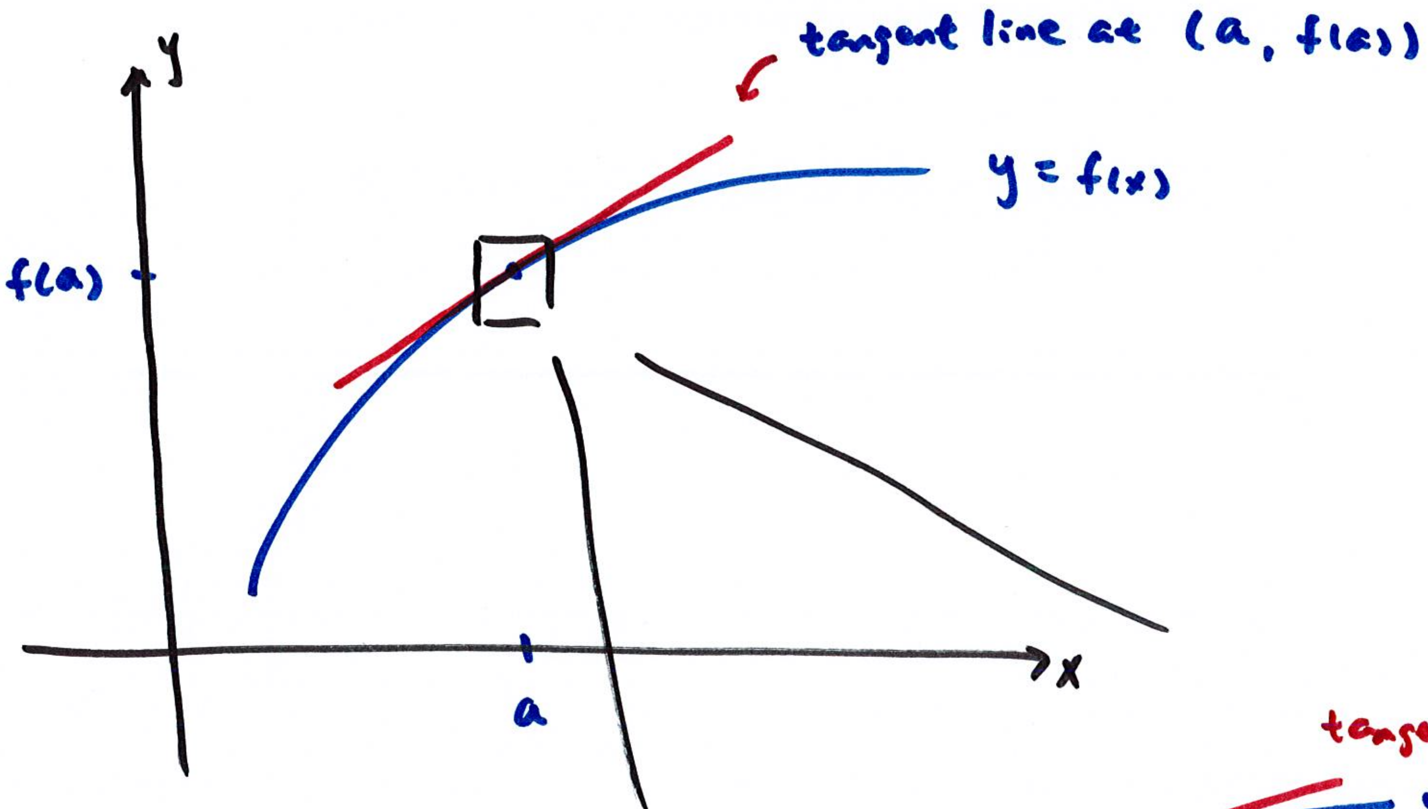
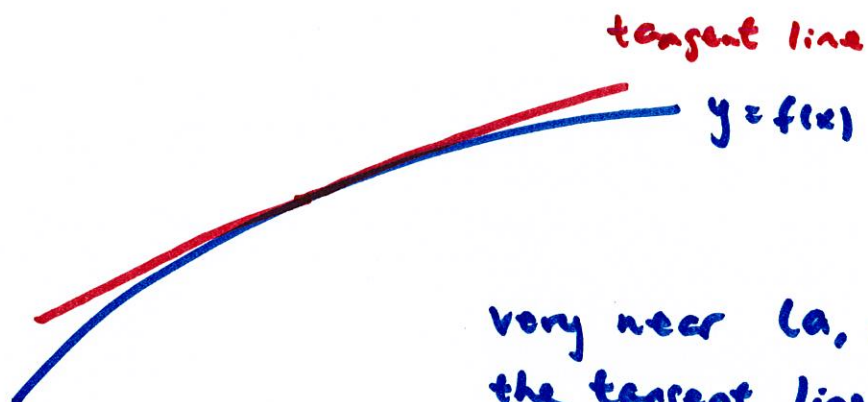
4.6 Linear Approximations and Differentials





very near (a, f(a)) the tangent line and the real curve f(x) are very close to each other if x is very close to a, then we can replace the function with its tangent line without losing much accuracy.

→ linear approximation

(approximate a function with its tangent line)

the tangent line approximation gets better the closer we are to $x = a$

let's write out the equation of the tangent line at $x = a$, $y = f(a)$

slope: $f'(a)$

point it goes through: $(a, f(a))$

equation in point-slope form: $y - f(a) = f'(a)(x - a)$

then: $\boxed{y = f(a) + f'(a)(x-a) \approx f(x) \text{ if } x \text{ is near } a}$

this is the linear approximation of $f(x)$.

$$\boxed{\text{linear approximation: } L(x) = f(a) + \underbrace{f'(a)(x-a)} \approx f(x)}$$

this is called the <u>differential</u>

it gives an estimate of the change in $y = f(x)$ as x changes from a.

<u>example</u> Find a linear approximation of $f(x) = \sin(x)$ near $x = 0$

$\sin(x)$ is hard to evaluate without a calculator

here, we will replace it with a line which is much easier to work with.

make a tangent line at $x = 0 \to a = 0$

$L(x) = f(a) + f'(a)(x-a)$

$f(a) = f(0) = \sin(0) = 0$

$f'(x) = \cos(x)$

$f'(a) = f'(0) = \cos(0) = 1$

$L(x) = 0 + (1)(x - 0)$

$\boxed{L(x) = x}$ this means that when x is close $a = 0$, x and $\sin(x)$ are very close to each other

for example, $\sin(\underbrace{0.001}_{x}) = 0.00099999983$

we see $\sin(x)$ and x are very close

this is another way to explain why $\lim_{x \to 0} \frac{\sin x}{x} = 1$

another example with $\sin(x)$: estimate $\sin(30°)$

change to randians!

$30° = \frac{\pi}{6}$

$\sin(x) \approx x$ so, we can estimate $\sin(30°) = \sin(\frac{\pi}{6}) \approx \frac{\pi}{6} = 0.52$

true value: $\sin(\frac{\pi}{6}) = \frac{1}{2} = 0.5$

pretty close, let's calculate the error and percentage error

error = $|\text{approximation} - \text{true}| = |0.52 - 0.5| = 0.02$

percentage error = $100 \cdot \dfrac{\text{error}}{\text{true}} = 100 \cdot \dfrac{0.02}{0.5} = 4$

so, there is a 4 % error using linear approximation.

the estimate is good, since we didn't move too far from $x=0$

what happens if we go too far?

try $\sin(90°) = \sin(\frac{\pi}{2}) = 1$ (true value)

linear approx: $\sin(x) \approx x$

so estimate of $\sin(\frac{\pi}{2})$ is $\frac{\pi}{2} = 1.57$

% error = $\dfrac{0.57}{1} \cdot 100 = 57\ \%$

example Use a linear approximation to estimate $\sqrt{108}$

we can look at this as $f(x) = \sqrt{x}$ with $x = 108$

we can use a linear approximation with an appropriate "a" to estimate this

choose "a" such that we know $f(a)$ easily and that "a" is not too far from $x = 108$

find the closest number to 108 whose square root we know

so, here, $a = 100$ is a good choice

$$L(x) = f(a) + f'(a)(x-a)$$

$$f(a) = f(100) = \sqrt{100} = 10$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$f'(a) = f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20} = 0.05$$

$$L(x) = 10 + 0.05(x-100) \approx \sqrt{x}$$ when x is near 100

so, $\sqrt{108} \approx L(108) = 10 + 0.05(108-100) = 10.4$

how good is it? true value of $\sqrt{108}$ from calculator is 10.3923

<u>example</u> Use a linear approximation to estimate $\ln(3)$

$f(x) = \ln(x)$ with $x = 3$

now find an "a" such that $f(a)$ is known or easy to calculate

e is a good choice: $e \approx 2.7$ close to 3 and $\ln(e) = 1$

$L(x) = f(a) + f'(a)(x-a)$

$f(a) = f(e) = \ln(e) = 1$

$f'(x) = \frac{1}{x}$

$f'(e) = \frac{1}{e}$

$L(x) = 1 + \frac{1}{e}(x-e) \approx \ln(x)$ if x is close to e

$\ln(3) \approx L(3) = 1 + \frac{1}{e}(3 - e) = 1 + \frac{3}{e} - \frac{e}{e} = \frac{3}{e} \approx 1.1$

from a calculator, true value of $\ln(3)$ is 1.0983

once again, pretty close because 3 is close to e