3.7 The Chain Rule (part 2)

remember, all basic rules remain the same, just need the extra $\frac{du}{dx}$ to account for the fact that u is not just x

e.g. $\frac{d}{dx} u^n = n u^{n-1} \cdot \frac{du}{dx}$

$\frac{d}{dx} \sin(u) = \cos(u) \cdot \frac{du}{dx}$

$\frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}$

<u>example</u>

$$y = \sqrt{3x + 2e^{8x}} = (\underbrace{3x + 2e^{8x}}_{u})^{1/2}$$

in the form of u^n

deriv. is $n u^{n-1} \cdot \frac{du}{dx}$

$$y' = \frac{1}{2}(3x + 2e^{8x})^{-1/2} \cdot \frac{d}{dx}(3x + 2e^{8x})$$

$$= \frac{1}{2}(3x + 2e^{8x})^{-1/2} \cdot \left(3 + 2 \underbrace{\frac{d}{dx} e^{8x}}_{\text{another chain rule}}\right)$$

form e^u deriv. is $e^u \frac{du}{dx}$

$$= \frac{1}{2}(3x + 2e^{8x})^{-1/2} \cdot (3 + 2 \cdot e^{8x} \cdot 8)$$

$$= \frac{1}{2}(3x+2e^{8x})^{-1/2}(3+16e^{8x}) = \frac{3+16e^{8x}}{2\sqrt{3x+2e^{8x}}}$$

<u>example</u> $y = [(x+8)(x^2+6)]^8$ form: u^8 deriv. $8u^7 \cdot \frac{du}{dx}$

$$y' = 8[(x+8)(x^2+6)]^7 \cdot \underbrace{\frac{d}{dx}[(x+8)(x^2+6)]}_{\text{product rule or multiply out first then differentiate}}$$

$$= 8[(x+8)(x^2+6)]^7 \cdot \left[(x+8)\frac{d}{dx}(x^2+6) + (x^2+6)\frac{d}{dx}(x+8)\right]$$

$$= 8[(x+8)(x^2+6)]^7 \cdot [(x+8)(2x) + (x^2+6)(1)]$$

$$= 8[(x+8)(x^2+6)]^7 \cdot [(2x^2+16x+x^2+6]$$

$$= 8[(x+8)(x^2+6)]^7(3x^2+16x+6)$$

<u>example</u> If $f(-8) = -9$, $f'(-8) = 7$ and $g(x) = \sin(\pi f(x))$

Find $g'(-8)$

$$g(x) = \sin(\overbrace{\pi f(x)}^{u})$$ form: $\sin(u)$ deriv. is $\cos(u) \cdot \frac{du}{dx}$

let's find $g'(x)$

$$g'(x) = \cos(\pi f(x)) \cdot \frac{d}{dx}[\pi f(x)]$$

$$g'(x) = \cos(\pi f(x)) \cdot \pi f'(x)$$

$$g'(-8) = \cos(\pi \cdot \underbrace{f(-8)}_{-9}) \cdot \pi \underbrace{f'(-8)}_{7}$$

$$= \cos(-9\pi) \cdot 7\pi$$

$$= (-1) \cdot 7\pi = -7\pi$$

example

$$y = \tan^5(\sin(9x))$$

$$= \underbrace{[\tan(\sin(9x))]}_{u}^5$$

form: u^5 deriv. $5u^4 \cdot \frac{du}{dx}$

$$y' = 5[\tan(\sin(9x))]^4 \cdot \frac{d}{dx}[\underbrace{\tan(\sin(9x))}]$$

form: $\tan(u)$ deriv. $\sec^2(u) \cdot \frac{du}{dx}$

$$= 5[\tan(\sin(9x))]^4 \cdot \sec^2(\sin(9x)) \cdot \frac{d}{dx}\underbrace{\sin(9x)}$$

form: $\sin(u)$

deriv. $\cos(u) \cdot \frac{du}{dx}$

$$= 5[\tan(\sin(9x))]^4 \cdot \sec^2(\sin(9x)) \cdot \cos(9x) \cdot 9$$

$$= 45 \tan^4(\sin(9x)) \cdot \sec^2(\sin(9x)) \cos(9x)$$

example If $f(x) = e^{e^{e^x}}$ find $f'(0)$

$$f(x) = e^{\underbrace{(e^{e^x})}_{u}}$$

form: e^u deriv. is $e^u \cdot \frac{du}{dx}$

$$f'(x) = e^{e^{e^x}} \cdot \frac{d}{dx}\Big(\underbrace{e^{\overset{u}{(e^x)}}}_{e^u}\Big)$$

e^u deriv. $e^u \frac{du}{dx}$

$$= e^{e^{e^x}} \cdot e^{e^x} \cdot \frac{d}{dx}(e^x)$$

$$f'(x) = e^{e^{e^x}} e^{e^x} e^x$$

$$f'(0) = e^{e^{e^0}} e^{e^0} e^0 = e^e e = e^{e+1}$$

How do we know when to use the Chain Rule?

technical answer: with any composite functions (functions of functions)

more practical answer: ~~when~~ if you wonder if chain rule is needed, it <u>is</u>

because if you use the chain rule even when you don't have to, the answer is still right.

for example, $y = \sin(\overset{u}{\overbrace{x}})$ form: $\sin(u)$ deriv $\cos(u)\cdot\frac{du}{dx}$

$= \sin(u)$ with $u = x$

$y' = \cos(u)\cdot\frac{du}{dx} = \cos(u)\cdot\frac{d}{dx}(x)$

$= \cos(x)\cdot(1) = \cos(x)$ as expected

but if chain rule is needed but you don't use it, answer is wrong

for example, $y = \sin(x^2)$ $u = x^2$ $\sin(u)$

$y' = \cos(u)\cdot\frac{du}{dx} = \cos(x^2)\cdot 2x$

will be missing if Chain Rule not used

$\boxed{\text{When in doubt, Chain it out}}$

example

$$\cancel{y = \sec(9x^3) + x^2 e^{9x^3}}$$

$$y = \underbrace{\sec(9x^3 + x^2)}_{f(x)}\underbrace{(e^{9x^3})}_{g(x)}$$

form: product of two functions
$f(x) \cdot g(x)$

$$y' = \sec(9x^3+x^2) \cdot \frac{d}{dx}\underbrace{(e^{9x^3})}_{e^u} + e^{9x^3} \cdot \frac{d}{dx} \underbrace{\sec(9x^3+x^2)}_{\sec(u)}$$

$\hookrightarrow \sec(u)\tan(u)\frac{du}{dx}$

$$= \sec(9x^3+x^2) \cdot e^{9x^3} \cdot \frac{d}{dx}(9x^3) + e^{9x^3} \cdot \sec(9x^3+x^2)\tan(9x^3+x^2) \cdot \frac{d}{dx}(9x^3+x^2)$$

$$= \sec(9x^3+x^2) \cdot e^{9x^3} \cdot 27x^2 + e^{9x^3} \cdot \sec(9x^3+x^2)\tan(9x^3+x^2)(27x^2+2x)$$

<u>example</u>

$$y = \frac{e^{3x^2+2}}{\cos(\sin(x^2))}$$

quotient to start

$$y' = \frac{\cos(\sin(x^2)) \cdot \frac{d}{dx}(e^{3x^2+2}) - e^{3x^2+2}\, \frac{d}{dx}\overbrace{\cos(\sin(x^2))}^{\cos(u)}}{[\cos(\sin(x^2))]^2}$$

$$= \frac{\cos(\sin(x^2)) \cdot e^{3x^2+2} \cdot 6x - e^{3x^2+2} \cdot -\sin(\sin(x^2)) \cdot \frac{d}{dx}(\sin(x^2))}{[\ \dots\ ]^2}$$

$$\frac{d}{dx}(\sin(u))$$

$$= \cos(u) \cdot \frac{du}{dx}$$

$$= \cos(x^2) \cdot 2x$$