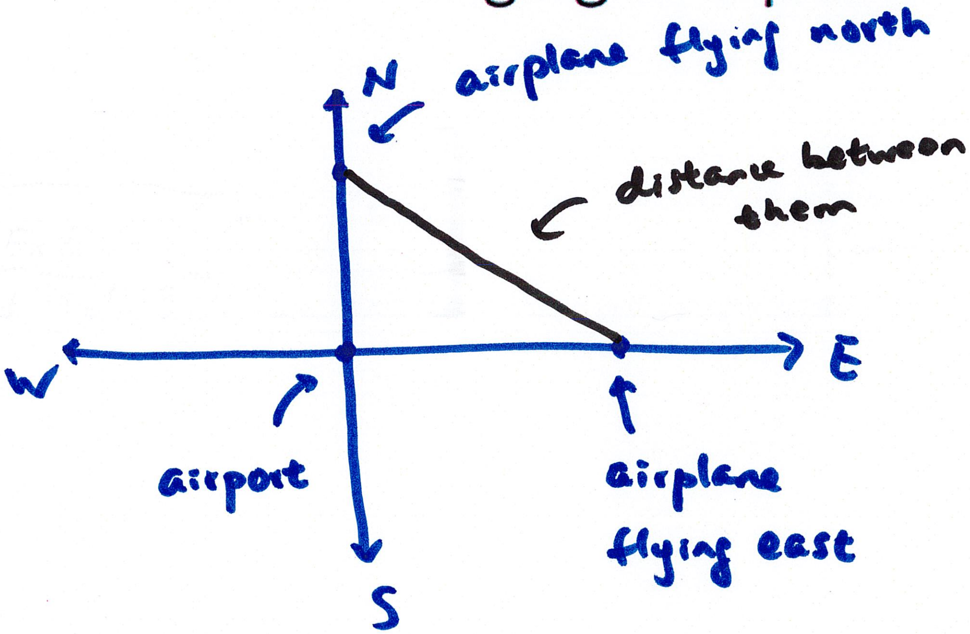
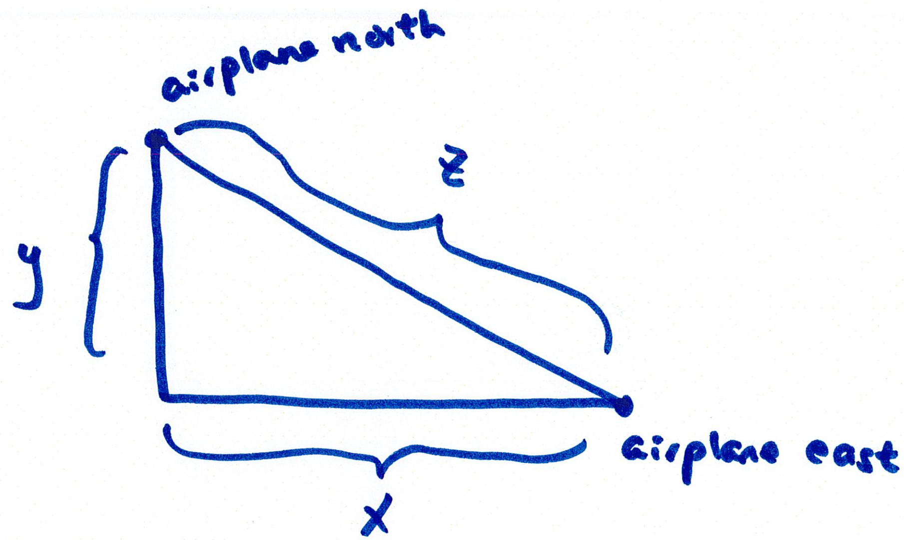
3.11 Related Rates (part 2)

- <u>Example 1.</u> Two airplanes take off from an airport. One leaves at noon flying 500 mi/hr due east. The other one leaves at 1 pm flying 600 mi/hr due north. Assuming both planes fly at the same and constant altitude, how fast is the distance between them changing at 3 pm?





x: dist of the plane flying east ~~to~~ from airport

y: dist of the plane flying north from airport

z: dist between them

Known: $\frac{dx}{dt} = 500$, $\frac{dy}{dt} = 600$

want: $\frac{dz}{dt}$ at 3 pm

$z^2 = x^2 + y^2$ $\quad$ x, y, z are functions of t

$$\frac{d}{dt}(z^2) = \frac{d}{dt}(x^2+y^2)$$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}$$

at 3pm, $\frac{dx}{dt} = 500$, $\frac{dy}{dt} = 600$

$x = 500 \cdot 3$ (left at noon, flying at 500 for 3 hrs)

$= 1500$

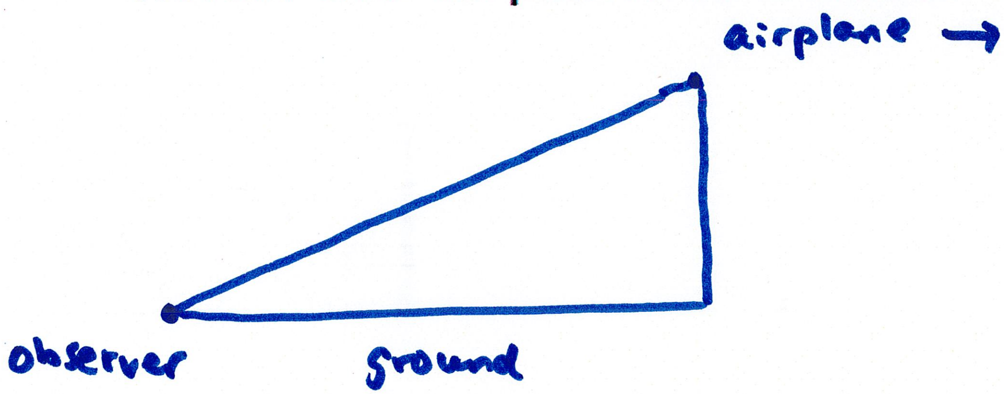
$y = 600 \cdot 2 = 1200$

$z = \sqrt{x^2+y^2} = \sqrt{1500^2 + 1200^2}$

$$= \frac{1500(500) + 1200(600)}{\sqrt{1500^2+1200^2}}$$

$$= \boxed{765.25} \text{ miles/hour}$$

- <u>Example 2.</u> An airplane is flying horizontally at an altitude of 1 mile and a speed of 500 mph. An observer on the ground is looking at the plane. At what rate is the angle between the line from the observer to the airplane and the ground changing when the airplane is 2 miles from the observer?

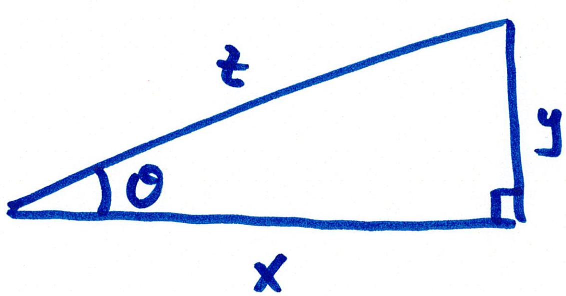


z: dist. from observer to plane

x: dist from observer to point below airplane on the ground

y: altitude $(=1)$

θ: angle of elevation



Known: $y = 1$

$\frac{dx}{dt} = 500$

want: $\frac{d\theta}{dt}$ when $z = 2$

we want to relate θ, x, z (adjacent and hypotenuse)

$\cos\theta = \frac{adj}{hyp} = \frac{x}{z}$

differentiate with respect to t

we don't know this ~~want~~

$$-\sin\theta \cdot \frac{d\theta}{dt} = \frac{z \cdot \frac{dx}{dt} - x \cdot \frac{dz}{dt}}{z^2}$$ (quotient rule used)

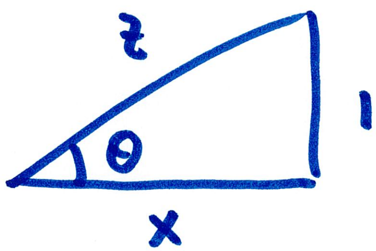
want

we don't know $\frac{dz}{dt}$ so relating x and z is not useful

cosine is not good

try a different one without hypotenuse

let's try tangent



$\tan\theta = \frac{1}{x} = x^{-1}$

differentiate

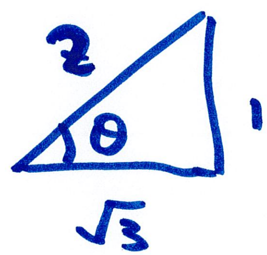
$\sec^2\theta \cdot \frac{d\theta}{dt} = -x^{-2} \cdot \frac{dx}{dt}$

$$\frac{d\theta}{dt} = \frac{-x^{-2}\frac{dx}{dt}}{\sec^2\theta} = \frac{-\frac{dx}{dt}}{x^2 \frac{1}{\cos^2\theta}} = -\frac{\cos^2\theta \cdot \frac{dx}{dt}}{x^2}$$

when $z = 2$

$z^2 = 1 + x^2$

$x = \sqrt{z^2 - 1} = \sqrt{3}$

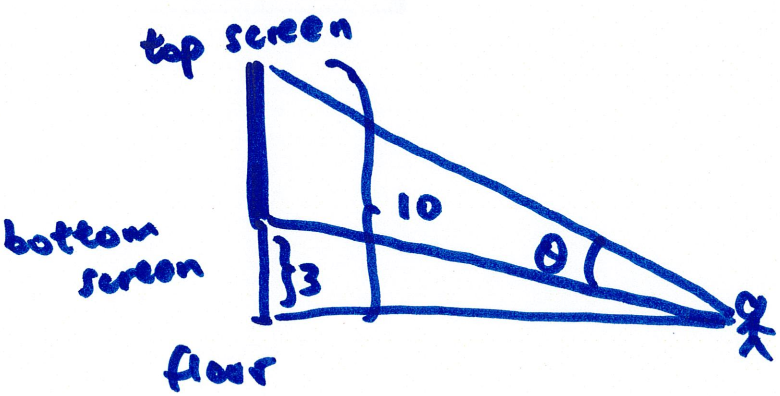


need $\cos\theta$:

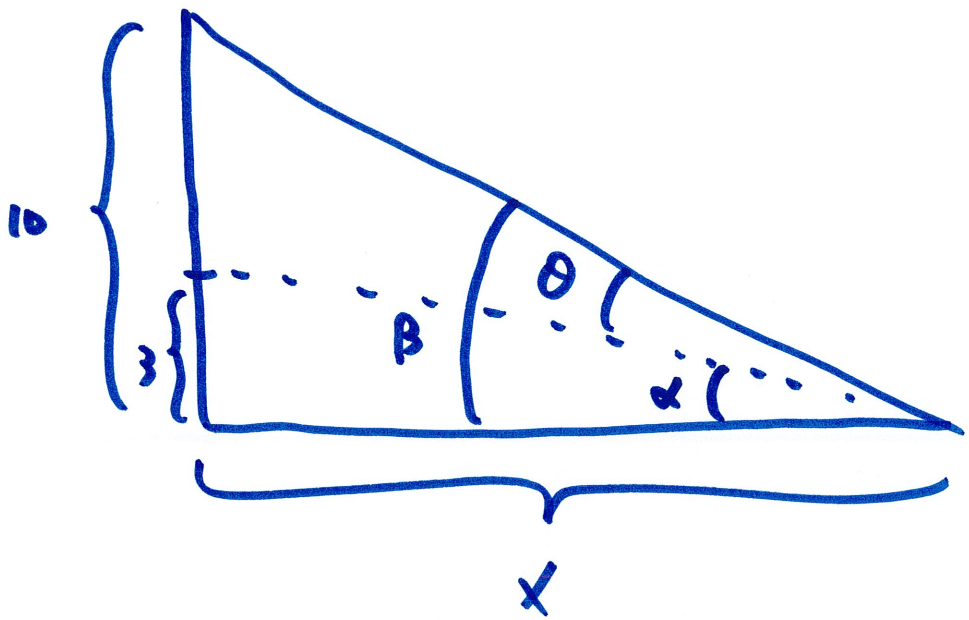
$\cos\theta = \frac{\sqrt{3}}{2}$

$$\frac{d\theta}{dt} = -\frac{\left(\frac{\sqrt{3}}{2}\right)^2 \cdot 500}{(\sqrt{3})^2} = \boxed{-125} \text{ radians/hr}$$

- <u>Example 4.</u> The bottom of a large theater screen is 3 ft above your eye level and the top of the screen is 10 ft above your eye level. Assume you walk toward the screen (perpendicular to the screen) at a rate of 7 ft/s while looking at the screen. What is the rate of change of the viewing angle (from the bottom of the screen to the top of the screen) when you are 70 ft from the wall, assuming the floor is flat?



θ: viewing angle changing as you walk toward/away from screen



α: angle from ground to bottom of screen

β: angle from ground to top of screen

X: dist. from you to wall w/ screen

known: $\frac{dx}{dt} = -7$

↑ negative because walking toward

want: $\frac{d\theta}{dt}$ when $x = 70$

notice $\theta = \beta - \alpha$

$\tan\beta = \frac{10}{x}$ $\quad \tan\alpha = \frac{3}{x}$

$\beta = \tan^{-1}\left(\frac{10}{x}\right)$ $\quad \alpha = \tan^{-1}\left(\frac{3}{x}\right)$

$\theta = \tan^{-1}\left(\frac{10}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)$

recall $\frac{d}{dx}\tan^{-1}(u) = \frac{1}{1+u^2}\frac{du}{dx}$

take derivative

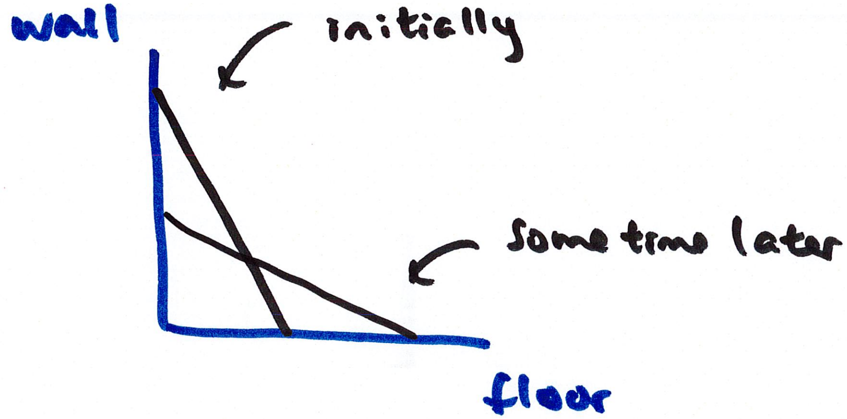
$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{10}{x}\right)^2}\frac{d}{dt}\overset{10x^{-1}}{\left(\frac{10}{x}\right)} - \frac{1}{1+\left(\frac{3}{x}\right)^2}\frac{d}{dt}\overset{3x^{-1}}{\left(\frac{3}{x}\right)}$$

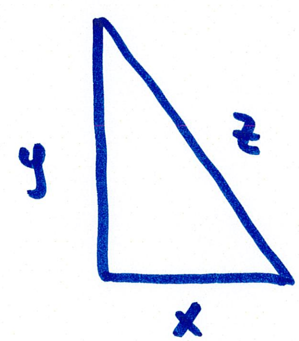
$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{10}{x}\right)^2}\left(-10x^{-2}\cdot\frac{dx}{dt}\right) - \frac{1}{1+\left(\frac{3}{x}\right)^2}\left(-3x^{-2}\cdot\frac{dx}{dt}\right)$$

sub in $x = 70$, $\frac{dx}{dt} = -7$

$$\frac{d\theta}{dt} = \dots = \boxed{0.0097} \text{ radians/s}$$

- Example 3: The top of a ladder leaning against a wall is sliding down at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 0.2 m/s. How long is the ladder?





z: length of ladder

x: dist. ~~fro~~ from bottom of ladder to wall

y: dist of top of ladder to ground

x, y are functions of time t

z cannot change

known: $\frac{dy}{dt} = -0.15$ $\quad \frac{dx}{dt} = 0.2$ when $x = 3$

want: z

$x^2+y^2=z^2$

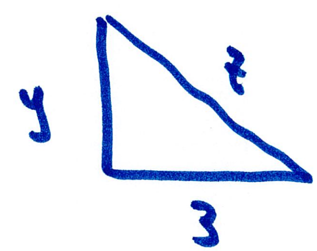
differentiate with respect to t

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

when $x=3$

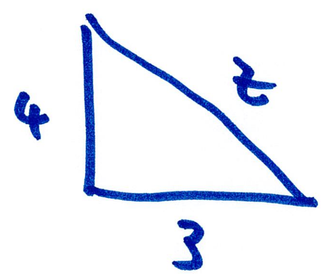
$$(2)(3)(0.2)+(2)(y)(-0.15)=0$$

need y:



$y=4$

now we know y:



$z=\sqrt{3^2+4^2}=\boxed{5}$ ladder is 5 m long