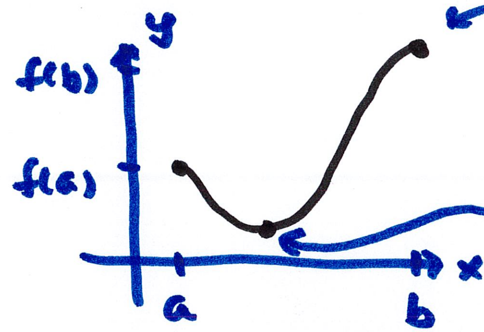
4.1 Maximum and Minimum Values

goal: find max/min values of a continuous function over some interval $a \le x \le b$

possible situations:

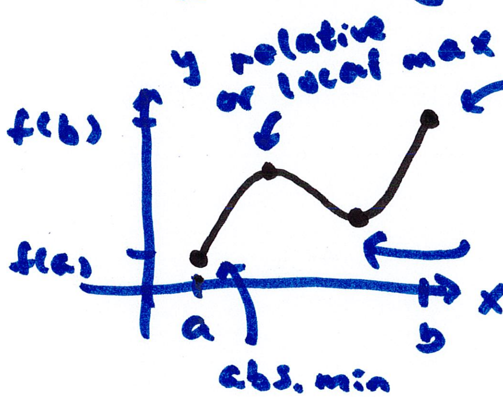


highest point on $a \le x \le b$

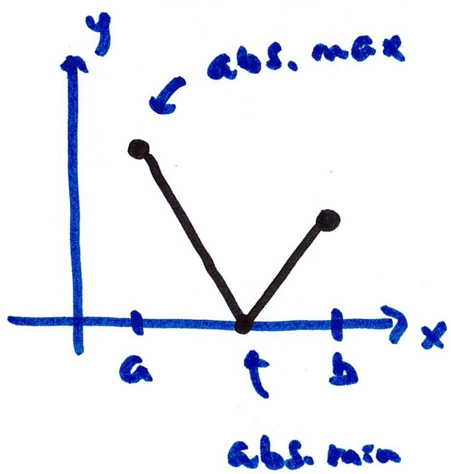
→ absolute (or global) maximum

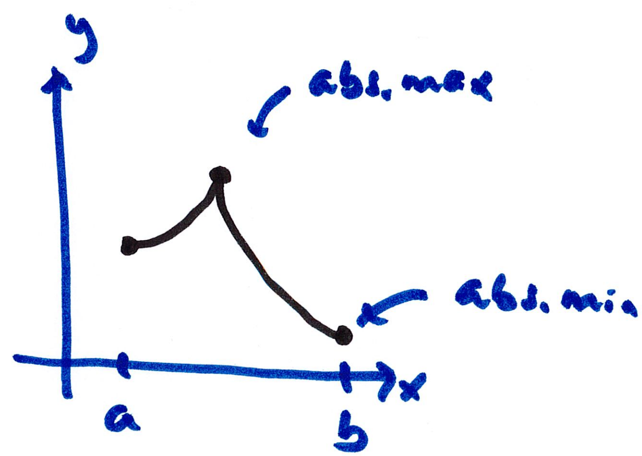
lowest point on $a \le x \le b$

→ absolute (or global) minimum



relative or local min





we see that the possible locations of abs. max/min are:

end points of the interval

where tangent line is horizontal ($f' = 0$)

where the slope of tangent line is not defined (f' DNE)

so, given $f(x)$ and interval $a \le x \le b$

find the above locations then compare values of $f(x)$

example Find the abs. max/min values of

$$f(x) = 2x^3 - 6x^2 - 18x + 4 \quad \text{on} \quad -2 \le x \le 4$$

find values of x where $f' = 0$ or f' DNE

$f'(x) = 6x^2 - 12x - 18$

$f' = 0 \rightarrow 6x^2 - 12x - 18 = 0$

$x^2 - 2x - 3 = 0 \qquad (x-3)(x+1) = 0$

$\boxed{x = -1, \quad x = 3}$ critical numbers (where $f' = 0$ or f' DNE) possible locations of max/min

f' DNE does not happen since $f(x)$ and $f'(x)$ are polynomials

possible locations of max/min values:

$\left.\begin{array}{l} x = -2 \\ x = 4 \end{array}\right\}$ end points of interval

$\left.\begin{array}{l} x = -1 \\ x = 3 \end{array}\right\}$ critical numbers

now we compare $f(x)$ at these locations

$f(-2) = 2(-2)^3 - 6(-2)^2 - 18(-2) + 4 = 0$

$f(-1) = 14$ $\rightarrow$ largest, so this means abs. max value of f(x) is 14 at $x = -1$

$f(3) = -50$ $\rightarrow$ smallest, so abs. min $= -50$ at $x = 3$

$f(4) = -36$

example

$f(x) = x + \frac{1}{x}$ on $\boxed{1 \leq x \leq 4}$ interval we care about

$= x + x^{-1}$

find critical numbers: $f' = 0$ or f' DNE

$f' = 1 - x^{-2} = 1 - \frac{1}{x^2}$

$f' = 0 \rightarrow 1 - \frac{1}{x^2} = 0 \qquad 1 = \frac{1}{x^2} \qquad x^2 = 1$ keep $\boxed{x = 1}$, $\cancel{x = -1}$

falls outside interval discard

f' DNE $\rightarrow 1 - \frac{1}{x^2} = f'$

DNE at $\cancel{x = 0}$ discard because:

1) outside interval

2) $f(x)$ not defined at $x = 0$

so we keep only one critical number: $x = 1$ (which happens to be left end point)

now compare $f(x)$ at these and end points

$f(1) = 1 + \frac{1}{1} = 2$ abs. min at $x = 1$

$f(4) = 4 + \frac{1}{4} = \frac{17}{4}$ abs. max at $x = 4$

example $f(x) = 2x^2 + 2\cos^{-1}x$ on $-1 \le x \le 1$

find critical numbers: $f' = 0$ or f' DNE

$$f'(x) = 4x + 2 \cdot \frac{-1}{\sqrt{1-x^2}} = 4x - \frac{2}{\sqrt{1-x^2}} = \frac{4x\sqrt{1-x^2} - 2}{\sqrt{1-x^2}}$$

$f' = 0 \rightarrow$ fraction $= 0$ means numerator $= 0$

$$4x\sqrt{1-x^2} - 2 = 0$$

$$4x\sqrt{1-x^2} = 2$$

$$(4x\sqrt{1-x^2})^2 = 2^2$$

$$16x^2(1-x^2) = 4$$

$$16x^2 - 16x^4 - 4 = 0 \quad \text{divide by } -4$$

$$4x^4 - 4x^2 + 1 = 0$$

$$4(x^2)^2 - 4(x^2) + 1 = 0 \quad \text{quadratic eq.}$$

$$(2x^2 - 1)(2x^2 - 1) = 0$$

$2x^2-1=0$

$x^2=\frac{1}{2}$ $\boxed{x=\frac{1}{\sqrt{2}}}$, $\boxed{x=-\frac{1}{\sqrt{2}}}$

are they both ~~in~~ inside the interval we care about? yes, so keep them

f' DNE → fraction DNE → denominator $=0$

$\sqrt{1-x^2}=0$ $1-x^2=0$ $x=1$, $x=-1$

they are the end points of interval, so keep them

NOT $f'(x)$

now compare $\boxed{f(x)}$ at $x=-1$, $x=-\frac{1}{\sqrt{2}}$, $x=\frac{1}{\sqrt{2}}$, $x=1$

$f(-1)=2(-1)^2+2\cos^{-1}(-1) = 2+2(\pi)=2+2\pi$ abs. max at $x=-1$

$f(-\frac{1}{\sqrt{2}})=2(-\frac{1}{\sqrt{2}})^2+2\cos^{-1}(-\frac{1}{\sqrt{2}})=1+2\left(\frac{3\pi}{4}\right)=1+\frac{3\pi}{2}$

$f(\frac{1}{\sqrt{2}})=2\left(\frac{1}{\sqrt{2}}\right)^2+2\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=1+2\left(\frac{\pi}{4}\right)=1+\frac{\pi}{2}$

$f(1)=2(1)^2+2\cos^{-1}(1)=2+2(0)=2$ abs. min at $x=1$