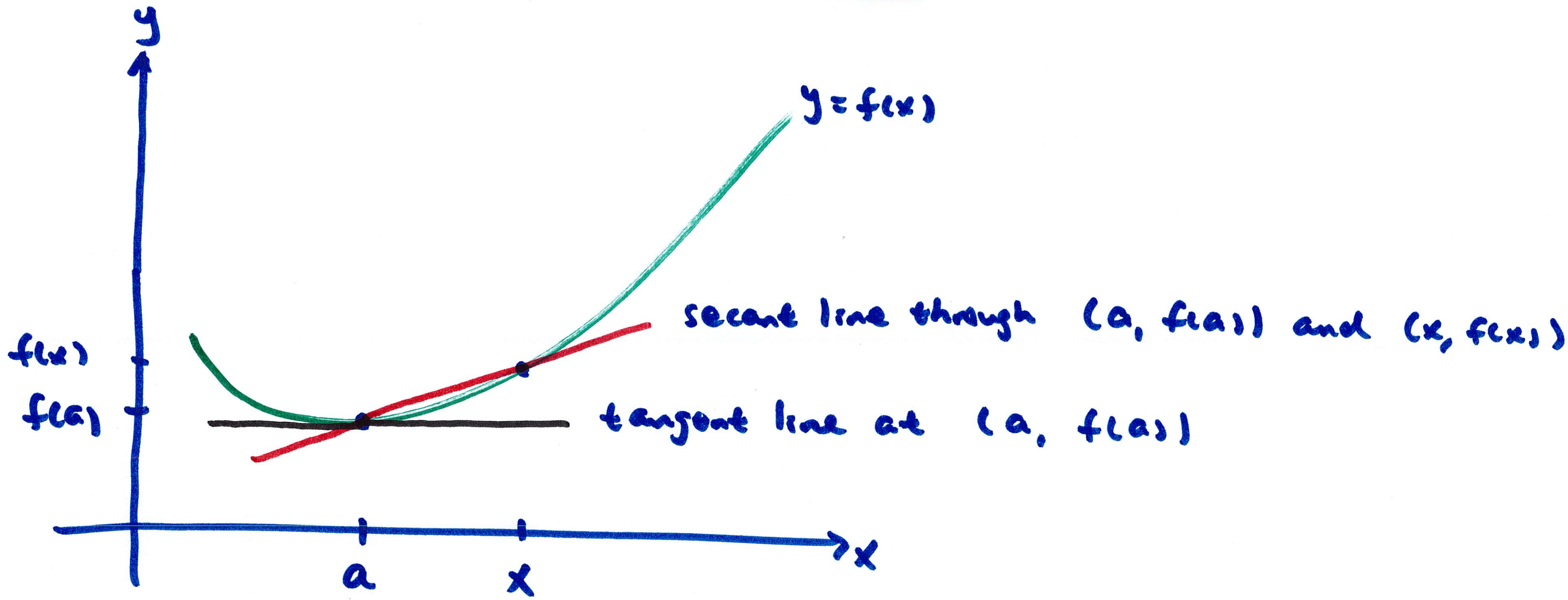
3.1 Introducing the Derivative



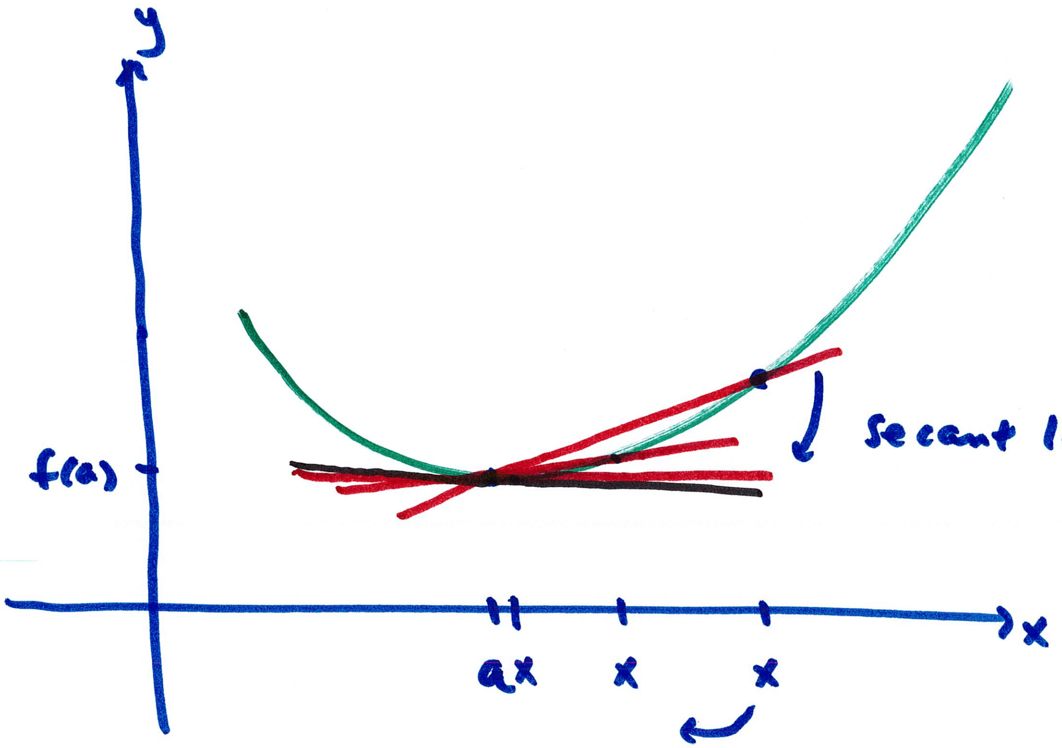
slope of secant line : $\dfrac{f(x) - f(a)}{x - a}$

slope of tangent line at $x = a$

tangent line : touches only $(a, f(a))$

slope of tangent line can't be found using basic geometry

but we can approximate it by using a secant line
whose two points are close together



secant line becomes like tangent line as $x \to a$

bring closer to a

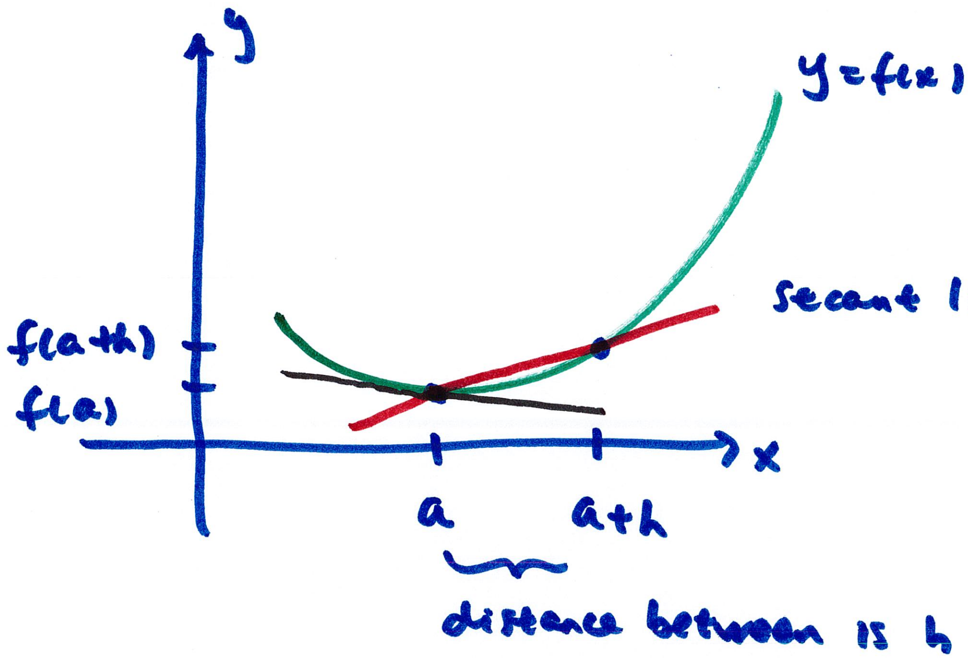
bring x close to a : limit as $x \to a$

tangent line slope of $f(x)$ at $x = a$ is

$$\boxed{\underbrace{f'(a)}_{\text{"f prime of a"}} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

this is also called the <u>derivative</u> of $f(x)$ at $x = a$

another form of the derivative formula



secant line slope $= \dfrac{f(a+h)-f(a)}{a+h-a} = \dfrac{f(a+h)-f(a)}{h}$

make second point come to first : shrink h $\to$ limit as $h \to 0$

so, the tangent line slope at $x = a$ is

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$$

secant line slope : average rate of change over an interval

tangent line slope : instantaneous rate of change at $x = a$
or over a very short interval ($x \to a$ or $h \to 0$)

example $f(x) = \dfrac{1}{4+3x}$ find $f'(2)$ ← a

we can use either form of $f'(a)$

let's use the 2nd form in this example

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ here, $a = 2$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$ if $h=0$, $\dfrac{f(2) - f(2)}{0} \to \dfrac{0}{0} = ?$

$$= \lim_{h \to 0} \frac{\frac{1}{4+3(2+h)} - \frac{1}{4+3(2)}}{h}$$ combine the top

$$= \lim_{h \to 0} \frac{\frac{1}{10+3h} - \frac{1}{10}}{h} = \lim_{h \to 0} \frac{\frac{1}{10+3h} \cdot \frac{10}{10} - \frac{1}{10} \cdot \frac{10+3h}{10+3h}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{10}{10(10+3h)} - \frac{10+3h}{10(10+3h)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{10-(10+3h)}{10(10+3h)}}{h} = \lim_{h \to 0} \left(\frac{-3\cancel{h}}{10(10+3h)} \cdot \frac{1}{\cancel{h}} \right)$$

$$= \lim_{h \to 0} \frac{-3}{10(10+3h)} = \boxed{-\frac{3}{100}} = f'(2)$$

slope of tangent line of $f(x) = \frac{1}{4+3x}$ at $x=2$

equation of tangent line?

it is a line through $(2, f(2)) = (2, \frac{1}{10})$ with slope $f'(2) = -\frac{3}{100}$

point-slope form: $y - y_1 = m(x - x_1)$

$$\boxed{y - \frac{1}{10} = -\frac{3}{100}(x-2)}$$

example

$f(x) = \sqrt{x-2}$ find $f'(3)$

this time, use $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x-a}$

here, $a = 3$

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x \to 3} \frac{\sqrt{x-2} - 1}{x-3} \to \frac{0}{0} \text{ as } x = 3$$

we see root so ratimelize

$$= \lim_{x \to 3} \frac{\sqrt{x-2} - 1}{x-3} \cdot \frac{\sqrt{x-2} + 1}{\sqrt{x-2} + 1}$$

$$= \lim_{x \to 3} \frac{(\sqrt{x-2})^2 - 1^2}{(x-3)(\sqrt{x-2}+1)} = \lim_{x \to 3} \frac{\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{x-2}+1)}$$

now let $x = 3$

$$\boxed{f'(3) = \frac{1}{2}}$$

example If $f'(a) = \lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h}$

what is $f(x)$ and what is a?

first, recognize the form of the derivative

we see $\lim_{h \to 0}$ so we must be using $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h} \quad 3 \to \sqrt{9}$$

line them up: $f(a+h) = \sqrt{9+h}$

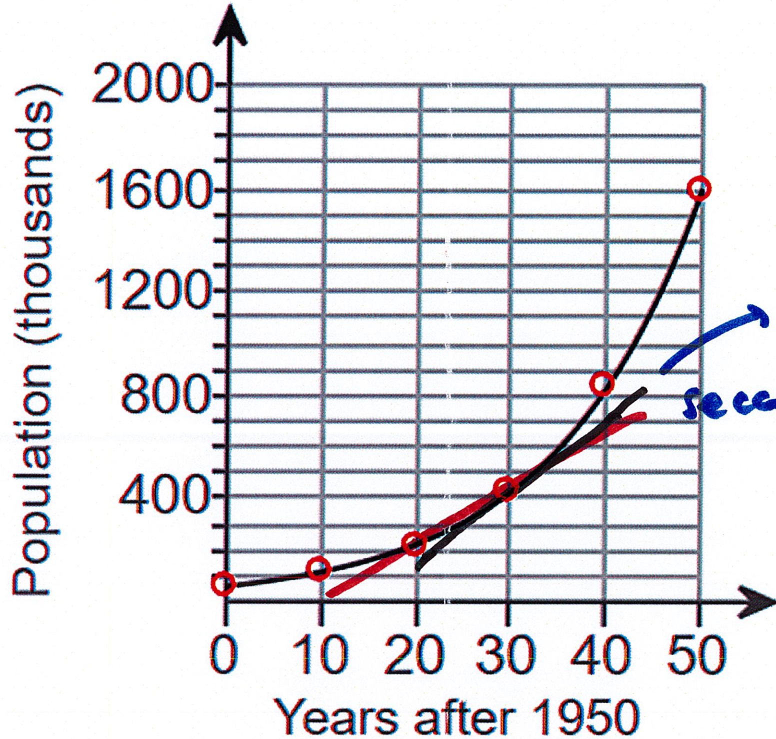
$f(a) = \sqrt{9} = 3$

now think of a $f(x)$ and a pair to make them work

try: $f(x) = \sqrt{x}$ and $a = 9$ $\rightarrow f(a) = \sqrt{a} = f(9) = \sqrt{9} = 3$

$f(a+h) = f(9+h) = \sqrt{9+h}$

so, $\boxed{f(x) = \sqrt{x} \text{ and } a = 9}$



tangent line slope at $t = 30$

secant line slope on $20 \le t \le 30$

Year	1950	1960	1970	1980	1990	2000
t	0	10	20	30	40	50
p(t)	59,000	114,153	220,862	427,322	826,779	1,599,646

Sometimes we can estimate the instantaneous rate of change by looking at the average rate of change

avg rate : secant line slope

inst rate : tangent line slope

notice the slopes of the secant line on $20 \leq t \leq 30$ and the tangent line slope at $t = 30$ are close

since we don't know what $p(t)$ looks like, we can't find $p'(30)$ but by comparing the slopes we can say

$$p'(30) \approx \text{avg. rate of change over } 20 \leq t \leq 30$$

$$\approx \frac{p(30) - p(20)}{30 - 20} \approx \frac{427322 - 220862}{10} \approx 20652$$