4.2 The Mean Value Theorem

Rolle's Theorem

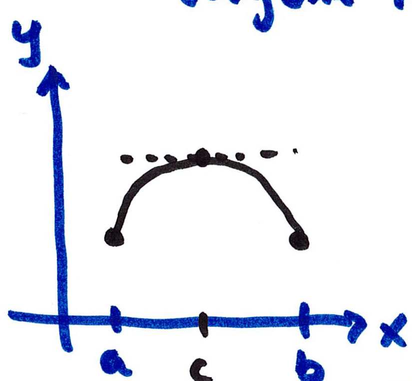
If $f(x)$ is

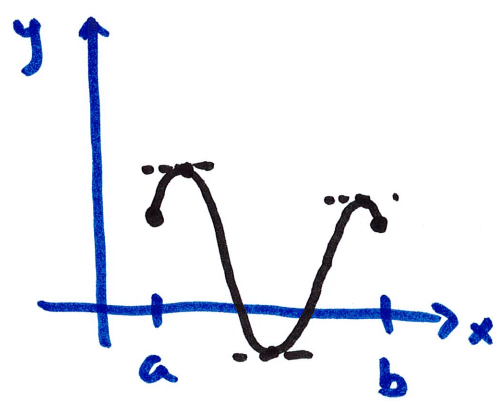
1) continuous on $[a, b]$ → no holes, no asymptotes, no gaps

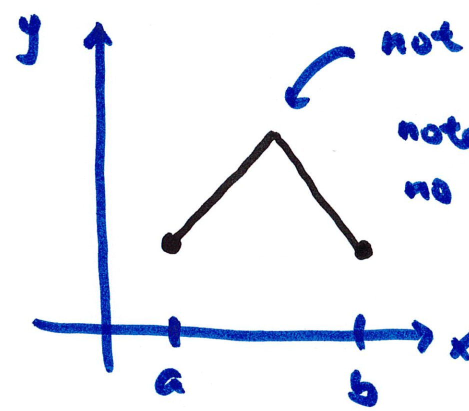
2) differentiable on (a, b) → smooth, no corners, no cusps

3) $f(a) = f(b)$ → same y value at ends of interval

then there is at least one value of x, call it c, $a < c < b$ where $f'(c) = 0$ → at least one place between ends where tangent line is horizontal

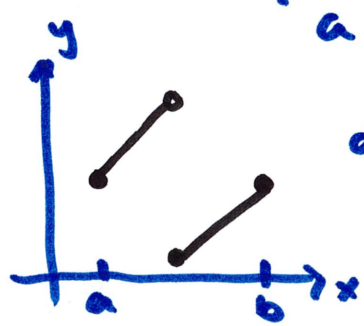






not differentiable

note there is no place where $f' = 0$



discontinuous so no place where $f' = 0$

example

$f(x) = e^{x^4}$ on $[-2, 2]$

verify Rolle's Theorem

$f(x)$ is exponential function so is continuous and differentiable everywhere, so first two requirements are met

$f(-2) = f(2)$?

$f(-2) = e^{(-2)^4} = e^{16}$ $f(2) = e^{(2)^4} = e^{16}$

so, 3rd requirement is met

Rolle's Theorem guarantees at least one place $-2 \le x \le 2$ where the tangent line is horizontal.

find it: $f'(x) = e^{x^4} \cdot 4x^3 = 4x^3 e^{x^4}$

$f' = 0 \rightarrow 4x^3 e^{x^4} = 0$ exponential $\neq 0$

$4x^3 = 0$, $\cancel{e^{x^4}} = 0$

$x = 0$

$f'(0) = 4(0)e^{(0)} = 0$ horiz. tangent line

Rolle's Theorem doesn't help us find where $f'=0$, but it assures us there is at least such location so we can spend the time and effort to find it.

(if Rolle's Theorem says there isn't one, then there is no point to look for it).

an extention to Rolle's Theorem is called

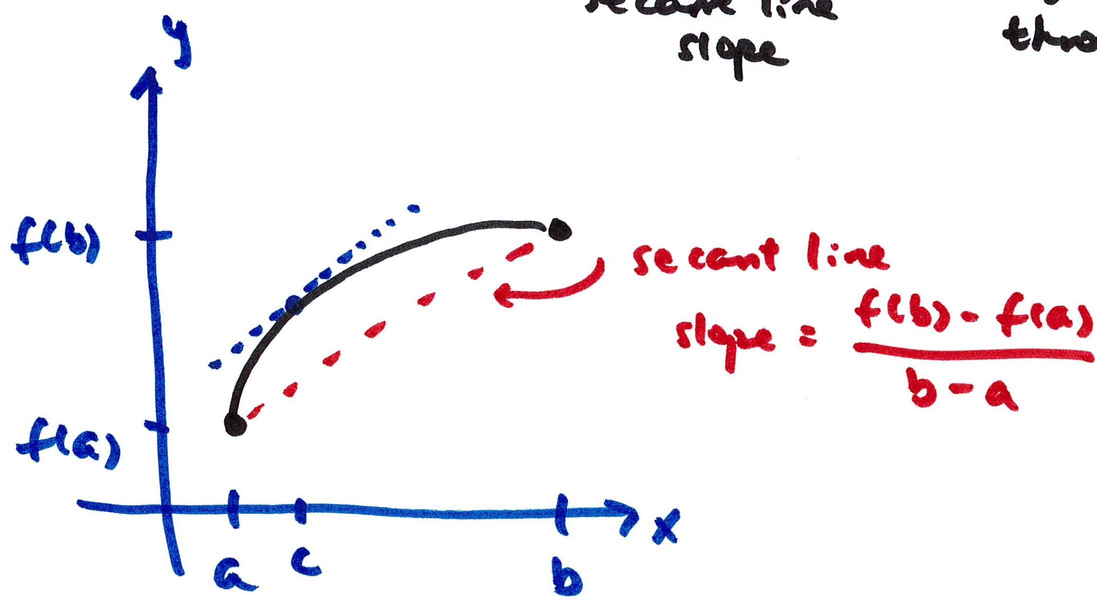
The Mean Value Theorem

If $f(x)$ is

1) continuous on $[a, b]$
2) differentiable on (a, b)

then there is at least one location c, $a < c \leq b$

where $f'(c) = \underbrace{\dfrac{f(b) - f(a)}{b - a}}_{\text{secant line slope}}$ → there is at one place where the tangent line slope is equal to the secant line slope through the two end points



example $f(x) = \cos x$ on $[0, \frac{\pi}{2}]$

$\cos x$ is continuous and differentiable everywhere so both requirements are met.

Slope of secant line through end points

$$\frac{f(\frac{\pi}{2}) - f(0)}{\frac{\pi}{2} - 0} = \frac{\cos(\frac{\pi}{2}) - \cos(0)}{\frac{\pi}{2}} = \frac{0 - 1}{\frac{\pi}{2}} = -\frac{2}{\pi}$$

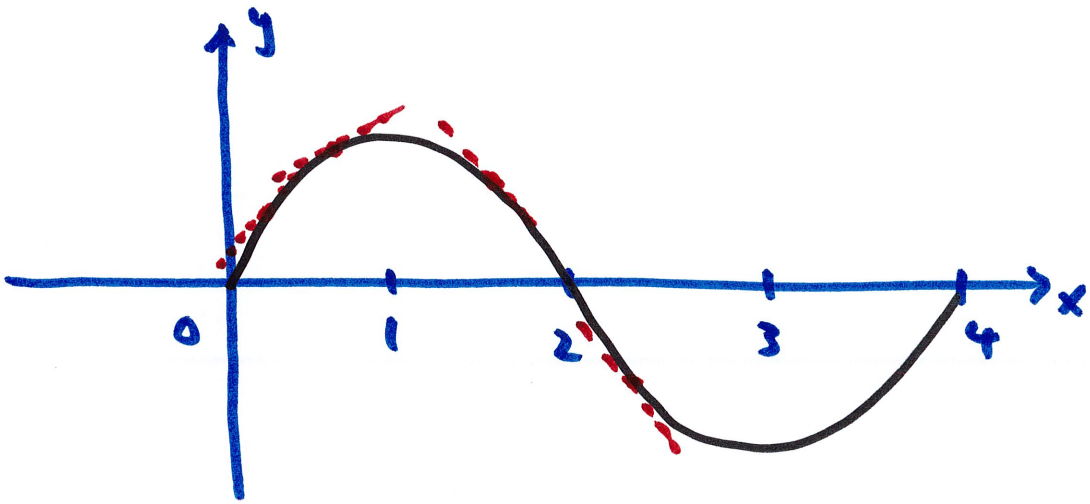
Mean Value Theorem says there is at least one c where $f'(c) = -\frac{2}{\pi}$

$$f'(x) = -\sin x = -\frac{2}{\pi}$$

$$\sin x = \frac{2}{\pi}$$

$$x = \sin^{-1}\left(\frac{2}{\pi}\right) \approx 0.7$$

4.3 What the Derivative Tells Us (part 1)



on interval $0 < x < 1$ (or $(0, 1)$) the function is increasing

notice on that interval the tangent line slope is always positive

on interval $1 < x < 3$ the function is decreasing

notice on that interval the tangent line slope is always negative

this tells us: if $f' > 0$ on some interval, f is increasing

if $f' < 0$ on some interval, f is decreasing there is a relative maximum at $x=1$ where $f'=0$

notice $f'>0$ before and $f'<0$ after $x=1$

there is a relative minimum at $x=3$ where $f'=0$

notice $f'<0$ before and $f'>0$ after $x=3$

this observation is the foundation of

First Derivative Test

If $x=c$ is a critical number ($f'(c)=0$ or $f'(c)$ DNE)
and if $f'(x)$ changes from $+$ to $-$ across $x=c$
then there is a relative max at $x=c$

if $f'(x)$ changes from $-$ to $+$ across $x=c$
then there is a relative min at $x=c$

if $f'(x)$ does not change sign, then there is
neither a relative max nor a relative min at $x=c$

example $f(x) = 3x^4 - 4x^3$

find critical numbers: $f' = 0$ or f' DNE

$f'(x) = 12x^3 - 12x^2$

$f' = 0 \rightarrow 12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0 \rightarrow \boxed{x = 0, \quad x = 1}$ critical numbers

f' DNE $\rightarrow$ ~~ever~~ never, since $f(x)$ is a polynomial, it is always differentiable

two critical numbers: $x = 0$, $x = 1$

draw a number line w/ these on it, then track sign of f'

Sign of $f'(x)$: - - - - - - 0 - - - - - - - - - 0 + + + + + + +

decreasing | $x=0$ | decreasing | $x=1$ | increasing

pick ANY # < 0 and check sign of f'

$f'(-1) = 12(-1)^3 - 12(-1)^2 < 0$

pick ANY # between 0 and 1 check sign of f'

$f'(\frac{1}{2}) = 12(\frac{1}{2})^3 - 12(\frac{1}{2})^2 < 0$

pick ANY # > 1 check sign of f'

$f'(2) = 12(2)^3 - 12(2)^2 > 0$

$f(x)$ decreasing on $(-\infty, 0)$ and $(0, 1)$

$f(x)$ increasing on $(1, \infty)$

f' changes from <u>$-$ to $+$</u> across $x = 1$

so there is a <u>relative min</u> at $x = 1$

f' does not change sign across $x = 0$

so there is neither a rel. max nor a rel. min at $x = 0$