4.9 Antiderivatives

NOT ON EXAM 3

antiderivative - reverse of derivative

→ given a function $f(x)$ find $F(x)$ such that $F'(x) = f(x)$

for example, $F(x) = -\cos x$ is the antiderivative of $f(x) = \sin x$

because $F'(x) = -(-\sin x) = \sin x = f(x)$

Antiderivative of x^n

an antiderivative of x^n is $\dfrac{x^{n+1}}{n+1}$ if $n \neq -1$

why? because $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = \dfrac{1}{n+1}(n+1)\,x^n = x^n$

for example, an antiderivative of x^2 is $\dfrac{x^{2+1}}{2+1} = \dfrac{x^3}{3} = \dfrac{1}{3}x^3$

check: $\dfrac{d}{dx}\left(\dfrac{1}{3}x^3\right) = \dfrac{1}{3}\cdot 3x^2 = x^2$

but notice $\frac{1}{3}x^3+1$ is also an antiderivative of x^2

because $\frac{d}{dx}\left(\frac{1}{3}x^3+1\right)=x^2$

in fact, so is $\underbrace{\frac{1}{3}x^3+5,\ \frac{1}{3}x^3+\pi,\ \frac{1}{3}x^3+e^{\pi}-10,\ \text{etc}}$

all are possible antiderivatives of x^2

they differ only by a constant

to represent ALL of those, we write $\frac{1}{3}x^3 + C$

some constant (constant of integration)

$$\boxed{\text{so, the most general antiderivative of } x^n \text{ is} \quad \frac{x^{n+1}}{n+1}+C, \quad n \neq -1}$$

<u>example</u> Find the antiderivative of $f(x) = -\frac{10}{x^{12}}$

(what function has this as the derivative?)

rewrite: $f(x) = \underbrace{-10}_{\substack{\text{constant}\\\text{multiple}\\\text{(doesn't participate}\\\text{just like in}\\\text{differentiation)}}} \underbrace{x^{-12}}_{x^n \text{ part}}$

$$F(x) = -10\left(\frac{x^{-12+1}}{-12+1}\right) + C$$

$$= -10\left(\frac{x^{-11}}{-11}\right) + C = \boxed{\frac{10}{11}x^{-11} + C}$$

the <u>indefinite integral</u> is used to denote the process of finding the antiderivative

$$\int f(x)\,dx = F(x) + C$$

integral sign (stretched out "S")

part of the indefinite integral symbol

so, from last example, we know $\int -\frac{10}{x^{12}}\,dx = \frac{10}{11}x^{-11} + C$

since finding antiderivative is closely related to differentiation, many rules and properties from derivative carry over.

for example, we can deal with terms ~~of expr~~ separated by + or − individually example

$$\int \left(\frac{2}{\sqrt{x}} + 2\sqrt{x} \right) dx$$

rewrite: $\int (\underbrace{2x^{-1/2} + 2x^{1/2}}_{}) dx$

+ or − separate them

deal with them one at a time

$$= 2\left(\frac{x^{-1/2+1}}{-1/2+1} \right) + 2\left(\frac{x^{1/2+1}}{1/2+1} \right) + C$$

just need one + C

$$= 2\left(\frac{x^{1/2}}{1/2} \right) + 2\left(\frac{x^{3/2}}{3/2} \right) + C$$

$$= \boxed{4x^{1/2} + \frac{4}{3}x^{3/2} + C}$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad \text{if } n \neq -1$$

what happens if $n = -1$?

$$\int x^{-1} \, dx = \int \underbrace{\frac{1}{x}}_{} \, dx$$

what function has this as derivative? $\rightarrow \ln x$

so, $$\boxed{\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln |x| + C}$$

need absolute value since x could be negative in $\int \frac{1}{x} \, dx$

with trig functions, think of the differentiation rules in reverse

$$\int \cos x \, dx = \sin x + C \qquad \text{because } \frac{d}{dx}(\sin x + C) = \cos x$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

and so on...

easy one: $\int e^x \, dx = e^x + C$

<u>example</u>

$$\underbrace{\int \left(\sin x + \sec x \tan x + \frac{2}{x} - \frac{10}{x^{1/3}} \right) dx}_{\text{separated by } +/- \text{ so one at a time}}$$

$$= \int \left(\sin x + \sec x \tan x + 2x^{-1} - 10x^{-1/3} \right) dx$$

$$= -\cos x + \sec x + 2\ln|x| - 10 \cdot \frac{x^{-1/3+1}}{-1/3+1} + C$$

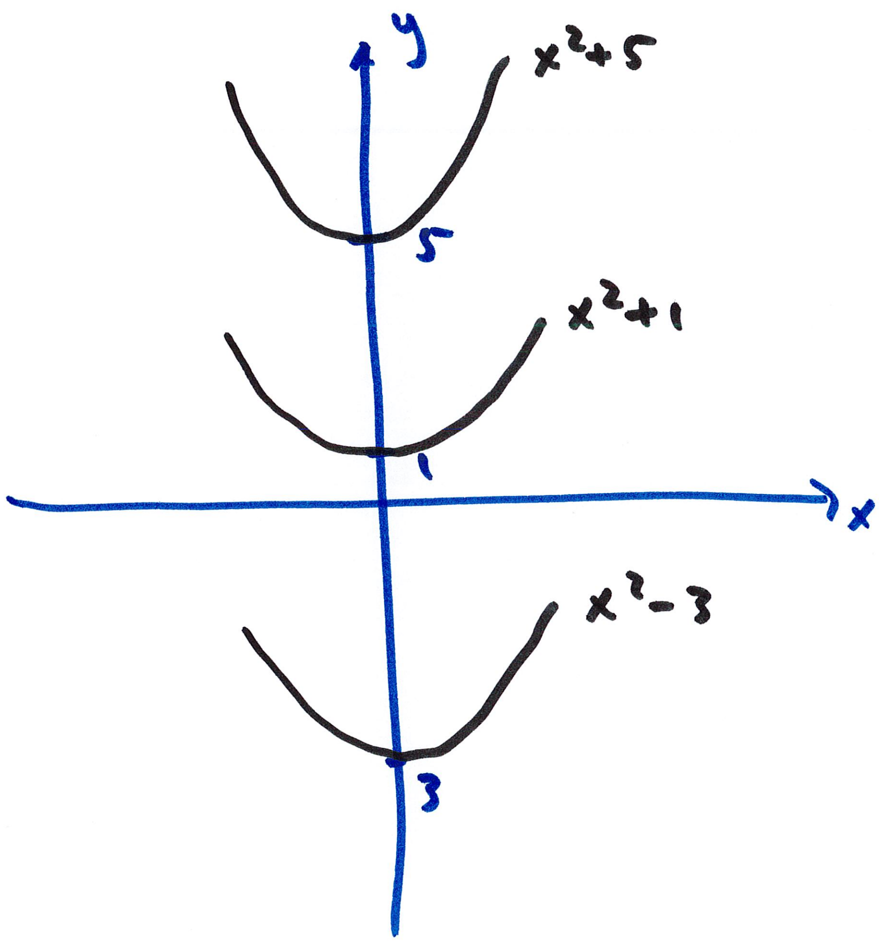
$$= -\cos x + \sec x + 2\ln|x| - 10 \cdot \frac{x^{2/3}}{2/3} + C$$

$$= \boxed{-\cos x + \sec x + 2\ln|x| - 15x^{2/3} + C}$$

to find the value of "c" we need to know one point on the antiderivative

for example, $\int 2x\,dx = \underbrace{x^2 + c}$

parabola with vertex at $(0, c)$

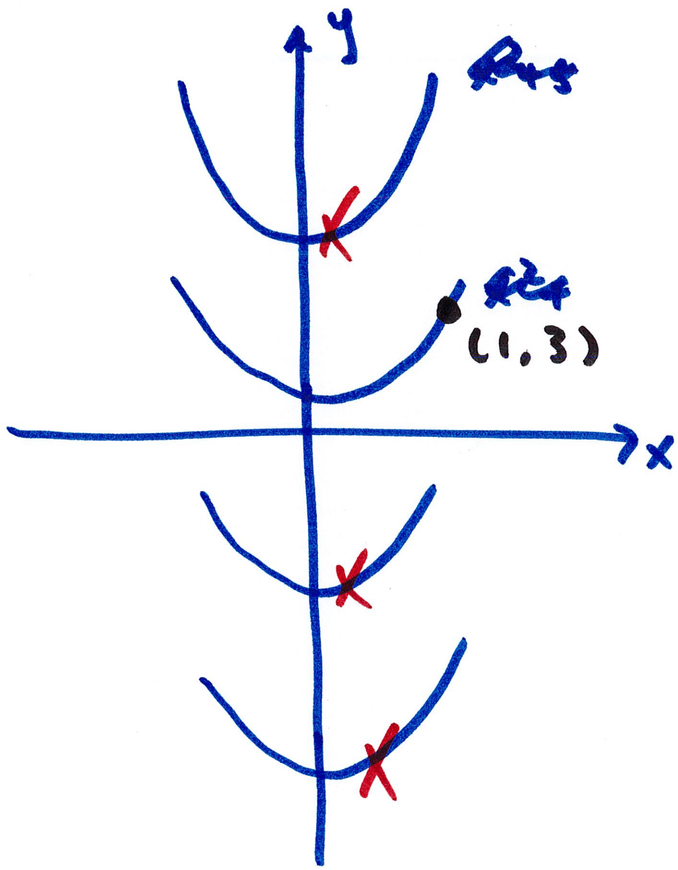


if we know one point that $F(x) = x^2 + c$ passes through, then we know which one it is

<u>example</u> $f(x) = 2x$

find $F(x)$ such that $F'(x) = f(x)$ and $\underbrace{F(1) = 3}_{(1,3) \text{ is on } F(x)}$

$$F(x) = \int 2x \, dx = x^2 + C$$



$F(x) = x^2 + C$

since $(1, 3)$ is on $F(x)$

sub in $x = 1$, $F(x) = 3$ to find C

$3 = 1^2 + C \qquad C = 2$

so, $\boxed{F(x) = x^2 + 2}$

we handle higher-order derivatives the same way, but each order of derivative introduces one constant of integration

<u>example</u> $F''(x) = \sin x$ find $F(x)$

$$F'(x) = \int \sin x \, dx = -\cos x + C$$

$$F(x) = \int (-\cos x + C) \, dx$$

second constant of integration

$$= -\sin x + Cx + D$$

$$\boxed{F(x) = -\sin x + Cx + D}$$

to find C and D, we need one point on $F(x)$ and one point on $F'(x)$

for example, if we know (for last example) $\underbrace{F'(0)=3}$, $\underbrace{F(0)=4}$

from last page, $F'(x) = -\cos x + C$

$(0,3)$ on F'

plug in $x=0$, $F'=3$

$3 = -\cos(0) + C$

$= -1 + C \rightarrow \boxed{C=4}$

from last page, $F(x) = -\sin x + Cx + D$

we now know $C=4$

$F(x) = -\sin x + 4x + D$

$(0,4)$ on F

plug in $x=0$, $F=4$

$4 = \underline{-\sin(0) + 4(0)} + D = D$ $\boxed{D=4}$

so, $\boxed{F(x) = -\sin x + 4x + 4}$

MA 16100
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