3.10 Derivatives of Inverse Functions

(NOT on exam 2)

$y = \sin x \rightarrow y' = \cos x$

$y = \sin^{-1} x \rightarrow y' = ?$

we can accomplish this by doing implicit differentiation

$y = \sin^{-1} x \longleftrightarrow \underbrace{\sin(y) = x}_{\text{differentiate implicitly}}$

$\frac{d}{dx} \sin(y) = \frac{d}{dx}(x)$

function of x
need chain rule : $\frac{dy}{dx}$

$\cos(y) \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos(y)}$

usually, if possible, we want the result explicitly (as function of x, not y)

we don't want that cos(y) on the right

we know: $\cos^2(y) + \underbrace{\sin^2(y)}_{} = 1$

from earlier, $\sin(y) = x$

so $\sin^2(y) = x^2$

$$\cos^2(y) + x^2 = 1$$

or $\cos^2(y) = 1 - x^2$ or $\cos(y) = \sqrt{1-x^2}$

sub into $\dfrac{dy}{dx} = \dfrac{1}{\cos(y)} = \dfrac{1}{\sqrt{1-x^2}}$

so, $$\boxed{\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}}$$

chain rule extension

$$\frac{d}{dx}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

following similar steps, we get

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

<u>example</u> $y = \sin^{-1}(6x^5)$

two options: use $\frac{d}{dx}\sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

or implicit differentiation

formula: $y = \sin^{-1}(\underbrace{6x^5}_{u})$

$$y' = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} = \frac{1}{\sqrt{1-(6x^5)^2}}\frac{d}{dx}(6x^5) = \boxed{\frac{30x^4}{\sqrt{1-36x^{10}}}}$$

implicit differentiation: $y = \sin^{-1}(6x^5)$

$$\sin(y) = 6x^5$$

$$\frac{d}{dx}\sin(y) = \frac{d}{dx}(6x^5)$$

$$\cos(y)\,y' = 30x^4$$

$$y' = \frac{30x^4}{\cos(y)}$$

now eliminate $\cos(y)$: $\cos^2(y) + \underbrace{\sin^2(y)} = 1$

from earlier: $\sin(y) = 6x^5$

so $\sin^2(y) = (6x^5)^2 = 36x^{10}$

so, $\cos^2(y) = 1 - 36x^{10}$

$\cos(y) = \sqrt{1-36x^{10}}$

then, $y' = \dfrac{30x^4}{\cos(y)} = \boxed{\dfrac{30x^4}{\sqrt{1-36x^{10}}}}$

<u>example</u>

$$y=\cos^{-1}(\underbrace{\sin(x^2)}_{u})$$

$$\frac{d}{dx}\cos^{-1}(u)=\frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$y'=\frac{-1}{\sqrt{1-\sin^2(x^2)}}\underbrace{\frac{d}{dx}\sin(x^2)}_{\sin(u)}$$

$$\frac{d}{dx}\sin(u)=\cos(u)\frac{du}{dx}$$

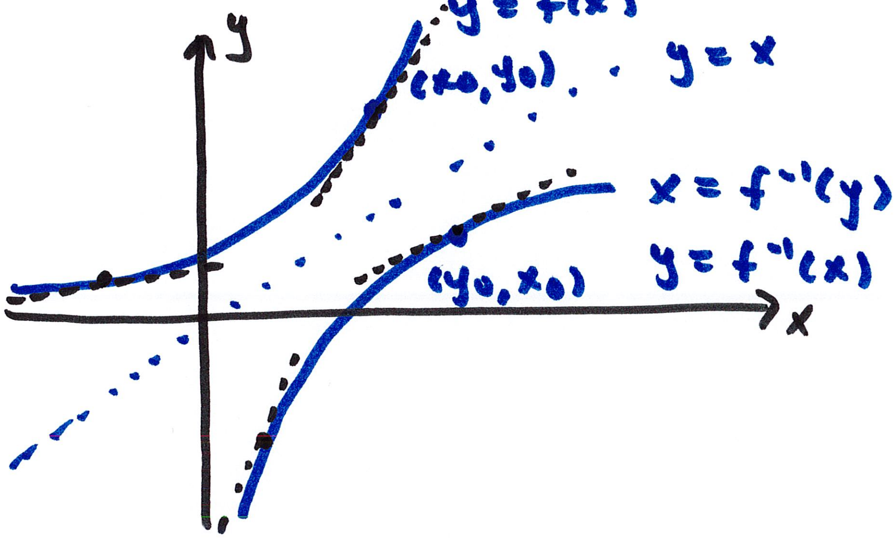
$$=\frac{-1}{\sqrt{1-\sin^2(x^2)}}\cdot\cos(x^2)\cdot 2x=\boxed{\frac{-2x\cos(x^2)}{\sqrt{1-\sin^2(x^2)}}}$$

we can simplify it a bit further: $\sin^2(x^2)+\cos^2(x^2)=1$

$$\cos^2(x^2)=1-\sin^2(x^2)$$

$$\frac{-2x\cos(x^2)}{\sqrt{1-\sin^2(x^2)}}=\frac{-2x\cos(x^2)}{\sqrt{\cos^2(x^2)}}=\boxed{\frac{-2x\cos(x^2)}{|\cos(x^2)|}}$$

↳ need absolute value because sign of $\cos(x^2)$ is not known what is the relationship between the tangent line slope on a function and its inverse?



when $f'(x_0)$ is steep, the corresponding tangent line slope on $f^{-1}(x)$ is shallow and when $f'(x_0)$ is ~~steep~~ shallow, the slope on $f^{-1}(x)$ is steep

this suggests the relationship is a reciprocal relationship what does it look like?

$$y = f(x)$$

$\hookrightarrow x = f^{-1}(y)$

differentiate implicitly

$$\frac{d}{dx}(x) = \frac{d}{dx} f^{-1}(y)$$

$$1 = (f^{-1})'(y) \cdot \frac{dy}{dx}$$

from chain rule since $y = f(x)$

since $\frac{dy}{dx} = f'(x)$, we can sub it in and solve for $(f^{-1})'(y)$

$$1 = (f^{-1})'(y) \cdot f'(x)$$

so, we get $\boxed{(f^{-1})'(y) = \frac{1}{f'(x)}}$

notice f' depends on x and $(f^{-1})'$ depends on the corresponding y

example $f(x)=e^x$ without finding $f^{-1}(x)$, find $(f^{-1})'(e)$

formula: $(f^{-1})'(y)=\dfrac{1}{f'(x)}$

$(f^{-1})'(e)=\dfrac{1}{f'(x)}$

$y=e$

corresponding x on $f(x)=e^x$ such that $y=e$

$f(x)=e^x$ and $f=y=e$

then $e=e^x \rightarrow x=1$

$f(x)=e^x$

$f'(x)=e^x$

$f'(1)=e$

then, $(f^{-1})'(e)=\dfrac{1}{f'(1)}=\boxed{\dfrac{1}{e}}$

let's verify that by finding the inverse and then its derivative at e

$f(x) = e^x \rightarrow f^{-1}(x) = \ln(x)$

$(f^{-1})'(x) = \frac{1}{x}$

so, $(f^{-1})'(e) = \frac{1}{e}$ same as using the other method