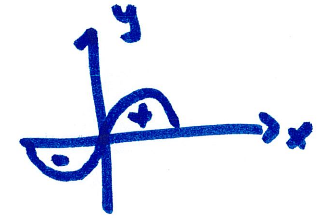
5.4 Working with Integrals

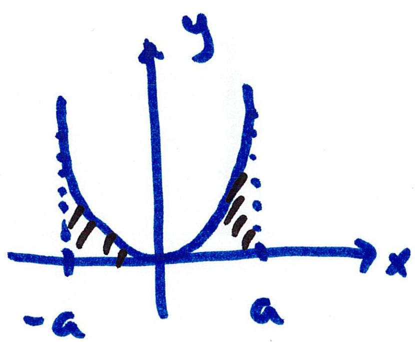
we know $\int_a^b f(x)\,dx$ gives us the area between $f(x)$ and x-axis on $[a, b]$

if $f(x) \geq 0$, then area is positive

if $f(x) < 0$, " " " negative



if $f(x)$ is even : $f(-x) = f(x) \rightarrow$ y-axis symmetry

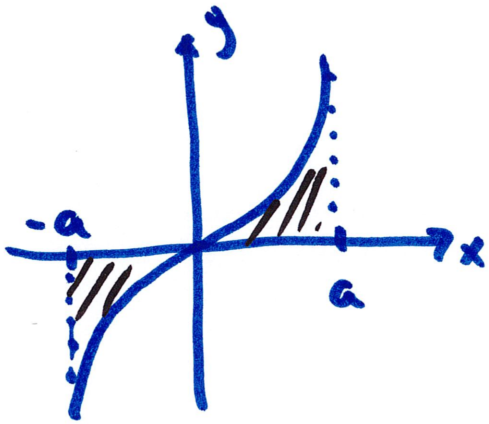


notice $\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(x)\,dx$

and since $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

so, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx = 2\int_{-a}^{0} f(x)\,dx$

so, if $f(x)$ is even, find area of half then multiply by 2 if $f(x)$ is odd : $\cancel{f(x) =}$ $f(-x) = -f(x)$ → origin symmetry



notice $\displaystyle\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx$

so, $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$= 0$

so, for odd function, the net area from $x = -a$ to $x = a$ is 0.

<u>example</u>

$\int_{-1}^{1} (x^4+3)\,dx$ area bounded by $f(x) = x^4+3$ and x-axis from $x=-1$ to $x=1$

notice $f(x) = x^4+3$ is even
because $f(-x) = (-x)^4+3 = x^4+3 = f(x)$

so we know $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

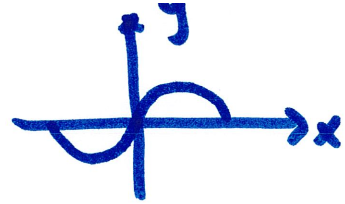
instead of calculating $\int_{-1}^{1} (x^4+3)\,dx$ we can do

$2\int_{0}^{1} (x^4+3)\,dx$ or $2\int_{-1}^{0} (x^4+3)\,dx$

$= 2\left(\frac{x^5}{5}+3x\right)\Big|_{0}^{1} = 2\left(\frac{1}{5}+3\right) - 2(0) = \boxed{\frac{32}{5}}$

example

$$\int_{-\pi}^{\pi} \sin x \, dx$$



since $\sin x$ is odd we know $\int_{-a}^{0} \sin x \, dx = -\int_{a}^{0} \sin x \, dx$

$$\int_{-\pi}^{\pi} \sin x \, dx = \int_{-\pi}^{0} \sin x \, dx + \int_{0}^{\pi} \sin x \, dx$$

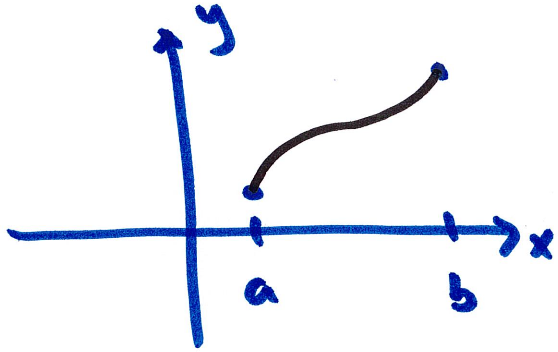
$$= 0$$

check: $\int_{-\pi}^{\pi} \sin x \, dx = -\cos x \Big|_{-\pi}^{\pi}$

$$= (-\cos \pi) - (-\cos(-\pi))$$

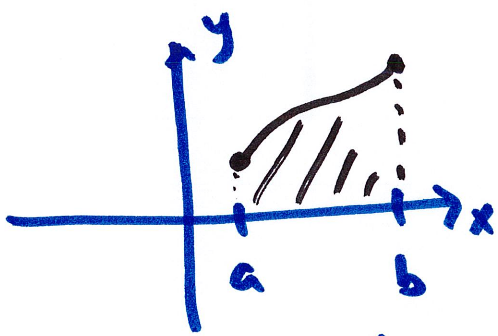
$$= (-1) - (-1) = -1 + 1 = 0$$

Average value of $f(x)$ on $[a, b]$



what is the average value of $f(x)$

idea: replace $f(x)$ with a constant function such that the area underneath is the same as area under $f(x)$



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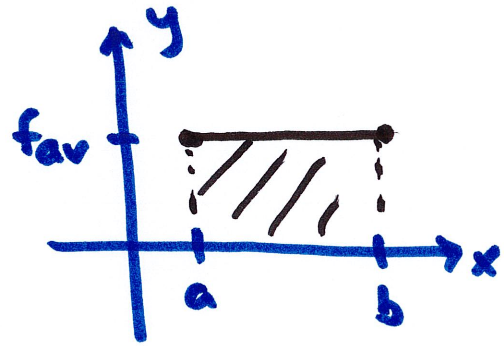
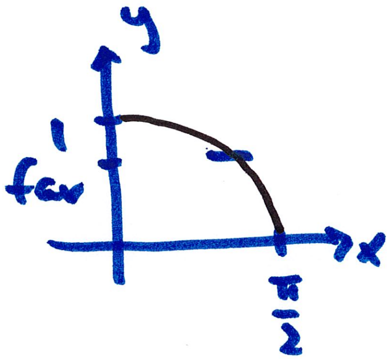


$$\int_a^b f(x)\,dx = f_{av}(b-a)$$

$$\boxed{f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

example $f(x) = \cos x$ on $[0, \frac{\pi}{2}]$



$$f_{av} = \frac{1}{b-a} \int_a^b \cos x \, dx$$

$$= \frac{1}{\frac{\pi}{2} - 0} \int_0^{\pi/2} \cos x \, dx$$

$$= \frac{1}{\frac{\pi}{2}} (\sin x) \Big|_0^{\pi/2}$$

$$= \frac{2}{\pi} \Big(\underbrace{\sin \tfrac{\pi}{2}}_{1} - \underbrace{\sin 0}_{0} \Big)$$

$$= \boxed{\frac{2}{\pi}}$$

note: average is not necessarily the middle between max and min.

5.5 Substitution Rule (part 1)

Integration by substitution is the reverse of Chain Rule.

$f(x) = \frac{1}{4}(x^2+3)^4$ when differentiating, we let $(x^2+3) = u$

$= \frac{1}{4}u^4 \qquad u = x^2+3$

$$f'(x) = \frac{1}{4} \cdot 4u^3 \cdot \frac{du}{dx} = u^3 \cdot 2x = (x^2+3)^3 \cdot 2x$$

when we want to find the antiderivative of $(x^2+3)^3 \cdot 2x$ we need to undo the Chain Rule

$$\int (x^2+3)^3 (2x)\,dx$$

let $u = x^2+3$ just like how we got here

then $\frac{du}{dx} = 2x$ and multiplying by dx we get

$$\underbrace{du = 2x\,dx}_{\text{"differential of } u\text{"}}$$

$$\int \underbrace{\boxed{(x^2+3)^3}}_{\displaystyle u^3}\;\underbrace{\boxed{(2x)\,dx}}_{\displaystyle du}$$

$$= \int u^3\,du$$ handle this just like $\int x^3\,dx$

$$= \frac{u^4}{4} + C$$ now undo the substitution $u = x^2+3$

$$= \boxed{\frac{1}{4}(x^2+3)^4 + C}$$

the main challenge of substitution is finding u

in this example we had Chain Rule to look at to know what u is, but generally we need to ~~p~~ find it

How do we choose u?

→ pick the part of the integrand (the things between $\int$ and dx) such that its derivative is a constant multiple of the other part

previous example: $\int (\underline{x^2+3})^3 (\underline{2x})\, dx$

deriv. of x^2+3 is a constant (1) multiple of the other part $(2x)$

easier Rule of Thumb: choose the more complicated part (or part with higher power to be u).

example

$$\int x^2 \sqrt{x^3+4}\, dx$$

$$= \int (\boxed{x^2})(\boxed{x^3+4})^{1/2}\, dx$$

compare x^2 and x^3+4

x^3+4 is more complicated, so let's pick it to be u

$$\boxed{u = x^3+4}$$

then take its derivative : $\frac{du}{dx} = 3x^2$

then find the differential of u : $\boxed{du = 3x^2 dx}$

now remove all x and dx from original integral

$$du = 3x^2 dx$$

$$\frac{1}{3} du = x^2 dx$$

$$\int \underbrace{\boxed{(x^3+4)^{1/2}}}_{u^{1/2}} \underbrace{\boxed{(x^2)\, dx}}_{\frac{1}{3} du}$$

$= \int u^{1/2} \cdot \frac{1}{3}\, du = \frac{1}{3} \int u^{1/2}\, du$ treat it like $\int x^{1/2}\, dx$

$= \frac{1}{3} \left(\frac{u^{3/2}}{3/2} \right) + C = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C$ $u = x^3 + 4$

undo the subs

$$= \boxed{\frac{2}{9}(x^3+4)^{3/2} + C}$$

example $\int e^{2x+3}\,dx$

$$= \int e^{\boxed{2x+3}} \boxed{(1)}\,dx$$

compare to power of e to the other part

$2x+3$ vs. 1

here, let $\boxed{u = 2x+3}$ because it's more complicated.

$$\frac{du}{dx} = 2$$

$$\boxed{du = 2\,dx}$$

we only have dx so divide by 2

$$\frac{1}{2}du = dx$$

$$\int \boxed{e^{2x+3}}\ \boxed{dx}$$

e^{u} $\quad$ $\frac{1}{2}du$

$$= \int \frac{1}{2}e^{u}\,du = \frac{1}{2}\int e^{u}\,du = \frac{1}{2}e^{u} + C = \boxed{\frac{1}{2}e^{2x+3} + C}$$

Definite integrals: change back to x then evaluate

<u>example</u>

$$\int_0^1 (x^2+3)^3 (2x)\,dx$$

pick u as usual: $u = x^2+3$

$$\frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

$$\int_{x=0}^{x=1} u^3\,du$$

Integration limits are for x NOT u

$$= \frac{1}{4}u^4 \Big|_{x=0}^{x=1}$$

change back to x by removing u

$$= \frac{1}{4}(x^2+3)^4 \Big|_{x=0}^{x=1} = \frac{1}{4}(4)^4 - \frac{1}{4}(3)^4 = \boxed{\frac{175}{4}}$$

other way to do definite integrals: change limits ~~for~~ from x to u

$$\int_0^1 (x^2+3)^3 (2x)\,dx$$

$u = x^2+3$

$du = 2x\,dx$

$$\int_{x=0}^{x=1} \underbrace{(x^2+3)^3}_{u^3} \underbrace{(2x)\,dx}_{du}$$

now we adjust limits so the numbers refer to u

from $u = x^2+3$

old upper limit : $x = 1 \rightarrow u = (1)^2+3 = 4$

old lower limit : $x = 0 \rightarrow u = (0)^2+3 = 3$

$$= \int_{u=3}^{u=4} u^3\,du = \frac{u^4}{4}\Big|_3^4 = \frac{4^2}{4} - \frac{3^4}{4} = \frac{175}{4}$$