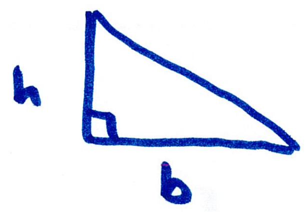
3.11 Related Rates (part 1)

relating the rate of change (derivative) of one thing to that of another

skills needed: chain rule, implicit differentiation

example



If the base is increasing at 1 cm/s when the base is 3 cm and height is 4 cm. How should the height change ~~of~~ for the area to remain unchanged?

here, both b and h are changing with time

→ functions of time : $h = h(t)$, $b = b(t)$

at a particular instant $b = 3$ and $h = 4$ and $\frac{db}{dt} = 1$

find $\frac{dh}{dt}$ such that the area A is unchanged ($\frac{dA}{dt} = 0$)

first, relate base b, height h, and the area A

for a triangle, area $= \frac{1}{2} \cdot$ base $\cdot$ height

$$A(t) = \frac{1}{2} \cdot b(t) \cdot h(t)$$

now differentiate both sides with respect to t

$$\frac{d}{dt} A = \frac{d}{dt}\left(\frac{1}{2} \cdot b \cdot h\right)$$

both are functions of t so need product rule

$$\frac{dA}{dt} = \frac{1}{2} \cdot \left(b \cdot \frac{dh}{dt} + h \cdot \frac{db}{dt}\right)$$

at the moment when $\underbrace{b = 3,\ h = 4,\ \frac{db}{dt} = 1, \text{ we want } \frac{dA}{dt} = 0}$

plug in these numbers
AFTER we related the rates

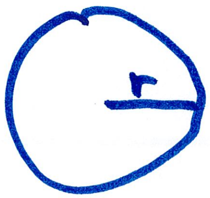
$$0 = \frac{1}{2}\left(3 \cdot \frac{dh}{dt} + 4 \cdot 1\right) = \frac{3}{2} \frac{dh}{dt} + 2 = 0$$

solve for $\frac{dh}{dt}$: $\quad \frac{dh}{dt} = \frac{-2}{3/2} = \boxed{-\frac{4}{3}}$

negative rate
so it means h is decreasing
at $4/3$ cm/s

example

How fast is the area of a circle changing if the circumference of the circle is decreasing at the rate of 2 cm/s when the circumference is 3 cm?



$A = \pi r^2$ relates area to radius

want to relate area to circumference

circumference: $C = 2\pi r$

$$r = \frac{C}{2\pi}$$

$$A = \pi \left(\frac{C}{2\pi}\right)^2 = \pi \cdot \frac{C^2}{4\pi^2} = \frac{1}{4\pi} C^2$$

$A = \frac{1}{4\pi} C^2$ both A and C are functions of t

differentiate with respect to t

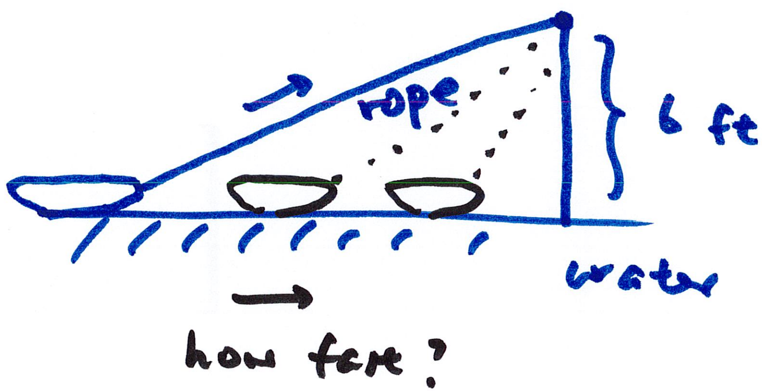
$$\frac{d}{dt} A = \frac{d}{dt}\left(\frac{1}{4\pi} C^2\right)$$

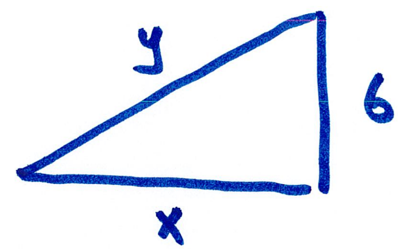
$$\frac{dA}{dt} = \frac{1}{4\pi} \cdot 2C \cdot \frac{dC}{dt}$$

now we related rates we can plug in numbers: $\frac{dC}{dt} = -2$, $C = 3$

$$= \frac{1}{2\pi} \cdot 3 \cdot -2 = \boxed{-3/\pi}$$ area decreasing

example A boat is being pulled toward a dock at a rate of ~~4 ft/~~ 4 ft/s, by a rope through a point 6 ft above water on the dock. How fast is the boat traveling when the rope is 14 ft?





x: dist. between boat and dock
y: rope length

we know: $\frac{dy}{dt} = -4$ negative rate because we pull in rope

we want: $\frac{dx}{dt}$ when $y = 14$

we need to relate y to x

Pythagorean Theorem is the easiest: $y^2 = x^2 + 6^2$

both x and y depend on t

now differentiate

$$\frac{d}{dt}(y^2) = \frac{d}{dt}(x^2 + 6^2)$$

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

Solve for what we want: $\frac{dx}{dt}$

$$\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$$

now rates are related, put in numbers

$y = 14$ rope is 14 ft $\quad \frac{dy}{dt} = -4$

$x = ?$ $\quad x^2 = y^2 - 36$

$$x = \sqrt{y^2 - 36} = \sqrt{14^2 - 36} = \sqrt{160}$$

$$\frac{dx}{dt} = \frac{14}{\sqrt{160}}(-4) = \boxed{-4.43} \text{ ft/s}$$

boat is approaching dock at 4.43 ft/s