7.2 Exponential Growth and Decay

consider a quantity that grows or decays at a rate that is proportional to its own size

y : quantity that is growing / decaying

constant of proportionality ("proportional to ___")

$$\frac{dy}{dt} = k \cdot y$$

size of y

the larger y is the faster it grows/decays

k is also the relative growth/decay rate

for example, if something is growing at a rate that is 2% of its size

then $\frac{dy}{dt} = \underbrace{0.02}_{2\%} y$ (for example, savings account w/ 2% interest rate)

$\frac{dy}{dt} = ky$ the right side (ky) is also called the absolute growth / decay rate

if $\frac{dy}{dt} = ky$ what is $y(t)$?

y is a type of function whose rate of change is a constant multiple of itself

(or whose derivative looks like itself)

↳ exponential : $y = Ce^{kt}$ C, k are constants

check: is $\frac{dy}{dt} = ky$ if $y = Ce^{kt}$?

↳ $\frac{dy}{dt} = C \cdot e^{kt} \cdot k$

$= k \cdot \underbrace{(Ce^{kt})}_{y} = ky$ so yes

so, growth / decay at rate proportional to its size is described by $y = Ce^{kt}$ → exponential growth / decay

if $k > 0$, then the rate of change is positive
so y is growing (population, interest, spread of disease)

if $k < 0$, then the rate of change is negative
so y is decaying (radioactive decay, metabolism of drugs)

$y = Ce^{kt}$ C: the initial size of y
so we usually use y_0 for that

$$\boxed{y = y_0 e^{kt} \longleftrightarrow \frac{dy}{dt} = ky}$$

example The average yearly inflation rate between 2016 and 2019 in the US is 2.1 %. If a loaf of bread cost \$2 in 2016, assuming the inflation rate stays the same, what will a loaf of bread cost in 2030?
When will the price double (\$4)

$$y(t) = y_0 e^{kt}$$

y : cost of bread

t : number of years since 2016
($t = 1 \rightarrow 2017$)

find : $y(14)$ and when $y = 4$, $t = ?$

2030 is 14
years after 2016
so $t = 14$

k is the key part

initially, cost is \$2 $\rightarrow$ $y_0 = 2$

$$\boxed{y(t) = 2e^{kt}}$$

now find k

$y(0) = 2$ and $y(1) = \overbrace{1.021}^{2\% \text{ higher} \rightarrow 1+0.021}\, y(0) = (1.021)(2) = 2.042$

now we sub $y(1) = 2.042$ into $y(t) = 2e^{kt}$

$$\underbrace{2.042}_{y(1)} = \underbrace{2}_{y_0} e^{k \cdot 1} \quad t = 1$$

$$\frac{2.042}{2} = e^{k} = 1.021$$

$$k = \ln(1.021) = \boxed{0.0208}$$

in 2030, the price is

$$y(14) = 2e^{0.0208(14)} = \boxed{2.68} \qquad \$2.68$$

when will price double : $y = 4$, $t = ?$

back to $y(t) = 2e^{0.0208t}$

$$4 = 2e^{0.0208t}$$

$$2 = e^{0.0208t}$$

$$\ln 2 = 0.0208t \rightarrow t = \frac{\ln 2}{0.0208} \approx 33$$ (yrs after 2016)

in $\boxed{\text{year } 2049}$

the time to double is the same for all y_0

$$y(t) = y_0 e^{kt}$$

time to double y_0 $\rightarrow$ $y(t) = 2y_0$

$$2\cancel{y_0} = \cancel{y_0} e^{kt}$$

$$2 = e^{kt}$$

$$\ln 2 = kt$$

$$\boxed{t = \frac{\ln 2}{k}}$$

time to double whatever y_0

example The half-life of caffeine in human body is around 5 hours. If 100 mg of caffeine was ingested by drinking coffee at 6 am, how long before 80 % of the caffeine is eliminated?

How much remains at 10 pm?

time to decay to half of initial size

($y = \frac{1}{2} y_0$, $t = ?$)

start w/ $y(t) = y_0 e^{kt}$

t : hours after 6 am ($t = 1 \to$ 7 am)

y : mg of caffeine in body

y_0 : initial amount $= 100$

equation: $y(t) = 100 e^{kt}$

find k from half-life (of 5 hours)

$$y(t) = y_0 e^{kt}$$

$$\underbrace{\tfrac{1}{2} \cancel{y_0}}_{\text{half left}} = \cancel{y_0} e^{k(5)}$$

t=5 is half-life

$$\tfrac{1}{2} = e^{5k} \qquad \ln \tfrac{1}{2} = 5k \qquad k = \frac{\ln \frac{1}{2}}{5} \approx \underbrace{-0.139}_{\text{relative decay rate}}$$

relative decay rate is 13.9 %

time to lose 80 % of initial 100 mg

$$y(t) = (0.2)(100) = 20$$

↳ lost 80 % so 20 % remains

$$y(t) = 100 e^{-0.139t}$$

$$20 = 100 e^{-0.139t}$$

$$\tfrac{1}{5} = e^{-0.139t}$$

$$\ln \tfrac{1}{5} = -0.139t$$

$$t = \frac{\ln \frac{1}{5}}{-0.139} \approx \boxed{11.6}$$ (hours after 6 am)

how much remains at 10 pm?

10 pm $\rightarrow$ $t = 16$ (16 hours after 6 am)

equation: $y(t) = 100e^{-0.139t}$

$$y(16) = 100e^{-0.139(16)} \approx \boxed{10.8} \text{ mg}$$