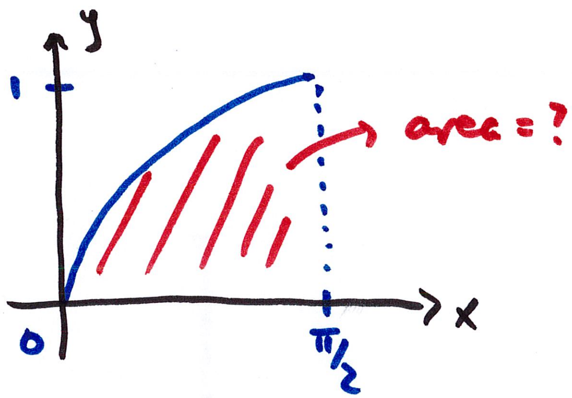
5.1 Approximating Areas under Curves

NOT on exam 3

How to find area under $f(x)$ $(f(x) > 0)$ on $a \le x \le b$

for example, what is the area under $f(x) = \sin x$ on $[0, \pi/2]$?



not an elementary shape (triangle, rectangle, etc) so we cannot use geometry

one way to estimate the area is to use a Riemann Sum

→ divide region into rectangles and sum the areas

$f(x) = \sin x$ on $[0, \pi/2]$ using a Riemann Sum

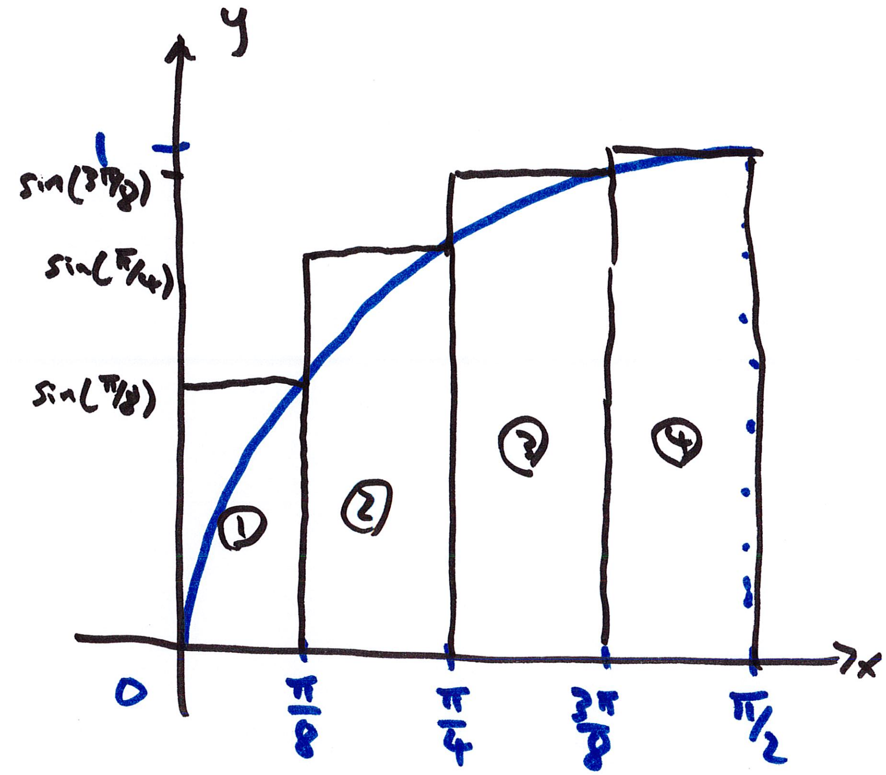
First, decide how many rectangles to use

as an example, let's use 4

so divide $[0, \pi/2]$ into 4 parts

this gives us the grid points

$0, \pi/8, \pi/4, 3\pi/8, \pi/2$



Then we decide where on each subinterval to sample the rectangle weight/height

typical choices: Right endpoint
Left endpoint
Midpoint

here, as an example, let's use Right endpoints

right endpoint

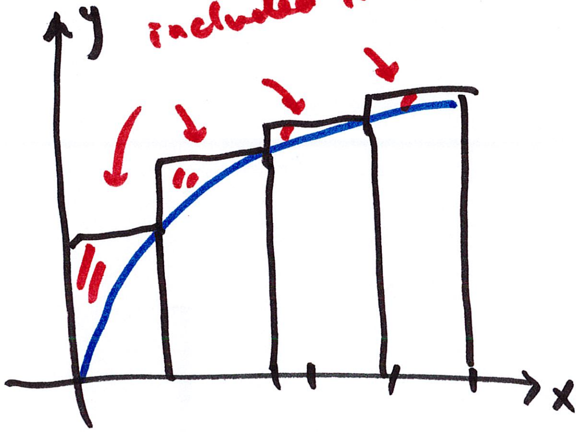
$$R_4 = \underbrace{\left(\frac{\pi}{8}\right)}_{\text{width of } 1} \underbrace{\sin\left(\frac{\pi}{8}\right)}_{\text{height of } 1} + \underbrace{\left(\frac{\pi}{8}\right)}_{\text{width of } 2} \underbrace{\sin\left(\frac{\pi}{4}\right)}_{\text{height of } 2} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{3\pi}{8}\right)}_{3} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{2}\right)}_{4} = \boxed{1.183}$$

four rectangles

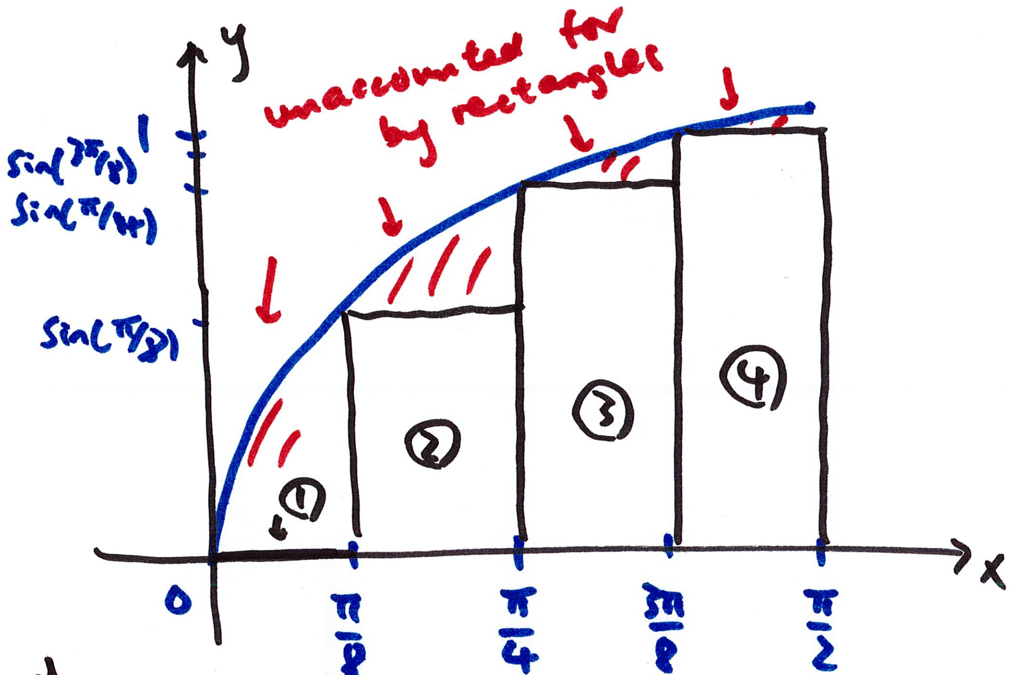
so we estimate area under $f(x) = \sin x$ on $[0, \pi/2]$ to be around 1.183

we know this is an <u>over</u> estimation because each rectangle has part sticking out over the curve

included in the 1.183 but shouldn't be



now let's try the left endpoints, still use 4 rectangles



note 1 has zero height

left end

4 rectangles

$$L_4 = \underbrace{\left(\frac{\pi}{8}\right)}_{\text{width of } 1} \underbrace{\sin(0)}_{\text{height of } 1} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}\right)}_{2} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{4}\right)}_{3} + \underbrace{\left(\frac{\pi}{8}\right)\sin\left(\frac{3\pi}{8}\right)}_{4} = \boxed{0.791}$$

so, 0.791 must be an <u>under</u> estimate since there are regions that the rectangles cannot cover now we can confidently say, whatever the true area under $f(x) = \sin(x)$ on $[0, \frac{\pi}{2}]$ is, it must be no smaller than 0.791 and no greater than 1.183

$$0.791 \leq \text{true area} \leq 1.183$$

if we used the midpoint with 4 rectangles, we would get an estimate of 1.006

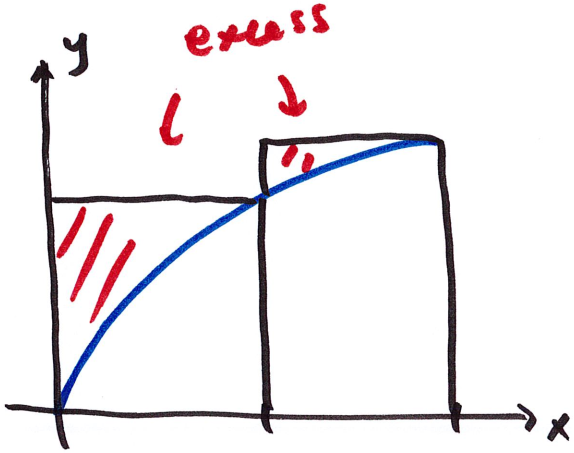
in comparison, the true area (we'll learn how to find it later in Ch. 4) is exactly 1.

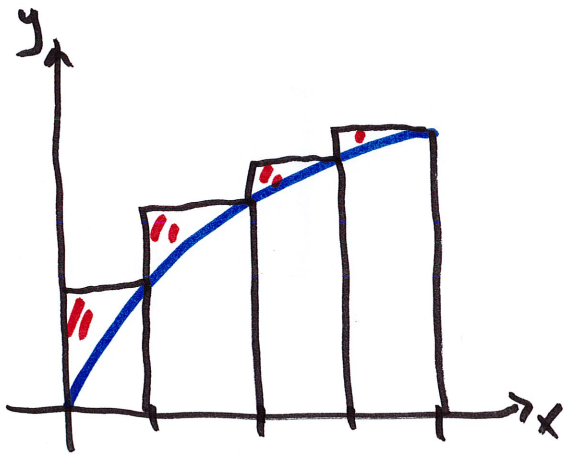
if $f(x)$ is an increasing function on (a, b), then using Right endpoints results in an over estimate

if $f(x)$ is a decreasing function on (a, b), then using Left endpoints results in an under estimate

(the opposite is true if $f(x)$ is decreasing)

whatever endpoint we use (R, L, M), the more rectangles we use the better the estimate why ?





we are including less of the excess by using more rectangles

In general, regardless of the choice of endpoints, we can write the approximation as

Δx is the width of each rectangle

$$\Delta x = \frac{b-a}{n}$$

$$A \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

Sample points (L, R, M)

upper limit

$$= \sum_{\underbrace{k=1}_{\text{lower limit}}}^{n} f(x_k)\Delta x$$

Summation (sigma) notation means we add up all terms starting at $k=1$ and ending at $k=n$ $(k = 1, 2, 3, \ldots, n)$

for example, $\displaystyle\sum_{k=1}^{5} k^2 = \underbrace{(1)^2}_{k=1} + \underbrace{(2)^2}_{k=2} + \underbrace{(3)^2}_{k=3} + \underbrace{(4)^2}_{k=4} + \underbrace{(5)^2}_{k=5}$

upper limit
stop

$= 1 + 4 + 9 + 16 + 25 = 55$

we also need to be able to write out sigma notation given the sum of terms

$$\cancel{\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11}}$$ ~~in sigma notation~~

$\cancel{k=1 \quad k=2 \quad k=3 \quad k=4 \quad k=5}$

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$ in sigma notation

$k=1 \quad k=2 \quad k=3 \quad k=4 \quad k=5$

lower limit is 1

upper limit is 5

relationship between k and each term

numerator is 1 regardless of k

denominator is 2 greater than k

$$= \sum_{k=1}^{5} \frac{1}{k+2}$$