2.3 Computing Limits

last time: find limits by graph or making table of values

today: by analyzing the functions algebraically

first, easy ones: $f(x) = 3$ then clearly, $\lim_{x\to5} f(x)$

$= \lim_{x\to5} 3 = 3$

because 3 is not affected by x

a number

"constant"

generalize: $\boxed{\lim_{x\to a} C = C}$

another easy one: if $f(x) = x$

then $\lim_{x\to5} f(x) = \lim_{x\to5} x = 5$

generalize: $\boxed{\lim_{x\to a} x = a}$

with those and some basic rules ("laws") we can calculate almost any limit

for example, $\lim_{x \to 5} (3x) = \lim_{x \to 5} [(3)(x)]$

$$= \left(\lim_{x \to 5} 3\right)\left(\lim_{x \to 5} x\right) = 3 \cdot 5 = 15$$ "Product Law"

generalize: $\lim_{x \to a} cx = c \cdot a$

likewise for quotients: $\lim_{x \to 5} \left(\frac{3}{x}\right) = \frac{\lim_{x \to 5} 3}{\lim_{x \to 5} x} = \frac{3}{5}$ "Quotient Law"

powers and roots: $\lim_{x \to 5} x^3 = \left(\lim_{x \to 5} x\right)^3 = 5^3 = 125$ "Power Law"

$$\lim_{x \to 5} \sqrt{x} = \sqrt{\left(\lim_{x \to 5} x\right)} = \sqrt{5}$$ "Root Law"

additions / subtractions work as expected, too

for example, $\lim_{x\to -2} (x^3+3x-5)$

$= \lim_{x\to -2} x^3 + \lim_{x\to -2} 3x - \lim_{x\to -2} 5$

$= \left(\lim_{x\to -2} x\right)^3 + \left(\lim_{x\to -2} 3\right)\left(\lim_{x\to -2} x\right) - \lim_{x\to -2} 5$

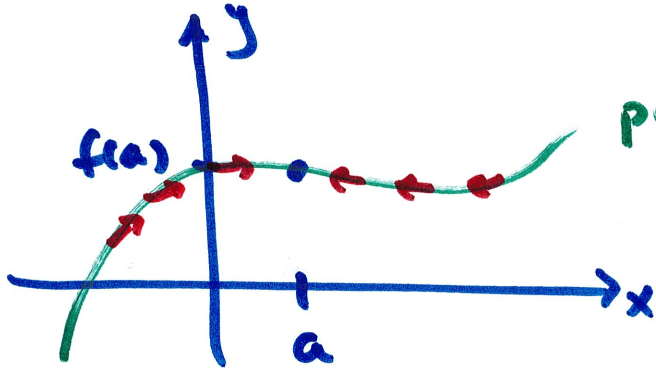
$= (-2)^3 + (3)(-2) - 5 = -19$

looks like we simply plugged in $x=-2$ (the number x is approaching) even though for limits we don't really care if x can be that number.

→ we can make the function as close to -19 as we want by making x sufficiently close to -2

x^3+3x-5 is a <u>polynomial</u> function ~~ah~~ → domain is all reals

so, for polynomials, because there no breaks, the limit is easy: ~~so~~ simply plug in the number x is approaching



polynomial has no breaks

as $x \to a$, $f(x) \to f(a)$

in fact, $\lim_{x \to a} f(x) = f(a)$ for polynomials

for rational functions (polynomial over polynomial)

~~the lim.~~ we can do the same if a in $x \to a$ is in domain

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} \quad \text{if } a \text{ is domain}$$

<u>example</u> $\lim_{x \to 1} \frac{x^2 + x - 20}{x - 4}$

$\frac{x^2+x-20}{x-4}$ is a rational function whose domain is

$(-\infty, 4) \cup (4, \infty)$

$x \to 1$ and 1 is in domain, so

$$\lim_{x \to 1} \frac{x^2+x-20}{x-4} = \frac{1^2+1-20}{1-4} = \frac{-18}{-3} = 6$$

what about $\lim_{x \to 4} \dfrac{x^2+x-20}{x-4}$?

clearly, $\dfrac{x^2+x-20}{x-4}$ is not defined at $x=4$

but the limit of $\dfrac{x^2+x-20}{x-4}$ may still exist as $x \to 4$

($\dfrac{x^2+x-20}{x-4}$ may still approach some number as x gets close to 4)

what if we plug in $x=4$? $\dfrac{4^2+4-20}{4-4} = \dfrac{0}{0} = ?$ "indeterminate form"

$\dfrac{0}{0}$ means $\dfrac{\text{very small number}}{\text{very small number}}$

we handle this by doing some factoring and canceling

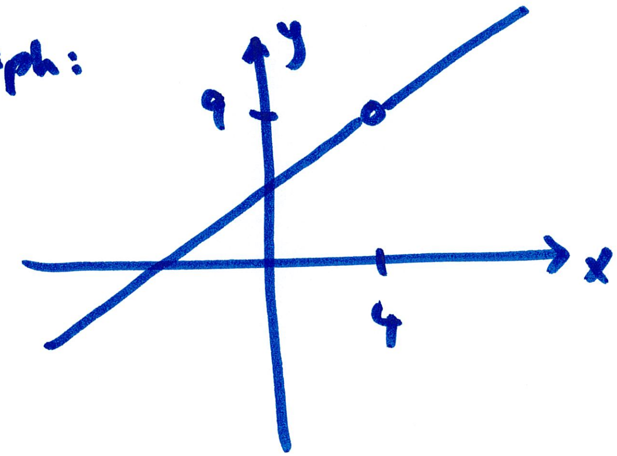
$$\lim_{x \to 4} \frac{x^2+x-20}{x-4} = \lim_{x \to 4} \frac{(x+5)(x-4)}{(x-4)}$$

this is what causes $\frac{0}{0}$

we can cancel out $x-4$ since $x-4$ is never zero (remember, $x \to 4$ means x is close to 4 but NOT equal to 4)

$$\lim_{x \to 4} \frac{(x+5)\cancel{(x-4)}}{\cancel{(x-4)}} = \lim_{x \to 4} \underbrace{(x+5)}_{\text{polynomial}} = 4+5 = 9$$

graph:



if $x \neq 4$, $\frac{x^2+x-20}{x-4} = x+5$

at $x=4$, $\frac{x^2+x-20}{x-4}$ is not defined

that's why there is a hole

it is NOT correct to say $\frac{x^2+x-20}{x-4} = x+5$

because for two functions to be equal, they must be equal at ALL x

another way to handle $\frac{0}{0}$

example $\lim\limits_{x\to 0} \frac{\sqrt{4+x}-2}{x}$ if $x=0$ $\frac{\sqrt{4}-2}{0} = \frac{0}{0} = ?$

never an acceptable answer to limit question

there is nothing to factor / cancel

root : rationalize

$$= \lim_{x\to 0}\left(\frac{\sqrt{4+x}-2}{x}\cdot\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}\right)$$

$$= \lim_{x\to 0}\frac{(\sqrt{4+x})^2 \cancel{+2\sqrt{4+x}} \cancel{-2\sqrt{4+x}} - 4}{x(\sqrt{4+x}+2)}$$

$$= \lim_{x\to 0}\frac{4+x-4}{x(\sqrt{4+x}+2)} = \lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}(\sqrt{4+x}+2)} = \lim_{x\to 0}\frac{1}{\sqrt{4+x}+2} = \frac{1}{\sqrt{4}+2}$$

$$= \frac{1}{4}$$

example

$$\lim_{x\to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x-2}$$

if $x=2$ $\quad \dfrac{\frac{1}{2}-\frac{1}{2}}{2-2} = \dfrac{0}{0} = ?$

do NOT stop at $\frac{0}{0}$

$$= \lim_{x\to 2} \frac{\frac{2}{2x} - \frac{x}{2x}}{x-2} = \lim_{x\to 2} \frac{\frac{2-x}{2x}}{x-2}$$

$2-x = -(x-2)$

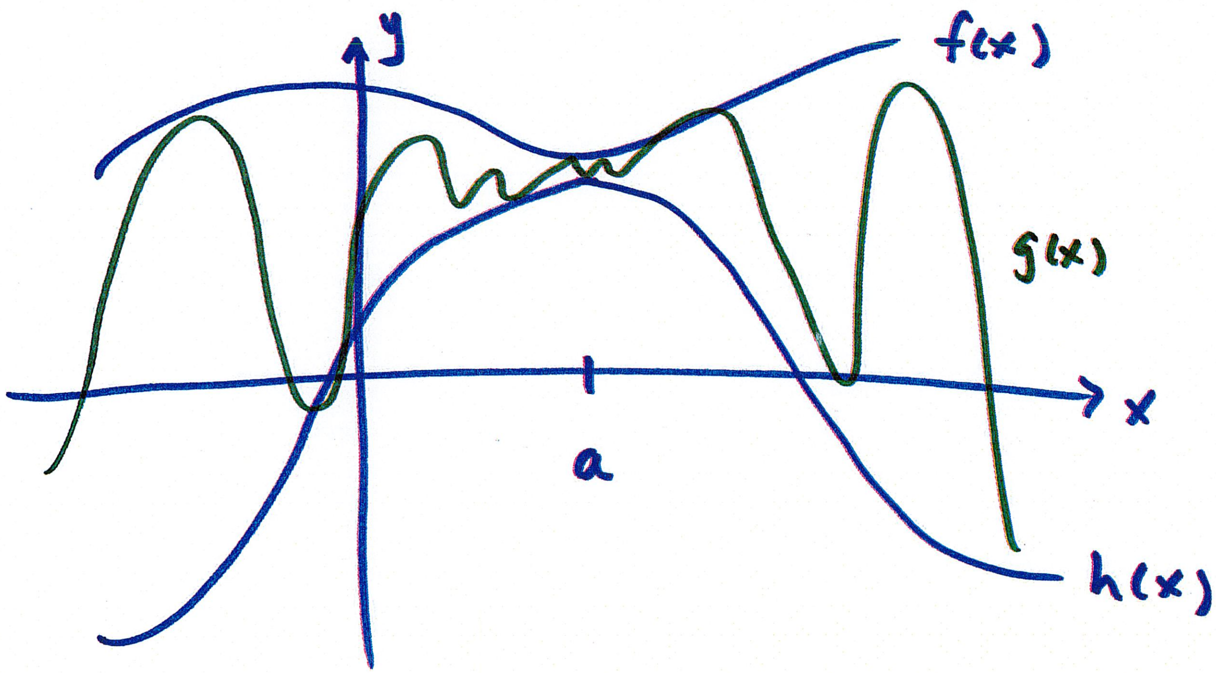
$$= \lim_{x\to 2} \frac{2-x}{2x}\cdot\frac{1}{x-2} = \lim_{x\to 2} \frac{-\cancel{(x-2)}}{2x}\cdot\frac{1}{\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{-1}{2x} = -\frac{1}{4}$$

Squeeze / Sandwich Theorem

if $f(x) \le g(x) \le h(x)$ for some x near $x = a$,
except possibly at $x = a$

then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x) \le \lim_{x \to a} h(x)$



if we know how f(x) and h(x) behave near $x = a$, then we know how g(x) behaves near $x = a$, even if g(a) is not defined.