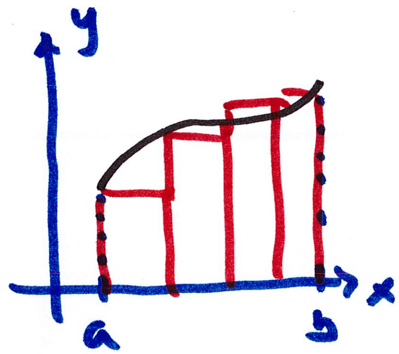
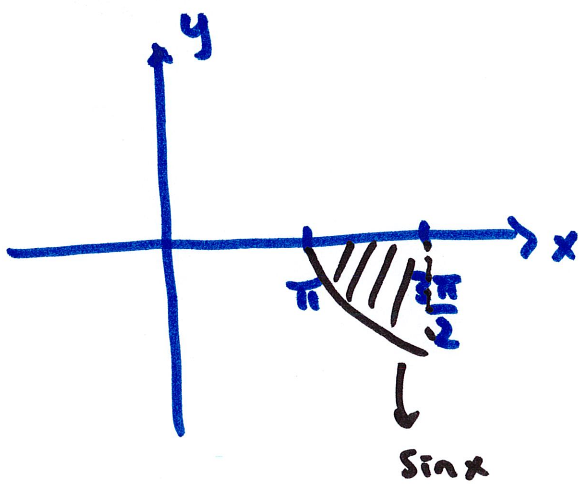
5.2 Definite Integrals

last time: if $f(x) \geq 0$, then the area underneath $f(x)$ from $x=a$ to $x=b$ could be approximated by a Riemann Sum



what if $f(x) < 0$?

for example, $f(x) = \sin x$ on $[\pi, \frac{3\pi}{2}]$

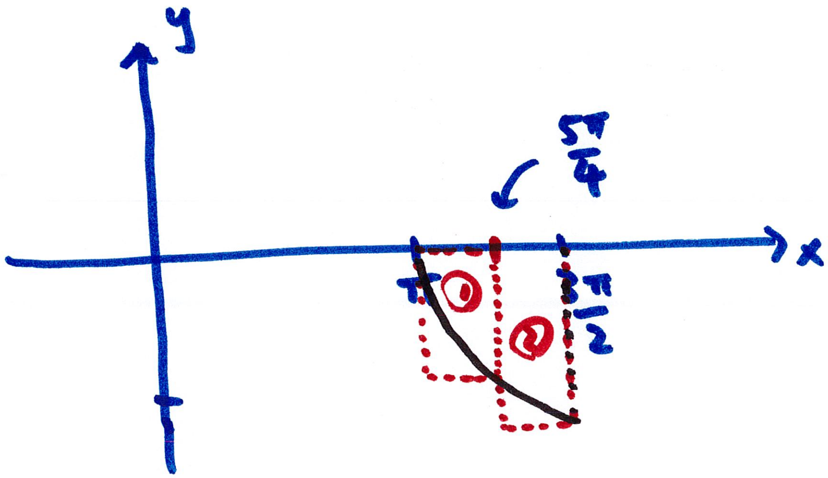


can't talk about area <u>under</u> the curve (it goes to $-\infty$)

but we can still calculate the area between the curve and x-axis (see shaded region on left)

example $f(x) = \sin x$ on $[\pi, \frac{3\pi}{2}]$

right end point, $n = 2$ rectangles



right end of first subinterval : $x = \frac{5\pi}{4}$

right end of second subinterval : $x = \frac{3\pi}{2}$

width of each rectangle is $\frac{b-a}{n} = \frac{3\pi/2 - \pi}{2} = \frac{\pi}{4}$

$$R_2 = \underbrace{\left(\frac{\pi}{4}\right)}_{\text{width of 1}} \underbrace{\sin\left(\frac{5\pi}{4}\right)}_{\text{height of 1}} + \left(\frac{\pi}{4}\right)\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{2}$$

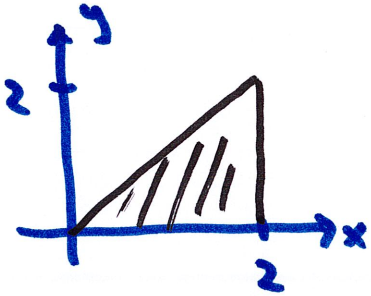
$$= \left(\frac{\pi}{4}\right)\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{\pi}{4}\right)(-1) = \boxed{-1.341}$$

note the "height" is negative for region below x-axis

notice if $f(x) < 0$ the area between the curve and x-axis is negative

Approximation of the area improves as more rectangles are used.

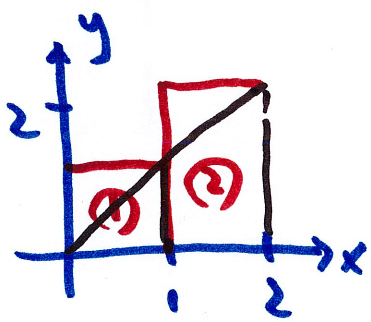
example Area under $y = x$ on $[0, 2]$



note this is a ~~rect~~ triangle so we know the true area $= \frac{1}{2}(2)(2) = 2$

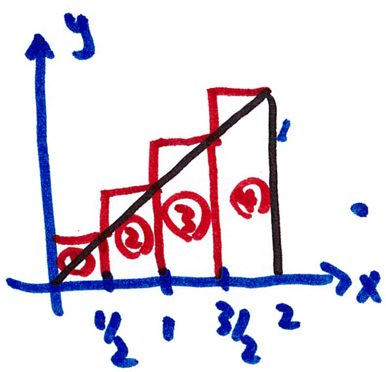
now we'll see how more rectangles can improve approximation

start w/ $n = 2$, right end point



$$R_2 = \underbrace{(1)(1)}_{1} + \underbrace{(1)(2)}_{2} = 3$$

try R_4



$$R_4 = \underbrace{\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)}_{1} + \underbrace{\left(\tfrac{1}{2}\right)(1)}_{2} + \underbrace{\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)}_{3} + \underbrace{\left(\tfrac{1}{2}\right)(2)}_{4}$$

$$= 2.5$$

$$R_8 = \left(\tfrac{1}{4}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right) + \left(\tfrac{1}{4}\right)(1) + \left(\tfrac{1}{4}\right)\left(\tfrac{5}{4}\right) + \left(\tfrac{1}{4}\right)\left(\tfrac{3}{2}\right) + \left(\tfrac{1}{4}\right)\left(\tfrac{7}{4}\right) + \left(\tfrac{1}{4}\right)(2) = 2.25$$

so, as we can see, as $n \to \infty$, the approx $\to$ true value

and, as $n \to \infty$, the sample point we choose to use (right/left/mid point) becomes irrelevant.

$\to$ they all lead to the same true value as $n \to \infty$

the approximation can be expressed as

$$\sum_{K=1}^{n} f(x_K)\,\Delta_K$$

sample point
(L/R/M)

width of rectangle
$\Delta_K = \dfrac{b-a}{n}$

it becomes <u>exact</u> as $n \to \infty$

so, $\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta k$ = true value of the area

this cumbersome expression can be neatly expressed as

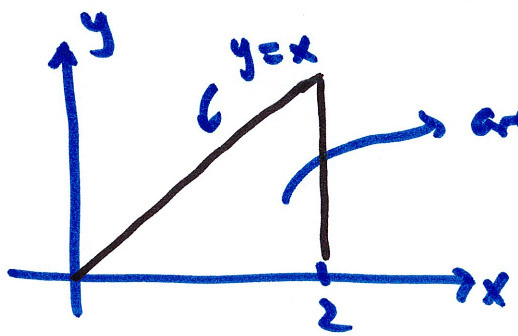
a <u>Definite Integral</u>

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta k = \int_a^b f(x)\,dx}$$

this represents the area between the curve $f(x)$ and the x-axis from $x=a$ to $x=b$

for example, $\int_0^2 x\,dx$ means the area between $y=x$

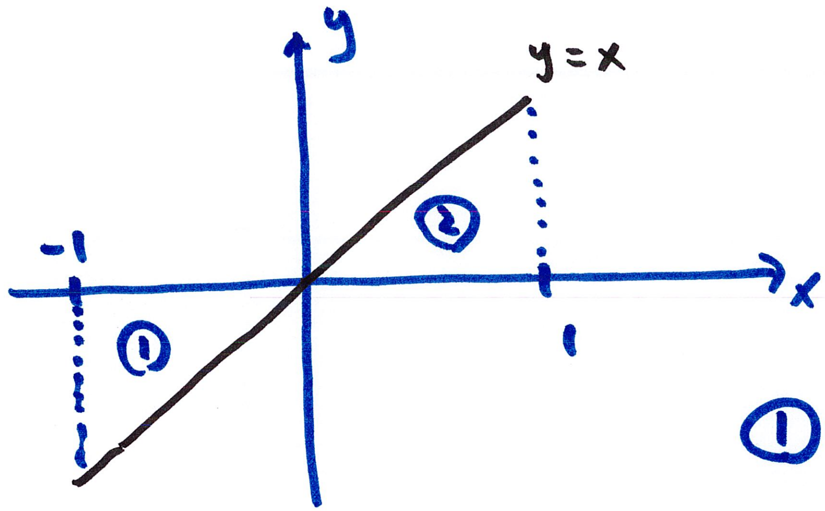
from $x=0$ to $x=2$



area $= \int_0^2 x\,dx$

<u>example</u> Interpret $\int_{-1}^{1} x\,dx$ in terms of areas and use them to evaluate

this definite integral is the area between $y = x$ and x-axis from $x = -1$ to $x = 1$



notice $\int_{-1}^{1} x\,dx$ = area of ① + area of ②

both ① and ② are triangles

① : $-\frac{1}{2}(1)(1) = -\frac{1}{2}$

below x-axis

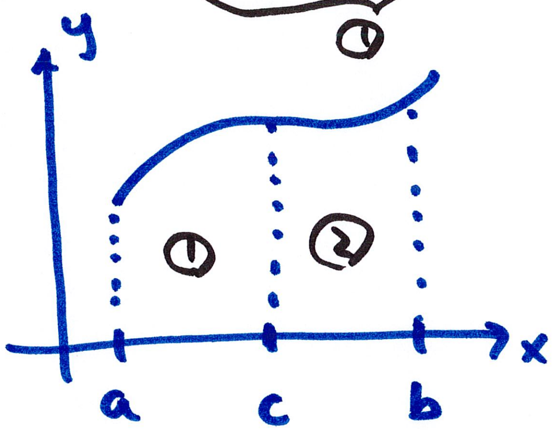
② : $\frac{1}{2}(1)(1) = \frac{1}{2}$

so, we see $\int_{-1}^{1} x\,dx = \underbrace{-\frac{1}{2}}_{①} + \underbrace{\frac{1}{2}}_{②} = 0$

also notice $\displaystyle\int_{-1}^{1} x\,dx = \underbrace{\int_{-1}^{0} x\,dx}_{1} + \underbrace{\int_{0}^{1} x\,dx}_{2}$

this can be generalized into a useful property

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{1} + \underbrace{\int_c^b f(x)\,dx}_{2}$$



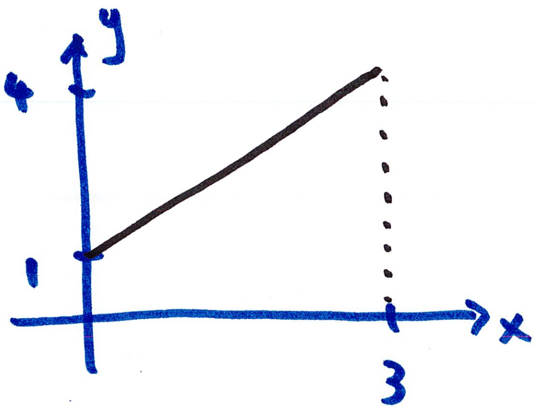
the whole region: $\displaystyle\int_a^b f(x)\,dx$

other useful properties: $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

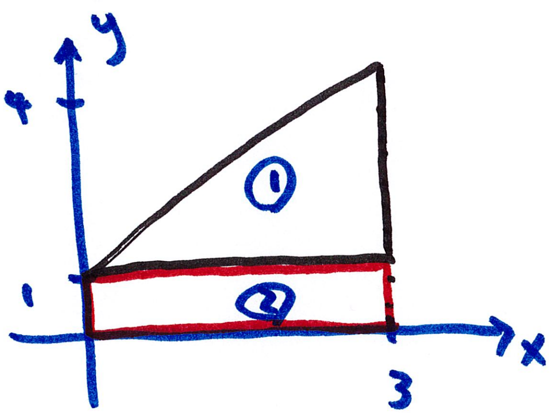
$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

example $\displaystyle\int_0^3 (x+1)\,dx$

this is the area between $y=x+1$ and x-axis from $x=0$ to $x=3$



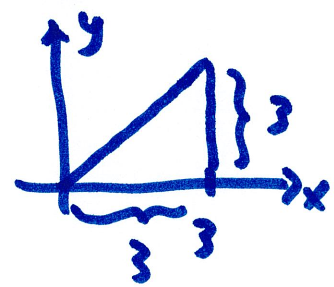
but notice we can chop it up into two pieces



1 is a triangle base 3 height 3

which is the same as

$\displaystyle\int_0^3 x\,dx$



2 is a rectangle length 3

width 1

same as $\displaystyle\int_0^3 1\,dx$

the property says $\int_0^3 (x+1)\,dx = \underbrace{\int_0^3 x\,dx}_{\text{triangle part}} + \underbrace{\int_0^3 1\,dx}_{\text{rectangle part}}$

this demonsterates why $\int_a^b [f(x)+g(x)]dx$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

is true

a couple other useful properties

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b k \cdot f(x)\,dx = k \int_a^b f(x)\,dx$$

constant

$$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

can switch places of a and b by making the integral negative