)$$

$$= -\frac{3}{4}(3x^2+5)^{-7/4}(6x)$$

$$= -\frac{9}{2}x\,(3x^2+5)^{-7/4}$$

or! $\frac{d}{dx}\underbrace{(3x^2+5)}_{\text{"}\square\text{"}}^{-3/4} \rightarrow \frac{d}{dx}(\square)^n = n(\square)^{n-1}\cdot\frac{d}{dx}(\square)$

$$= -\frac{3}{4}(3x^2+5)^{-7/4}\cdot 6x$$

example

$y = \sin^{50} x$

$y = (\sin x)^{50}$ NOT the same as $\sin(x^{50})$

think of it as $y = (\square)^{50}$ $y' = 50(\square)^{49} \cdot \frac{d}{dx}(\square)$

$y = 50(\sin x)^{49} \cdot \frac{d}{dx}(\sin x)$

$= 50(\sin x)^{49} \cdot \cos x = \boxed{50 \cos x \sin^{49} x}$

all the basic rules can be extended the same way

$$\frac{d}{dx} u^n = n u^{n-1} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} \sin(u) = \cos(u) \cdot \frac{du}{dx} \qquad \text{from} \quad \frac{d}{dx} \sin x = \cos x$$

example: $y = \sin(x^{50})$

$$y' = \cos(x^{50}) \cdot \frac{d}{dx}(x^{50}) = \cos(x^{50}) \cdot 50x^{49}$$

$$= 50x^{49} \cos(x^{50})$$

and so on.

<u>example</u> $y = \sec(6x^2)$

we know $\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$

so $\frac{d}{dx} \sec(u) = \sec(u)\tan(u) \cdot \frac{d}{dx}(u)$

$$y' = \sec(6x^2)\tan(6x^2) \cdot 12x$$

example

$y = e^{-2x^3}$

we know $\frac{d}{dx}(e^x) = e^x$

so $\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}$

$y' = e^{-2x^3} \cdot \frac{d}{dx}(-2x^3) = e^{-2x^3} \cdot -6x^2$

$= -6x^2 e^{-2x^3}$