5.5 Substitution Rule (part 2)

general idea: choose u such that its derivative is a constant multiple of the left over parts

example $\int \sin^3(2x)\cos(2x)\,dx$

ignore this while choosing u

$$= \int [\sin(2x)]^3 \cdot \cos(2x)\,dx$$

choose u from these

compare $\sin(2x)$ and $\cos(2x)$

notice the deriv. of $\sin(2x)$ is $2\cos(2x)$

which is two times the left over part ($\cos(2x)$)

so, let $\boxed{u = \sin(2x)}$

then $\frac{du}{dx} = 2\cos(2x)$

$$\boxed{du = 2\cos(2x)\,dx}$$

sub these into integral sub x and dx out

$$\int \underbrace{[\sin(2x)]^3}_{u^3} \cdot \underbrace{\cos(2x)\,dx}_{\frac{1}{2}du}$$

$u = \sin(2x)$

$du = 2\,\underline{\cos(2x)\,dx}$

I have this but without the 2

$\frac{1}{2}du = \cos(2x)dx$

$$= \int \frac{1}{2} u^3\,du = \frac{1}{2}\int u^3\,du = \frac{1}{2}\left(\frac{u^4}{4}\right) + C$$

now sub u out

$$= \frac{1}{8}[\sin(2x)]^4 + C = \boxed{\frac{1}{8}\sin^4(2x) + C}$$

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revisit the start again

$$\int [\sin(2x)]^3 \cos(2x)\,dx$$

compare: $\sin(2x)$ and $\cos(2x)$

we chose $u = \sin(2x)$ because its deriv. $2\cos(2x)$ is 2 times the other part

but notice deriv. of $\cos(2x)$ is $-2\sin(2x)$ which is also a constant multiple of the other part ($\sin(2x)$)

So what if we had chosen $u = \cos(2x)$ instead?

$$\int [\sin(2x)]^3 \cos(2x)\, dx$$

$$\boxed{u = \cos(2x)}$$

$$\frac{du}{dx} = -2\sin(2x)$$

$$\boxed{du = -2\sin(2x)\, dx}$$

$$= \int \underbrace{\cos(2x)}_{u} \cdot \underbrace{[\sin(2x)]^3}_{?} dx$$

what to do with this!
there is no $[\sin(2x)]^3$ in du

rewrite:

$$\int \underbrace{\cos(2x)}_{u} \cdot \underbrace{[\sin(2x)]^2}_{} \cdot \underbrace{\sin(2x)\, dx}_{-\frac{1}{2} du}$$

$$\sin(2x)\, dx = -\frac{1}{2} du$$

$$\sin^2(2x) + \cos^2(2x) = 1$$

$$\cos^2(2x) = u^2$$

$$\sin^2(2x) = 1 - u^2$$

new integral in u

$$\int u \cdot (1-u^2) \cdot -\frac{1}{2}\, du$$

$$= -\frac{1}{2} \int u(1-u^2)\, du = -\frac{1}{2} \int (u-u^3)\, du$$

$$= -\frac{1}{2}\left(\frac{u^2}{2} - \frac{u^4}{4}\right) + C = -\frac{1}{4}u^2 + \frac{1}{8}u^4 + C \qquad u = \cos(2x)$$

$$= \boxed{-\frac{1}{4}\cos^2(2x) + \frac{1}{8}\cos^4(2x) + C}$$

can use trig identities to make it look like the previous answer.

if by the time you try to get rid of x and sub in u things get messy, consider changing choice of u.

example

$$\int \frac{e^{2x}}{e^{2x}+3}\,dx$$

ignore for now

$$= \int (\boxed{e^{2x}+3})^{-1}\,(\boxed{e^{2x}})\,dx$$

compare the base parts to choose u

$\frac{d}{dx}(e^{2x}+3) = 2e^{2x}$ = 2 times left over (e^{2x})

but $\frac{d}{dx}(e^{2x}) = 2e^{2x} \neq$ constant multiple of $e^{2x}+3$

so, we pick $\boxed{u = e^{2x}+3}$ $\quad \frac{du}{dx} = 2e^{2x}$ $\quad \boxed{du = 2e^{2x}dx}$

$\frac{1}{2}du = e^{2x}dx$

$$\int \underbrace{(e^{2x}+3)^{-1}}_{u^{-1}}\,\underbrace{(e^{2x})\,dx}_{\frac{1}{2}du}$$

$$= \frac{1}{2}\int u^{-1}\,du = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|e^{2x}+3| + C}$$

example

$$\int \frac{\sec^2 x}{\tan^3 x}\,dx$$

$$= \int \underline{(\tan x)}^{-3}\,\underline{(\sec x)}^2\,dx$$

if we temporarily ignore the powers as we usually do,

we compare $\tan x$ and $\sec x$

$\frac{d}{dx}\tan x = \sec^2 x \neq$ constant multiple of left over (sec x)

$\frac{d}{dx}\sec x = \sec x \tan x \neq$ constant multiple of $\tan x$

no appropriate ~~of~~ choice of u

$$\int (\boxed{\tan x})^{-3}\,\boxed{(\sec^2 x)}\,dx$$

now let's keep the power of $\sec x$

$\frac{d}{dx}\tan x = \sec^2 x$ which matches the other part exactly

so, let $\boxed{u = \tan x}$

$$\frac{du}{dx} = \sec^2 x \qquad \boxed{du = \sec^2 x \, dx}$$

$$\int \underbrace{(\tan x)^{-3}}_{u^{-3}} \underbrace{(\sec^2 x)\, dx}_{du} = \int u^{-3}\, du$$

$$= \frac{u^{-2}}{-2} + C = -\frac{1}{2} u^{-2} + C = -\frac{1}{2u^2} + C$$

$$= \boxed{-\frac{1}{2\tan^2 x} + C}$$

__example__

$$\int_{e^{36}}^{e^{81}} \frac{1}{x\sqrt{\ln x}}\,dx$$

$$= \int_{e^{36}}^{e^{81}} \left(\boxed{\ln x}\right)^{-1/2} \left(\boxed{\frac{1}{x}}\right) dx$$

we notice $\frac{d}{dx}\ln x = \frac{1}{x}$ so we let $\boxed{u = \ln x}$

$\frac{du}{dx} = \frac{1}{x}$ $\boxed{du = \frac{1}{x}dx}$

now we adjust the ~~lim~~ integration limits to refer to u

old upper limit : $x = e^{81} \rightarrow u = \ln x = \ln e^{81} = \boxed{81}$ new upper limit

old lower limit : $x = e^{36} \rightarrow u = \ln x = \ln e^{36} = \boxed{36}$ new lower limit

now sub in u, du, and the new limits

$$\int_{36}^{81} u^{-1/2}\,du = \left.\frac{u^{1/2}}{1/2}\right|_{36}^{81}$$

DO __NOT__ go back to x

we changed the limits to refer to u so stay w/ u

$$= 2u^{1/2}\Big|_{36}^{81} = 2(81)^{1/2} - 2(36)^{1/2} = 2(9) - 2(6) = \boxed{6}$$