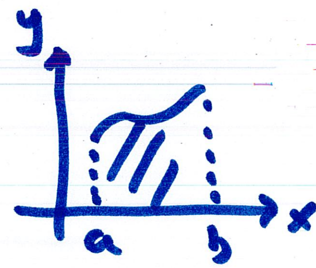
5.3 The Fundamental Theorem of Calculus

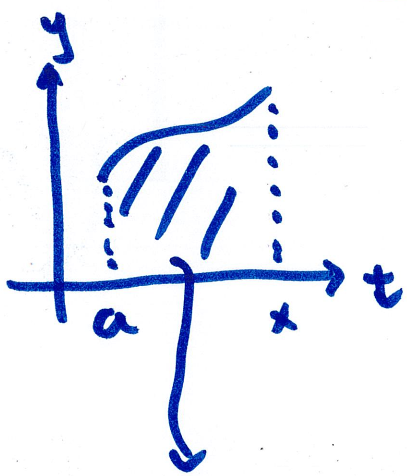
last time: $\int_a^b f(x)\,dx$ represents area between $f(x)$ and x-axis

from $x=a$ to $x=b$



now let's look at an area function $\quad g(x) = \int_a^x f(t)\,dt$

t is called a "dummy variable" we don't want to reuse x



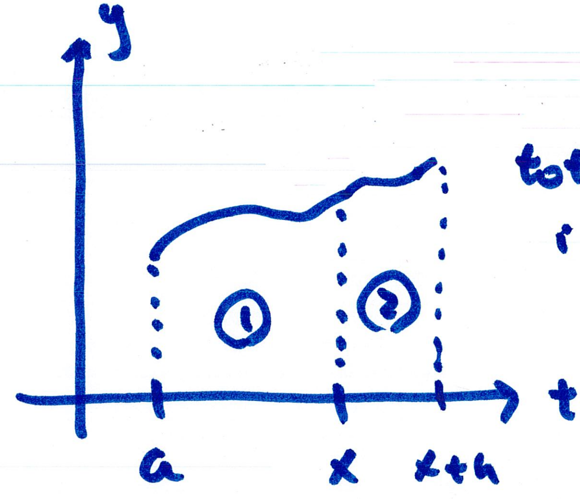
x can move around

area as function of x

$$\int_a^x f(t)\,dt = g(x)$$

$g(x) = \int_a^x f(t)\,dt$ what is $g'(x)$? what is the rate of change of the accumulated area?

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$



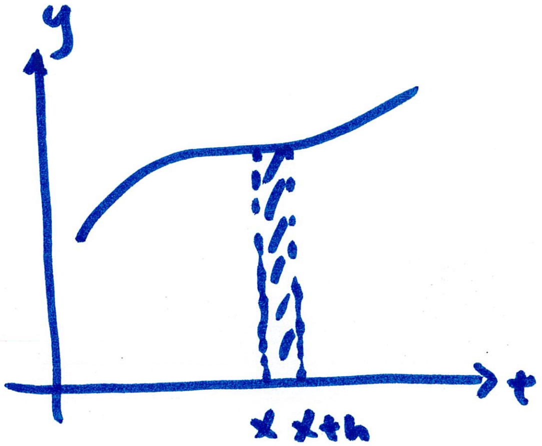
total area is $g(x+h)$

$$g(x) = \int_a^x f(t)\,dt \quad ①$$

$$g(x+h) = \int_a^{x+h} f(t)\,dt$$

$$g(x+h) - g(x) = \underbrace{\int_a^{x+h} f(t)\,dt}_{\text{total}} - \underbrace{\int_a^x f(t)\,dt}_{①} = ②$$

when $h \to 0$, ② becomes approximately a rectangle



height is $f(x)$

width is $x+h-x=h$

area $= f(x)\,h$

which is numerator of

$$g'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

so, $g'(x) = \lim_{h\to 0} \frac{f(x)\cdot h}{h} = \lim_{h\to 0} f(x) = f(x)$

this gives us

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

This is called the

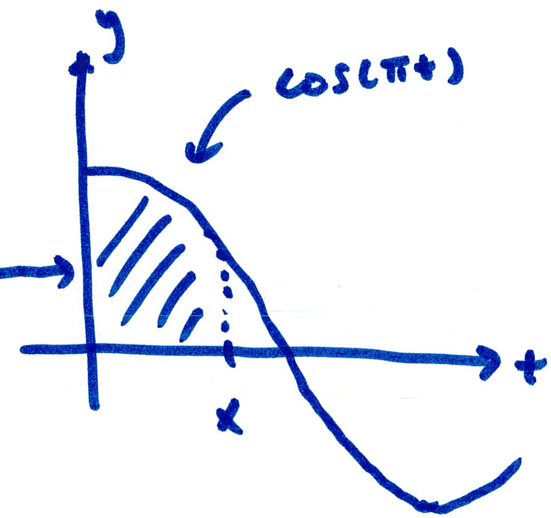
<u>Fundamental Theorem of Calculus</u>
<u>(part 1)</u>

(FTC 1)

it says the rate of change of accumulated area is equal to the function bounding above example

$$g(x) = \int_0^x \cos(\pi t)\,dt$$

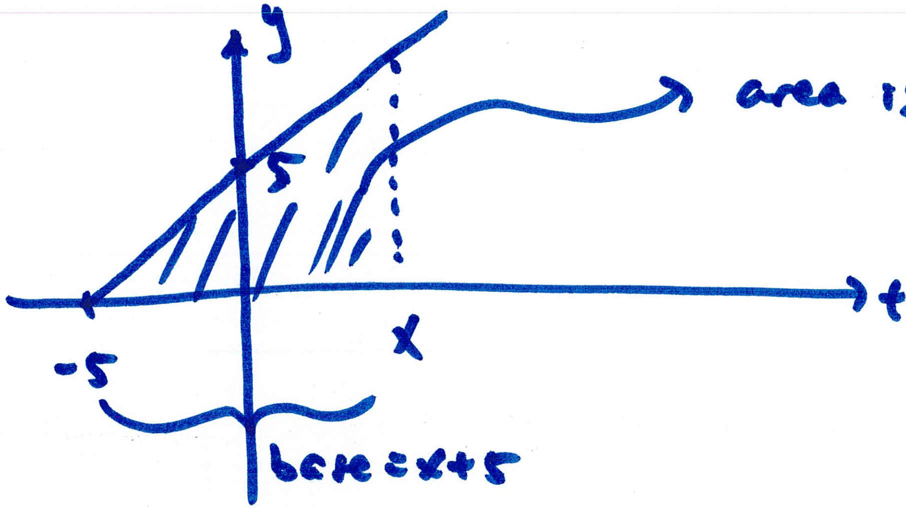
find $g'(x)$



FTC 1: $$\frac{d}{dx}\int_a^x \boxed{f(t)}\,dt = \boxed{f(x)}$$

so, $$\frac{d}{dx}\int_0^x \boxed{\cos(\pi t)}\,dt = \boxed{\cos(\pi x)}$$

example Find the rate of change of the function representing the area under $f(t) = t+5$ from $t=-5$ to $t=x$



area is $A(x) = \int_{-5}^{x} (t+5)\,dt$

$$A'(x) = \frac{d}{dx}\int_{-5}^{x} (t+5)\,dt = \boxed{x+5}$$

note the region is a triangle with base $x+5$ height $= f(x) = x+5$

$$A(x) = \frac{1}{2}(x+5)(x+5) = \frac{1}{2}(x+5)^2$$

$$A'(x) = \frac{1}{2}\cdot 2\cdot(x+5) = x+5$$ which matches result from FTC 1

FTC 1: $\frac{d}{dx}\int_a^{x} f(t)\,dt = f(x)$

only true if the variable x is the upper limit of integration

if it's not, we can use the property $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

example

$$\frac{d}{dx}\int_x^1 \sqrt{t^5+1}\,dt$$

note x is not the upper limit

$$= \frac{d}{dx}\left(-\int_1^x \sqrt{t^5+1}\,dt\right)$$

$$= -\underbrace{\frac{d}{dx}\int_1^x \sqrt{t^5+1}\,dt}_{\text{use FTC 1}} = \boxed{-\sqrt{x^5+1}}$$

what if x is not simply x?

<u>example</u>

$$y = \int_2^{e^{3x}} \sin^2(5t)\,dt \qquad y' = ?$$

FTC I needs the upper limit to be just a variable

let's use the Chain Rule

let $u = e^{3x}$

then y becomes $y = \int_2^{u} \sin^2(5t)\,dt = y(u)$ w/ $u = e^{3x}$

Chain Rule: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$$y = \int_2^u \sin^2(5t)\,dt \qquad u = e^{3x}$$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$= \underbrace{\left(\frac{d}{du}\int_2^u \sin^2(5t)\,dt\right)}\frac{d}{dx}e^{3x}$$

use FTC I: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$$\frac{d}{du}\int_a^u f(t)\,dt = f(u)$$

$$= \sin^2(5u)\cdot 3e^{3x}$$

$$u = e^{3x}$$

$$= \sin^2(5e^{3x})\cdot 3e^{3x} = \boxed{3e^{3x}\sin^2(5e^{3x})}$$

The second part of the Fundamental Theorem of Calculus allows us to calculate $\int_a^b f(x)dx$ exactly (no more Riemann sum)

FTC 1: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

so the antiderivative of $\frac{d}{dx}\int_a^x f(t)\,dt$ is the antiderivative of $f(x)$

$\rightarrow \int_a^x f(t)dt = F(x) + C$ where $F'(x) = f(x)$

we know $\int_a^a f(t)dt = 0$

so $\int_a^a f(t)dt = F(a) + C = 0 \rightarrow \boxed{C = -F(a)}$

from $\int_a^x f(t)\,dt = F(x) + C$

we get, by letting $x = b$

$$\int_a^b f(t)\,dt = F(b) + C \qquad \text{but} \quad C = -F(a)$$

$$= F(b) - F(a)$$

therefore, $$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where } F' = f}$$

Fundamental Theorem of Calculus (part 2)

FTC 2

example

$$\int_0^3 \underbrace{2x}_{f(x)}\,dx$$

FTC 2: $\int_a^b f(x)\,dx = F(b) - F(a)$

find $F(x)$ such that $F'(x) = f(x)$

(in other words, F is antiderivative of $f(x)$)

$f(x) = 2x$

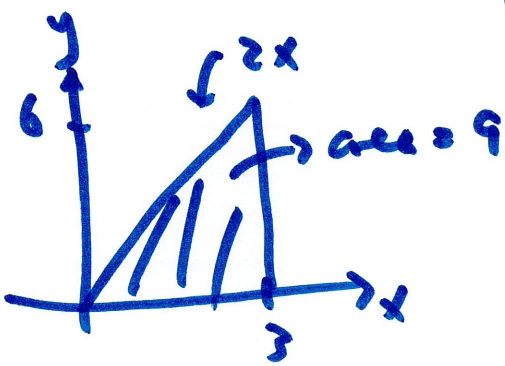
$$F(x) = 2\cdot\frac{x^2}{2} + C = \underbrace{x^2 + C}_{F(x)}$$

now use FTC 2

$$\int_{0 \,\leftarrow a}^{3 \,\leftarrow b} 2x\,dx = F(3) - F(0)$$

$$= (9 + C) - (0 + C) = \boxed{9}$$

notice C disappears



check: $\frac{1}{2}(3)(6) = 9$

notation: FTC 2 is written as

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

example

$$\int_{-1}^{2} \underbrace{(x^2 - 2x + 7)}\,dx$$

$f(x) = x^2 - 2x + 7$ $F(x) = \frac{1}{3}x^3 - x^2 + 7x + C$

$$= \underbrace{\frac{1}{3}x^3 - x^2 + 7x}_{\text{don't write } +C \text{ it goes away anyway}} \Big|_{-1}^{2} = \underbrace{\left[\frac{1}{3}(2)^3 - (2)^2 + 7(2)\right]}_{F(b)} - \underbrace{\left[\frac{1}{3}(-1)^3 - (-1)^2 + 7(-1)\right]}_{F(a)}$$

$$= \boxed{21}$$