4.3 What the Derivatives Tell us (part 2)

last time: if $f' > 0$ then f is increasing

if $f' < 0$ then f is decreasing

at $x = c$ where $f'(c) = 0$ or DNE

if f' changes from $+$ to $-$ $\rightarrow$ relative max at $x = c$

f' " " $-$ to $+$ $\rightarrow$ relative min at $x = c$

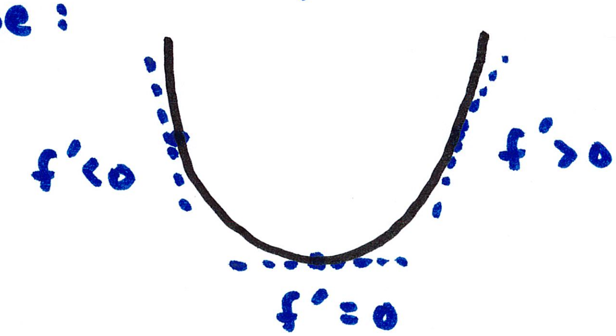
no sign change across $x = c$ $\rightarrow$ neither max/min at $x = c$

today: what does f'' tell us?

if $f'' > 0$, then $(f')' > 0$ $\rightarrow$ f' is increasing

slope of tangent line is increasing (from left to right)

this gives us this shape:

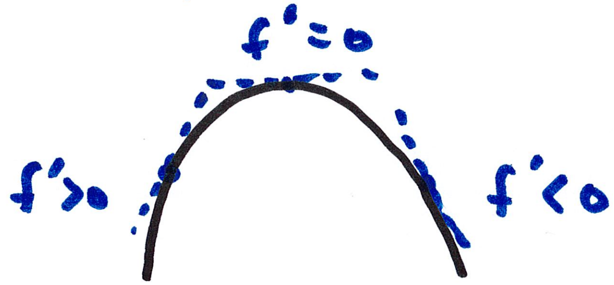


$\boxed{f'' > 0}$

we say the graph is $\boxed{\text{concave upward}}$

$f'' < 0$, then $(f')' < 0 \rightarrow f'$ is decreasing

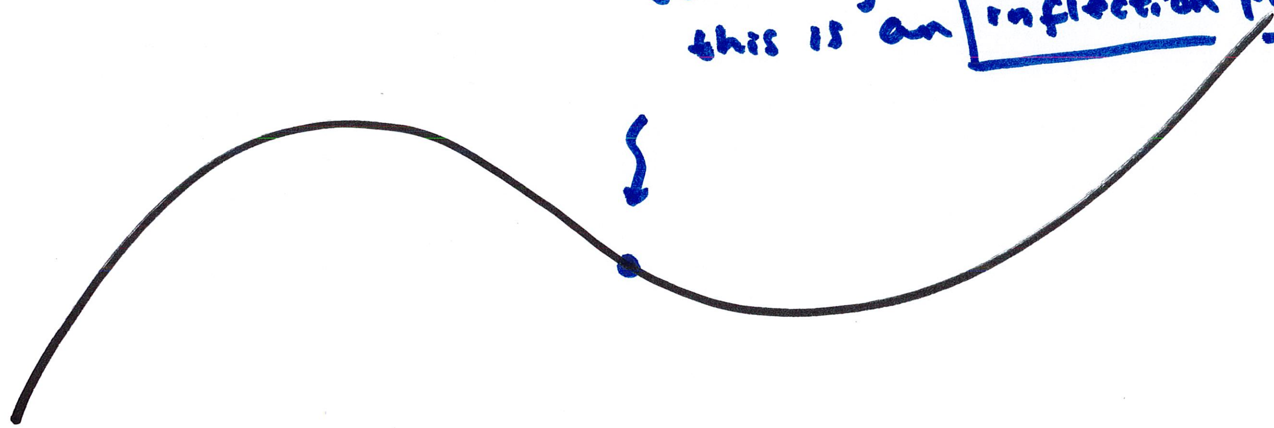
moving left to right, slope of tangent line gets smaller



$\boxed{f'' < 0}$

we say the graph is **concave downward**

concavity changes here
this is an **inflection point**



inflection points can occur at where
$f'' = 0$ or
f'' DNE

Similar to where critical points can occur ($f' = 0$ or DNE)
HOWEVER, ~~cut~~ inflection pts $\neq$ critical pts

concave down
$f'' < 0$

concave up
$f'' > 0$

<u>example</u> On what intervals is $f(x) = 3x^4 - 4x^3$ concave up/down, where are the inflection points?

need f''

$f'(x) = 12x^3 - 12x^2$

$f''(x) = 36x^2 - 24x$

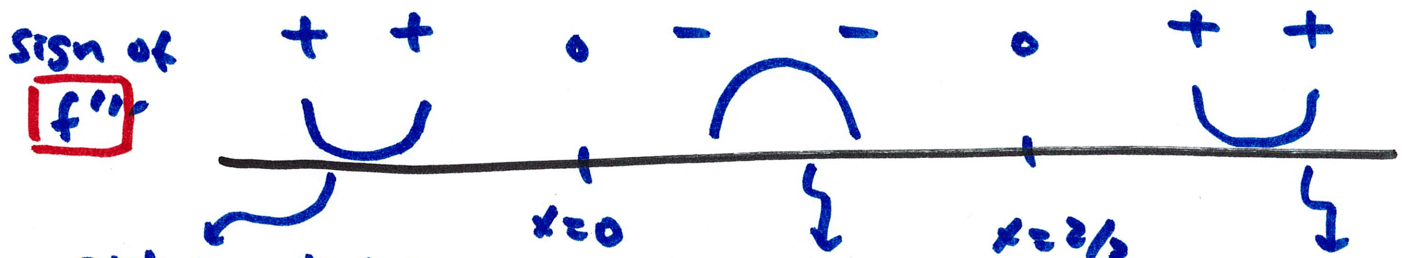
find where $f'' = 0$ or DNE (very much like finding critical pts)

$f'' = 0 \rightarrow 36x^2 - 24x = 0$

$12x(3x-2) = 0 \rightarrow x = 0, \quad x = 2/3$

f'' DNE $\rightarrow$ never because $f(x)$ is a polynomial

now make a sign chart for f''



pick any # < x=0
check sign of f''

$f''(-1) \geq 0$ (concave up)

$f''(\frac{1}{2}) < 0$
(concave down)

$f''(1) \geq 0$
(concave up)

concave up: $(-\infty, 0)$, $(2/3, \infty)$

Concave down: $(0, 2/3)$

f'' changes sign at $x=0$ and $x=2/3$

$y = f(2/3) = 3(2/3)^4 - 4(2/3)^3$

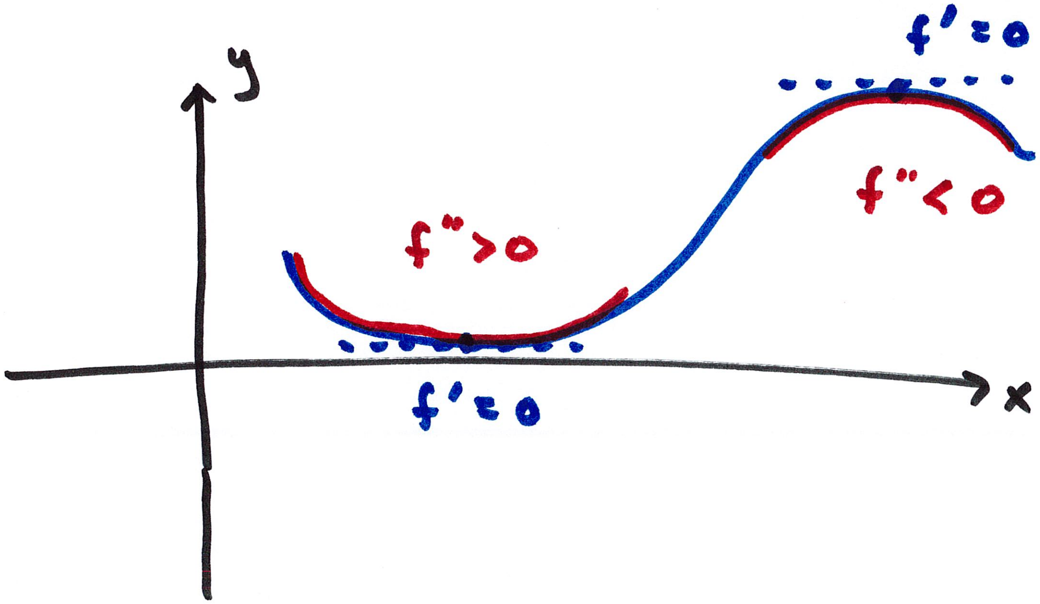
So, there are two inflection pts: $(0, 0)$, $(2/3, -16/27)$

x

$y = f(0)$

$= 3(0)^4 - 4(0)^3$

we can also use f'' to find relative max / min



relative max : $f' = 0$
$f'' < 0$

relative min : $f' = 0$
$f'' > 0$

<u>Second Derivative Test</u> (for finding relative max/min)

if $x = c$ is a critical number ($f' = 0$ or f' DNE)

and if <u>$f''(c) > 0$</u> then there is a <u>relative min</u> at $x = c$

<u>$f''(c) < 0$</u> than there is a <u>relative max</u> at $x = c$

<u>$f''(c) = 0$</u> then this test is <u>inconclusive</u>

($x = c$ could still be a relative max/min)

example $f(x) = e^x(x-1)$

again, need f''

product rule $f'(x) = e^x(1) + (x-1)e^x = e^x + xe^x - e^x = xe^x$

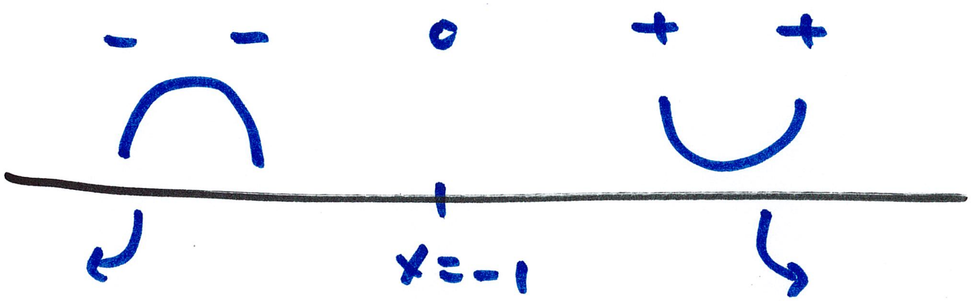
$f''(x) = xe^x + e^x(1) = xe^x + e^x = e^x(x+1)$

$f'' = 0 \rightarrow e^x = 0$ or $x+1=0$

$\downarrow$ never, since $e^x \neq 0$ $\boxed{x=-1}$

f'' DNE $\rightarrow$ never, since e^x and $x+1$ exist for all x

sign of $\boxed{f''}$



$f''(-2) = e^{-2}(-2+1) < 0$

concave up: $(-1, \infty)$

concave down: $(-\infty, -1)$

$f''(0) = e^0(0+1) > 0$

f'' changes sign at $x=-1$

so inflection pt @ $(-1, -2e^{-1})$

$\leftarrow f(-1) = e^{-1}(-1-1)$

example $f(x) = e^x(x-1)$ (previous example)

process to find max/min: find critical numbers first

$f'(x) = xe^x = 0 \rightarrow \boxed{x=0}$ → the only critical #

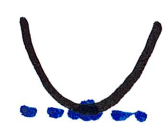
$f'(x) = xe^x$ DNE doesn't happen

now check sign of f'' at each critical number

$f''(x) = e^x(x+1)$

$f''(0) = e^0(0+1) > 0$

so, near this critical # $x=0$ the graph is concave up



So there is a relative minimum at $x=0$

<u>example</u> $f(x) = x^4 - 4x^3 + 1$

$f'(x) = 4x^3 - 12x^2$

find critical pts: $f' = 0 \rightarrow 4x^3 - 12x^2 = 0$

$$4x^2(x-3) = 0$$

$$\boxed{x = 0, \ x = 3}$$ two critical pts

$f''(x) = 12x^2 - 24x$

$f''(3) = 36 > 0 \rightarrow$ $\boxed{\text{relative min at } x = 3}$

$f''(0) = 0 \rightarrow$ inconclusive

we still need to find out what's going on

use first derivative test as back up

sign of $\boxed{f'}$

$- \quad - \quad 0 \quad - \quad - \quad 0$

$\searrow \qquad \searrow$

$x = 0 \qquad x = 3$

$f'(-1) < 0 \qquad f'(1) < 0$

no sign change across $x = 0$, so there is neither a rel. max nor a rel. min at $x = 0$