3.8 Implicit Differentiation

a function is in explicit form if y is on its own side and by itself

e.g. $y = x^2 - 5x + 10$

on its own side, not raised to power other than 1 and not part of another function (e.g. $\sin(y)$)

a function is in implicit form if it is not in explicit form.

e.g. $x^2 + y^2 = 9$

$xy = \sin(xy)$

sometimes we can turn implicit into explicit: $x^2 + y^2 = 9$

$$y = \sqrt{9 - x^2}$$

but it's not always possible: $xy = \sin(xy)$

y cannot be isolated

finding $\frac{dy}{dx}$ or y' when y is explicit is straight forward.

but if y is implicit (e.g. $xy = \sin(xy)$), we need <u>implicit differentiation</u>

implicit differentiation is built upon the <u>Chain Rule</u>

if we know $y = 3x^2+5$

then $$\frac{d}{dx}(y)^3 = \frac{d}{dx}\underbrace{(3x^2+5)}_{u}{}^3 = \underbrace{3\cdot(3x^2+5)^2}_{\substack{\cancel{3y^2} \\ 3u^2}}\cdot\underbrace{(6x)}_{\frac{du}{dx}}$$

the same applies if we don't know what y looks like
but we know <u>y is a function of x</u>

$$\frac{d}{dx}(y)^3 = 3y^2\cdot\boxed{\frac{dy}{dx}}$$ from chain rule

some function of ~~y~~ x

example $x^3+y^3=8$ find $\frac{dy}{dx}$

↳ y is some implicit function of x

take derivative with respect to x on both sides

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(8)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(\underline{\underline{y}}^3) = \frac{d}{dx}(8)$$

y is a function of x
chain rule applies

$$3x^2+3y^2\frac{dy}{dx} = 0$$

now solve for $\frac{dy}{dx}$

$$3y^2\frac{dy}{dx} = -3x^2$$

$$\boxed{\frac{dy}{dx} = -\frac{x^2}{y^2}}$$

we can actually rewrite $x^3+y^3=8$ in explicit form

$$y^3 = 8 - x^3$$

$$y = (8-x^3)^{1/3}$$ its derivative should match the other one

$$y' = \frac{dy}{dx} = \frac{1}{3}(8-x^3)^{-2/3} \cdot \frac{d}{dx}(8-x^3)$$

$$= \frac{1}{3}(8-x^3)^{-2/3}(-3x^2) = \frac{-x^2}{(8-x^3)^{2/3}}$$

this is also $\dfrac{-x^2}{\left[\underbrace{(8-x^3)^{1/3}}_{y}\right]^2} = \dfrac{-x^2}{y^2}$ matches the other form

<u>example</u> $\quad \frac{1}{x} + \frac{1}{y} = 2 \qquad$ find $\frac{dy}{dx}$

y is some function of x

rewrite : $\quad x^{-1} + y^{-1} = 2$

take deriv. of both sides with respect to x

$$\frac{d}{dx}(x^{-1}) + \frac{d}{dx}(y^{-1}) = \frac{d}{dx}(2)$$

↳ contains x so <u>chain rule applies</u>

$$-x^{-2} + \left(-y^{-2}\frac{dy}{dx}\right) = 0$$

solve for $\frac{dy}{dx}$

$$-y^{-2}\frac{dy}{dx} = x^{-2}$$

$$\frac{dy}{dx} = \frac{x^{-2}}{-y^{-2}} = \boxed{-\frac{y^2}{x^2}}$$

example

$$xy = \sin(xy)$$

contains x

differentiate both sides with respect to x

contains x

$$\frac{d}{dx}(xy) = \frac{d}{dx}[\sin(xy)]$$

product of two functions of x

product rule is required

$$(x)\cdot\frac{d}{dx}(y) + (y)\cdot\frac{d}{dx}(x) = \frac{d}{dx}[\sin(xy)]$$

$$x\cdot\frac{dy}{dx} + y\cdot 1 = \cos(xy)\cdot\underbrace{\frac{d}{dx}(xy)}_{\text{product rule}}$$

$$x\frac{dy}{dx} + y = \cos(xy)\cdot\left(x\cdot\frac{dy}{dx} + y\right)$$

solve for $\frac{dy}{dx}$

$$x\frac{dy}{dx}+y = x\cos(xy)\frac{dy}{dx} + y\cos(xy)$$

$$x\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = y\cos(xy) - y$$

$$\frac{dy}{dx}\left[x - x\cos(xy)\right] = y\cos(yx) - y$$

$$\boxed{\frac{dy}{dx} = \frac{y\cos(xy) - y}{x - x\cos(xy)}}$$

<u>example</u> $e^{2y} + x = 8y$ find $\frac{d^2y}{dx^2}$

contain x

derivative with respect to x on both sides

$$\frac{d}{dx}(e^{2y}) + \frac{d}{dx}(x) = \frac{d}{dx}(8y)$$

$$\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}$$

$$e^{2y} \cdot \frac{d}{dx}(2y) + 1 = 8\frac{dy}{dx}$$

$$e^{2y} \cdot 2\frac{dy}{dx} + 1 = 8\frac{dy}{dx}$$

solve for $\frac{dy}{dx}$

$$2e^{2y}\frac{dy}{dx} - 8\frac{dy}{dx} = -1$$

$$\frac{dy}{dx}(2e^{2y} - 8) = -1$$

$$\frac{dy}{dx} = \frac{-1}{2e^{2y} - 8}$$

now differentiate again to find $\frac{d^2y}{dx^2}$

$$\boxed{\frac{dy}{dx} = -(2e^{2y}-8)^{-1}}$$

form of u^{-1} deriv: $-u^{-2} \cdot \frac{du}{dx}$

$$\frac{d^2y}{dx^2} = -(-1)(2e^{2y}-8)^{-2} \cdot \frac{d}{dx}(2e^{2y}-8)$$

$$= (2e^{2y}-8)^{-2} \cdot 2 \cdot e^{2y} \cdot \frac{d}{dx}(2y)$$

$$= (2e^{2y}-8)^{-2} \cdot 2 \cdot e^{2y} \cdot 2\frac{dy}{dx}$$

$$= 4e^{2y}(2e^{2y}-8)^{-2} \cdot \boxed{\frac{dy}{dx}}$$

we know what this looks like

$$= 4e^{2y}(2e^{2y}-8)^{-2} \cdot -(2e^{2y}-8)^{-1}$$

$$= -4e^{2y} \cdot (2e^{2y}-8)^{-3} = \boxed{\frac{-4e^{2y}}{(2e^{2y}-8)^3}}$$