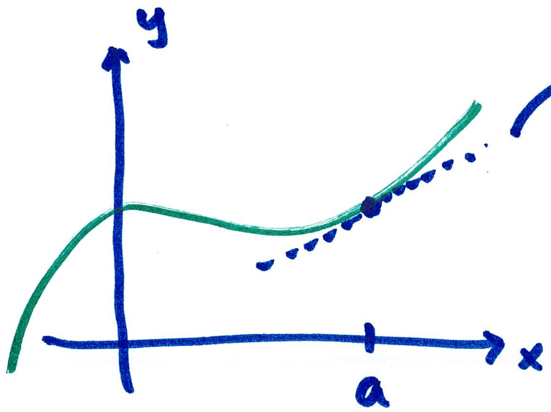
3.2 The Derivative as a Function



tangent line slope at $x=a$

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

instead of fixing x at a, take the 2nd form and change a to x

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}$$

this is a function of x

this gives the tangent line slope where this limit exists.

one condition needed:

function must be continuous

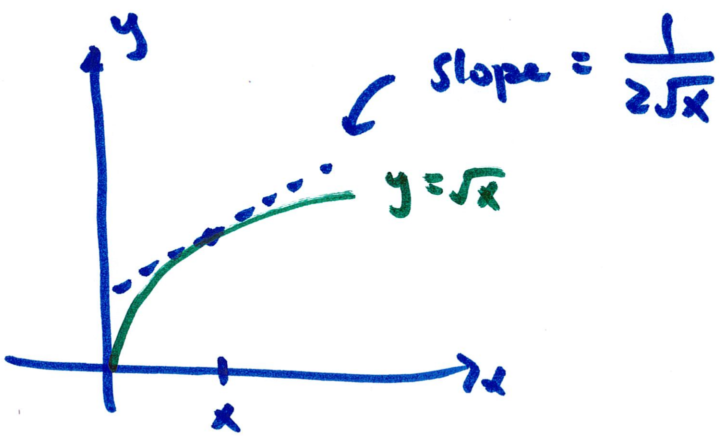
<u>example</u> $f(x) = \sqrt{x}$ domain: $[0, \infty)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$ rationalize

$$= \lim_{h \to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

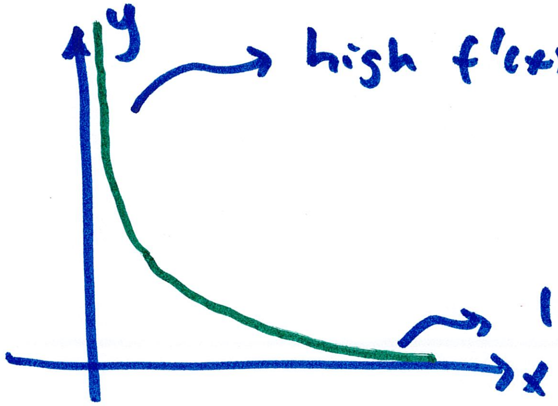
$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$



note $f'(x)$ has domain $(0, \infty)$
not same as $f(x)$

$f'(0)$ DNE

↳ $f(x)$ is <u>not differentiable</u> at 0 graph of $f'(x) = \frac{1}{2\sqrt{x}}$

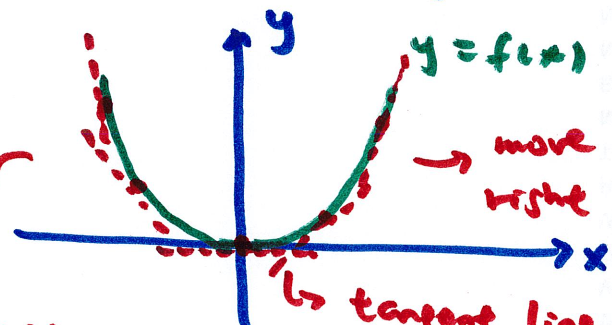


high $f'(x)$ → tangent line very steep ~~[illegible]~~

low $f'(x)$ → tangent line very shallow

we can sometimes graph $f'(x)$ by looking at graph of $f(x)$

<u>example</u>

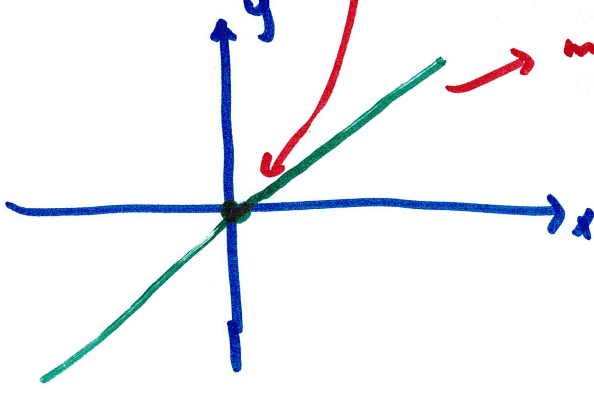


graph of $f(x)$

→ move to right → tangent line steeper → $f'(x)$ increases

more left → slope becomes more negative

↳ tangent line is horizontal → slope = $f'(x) = 0$



→ move right → higher $f'(x)$

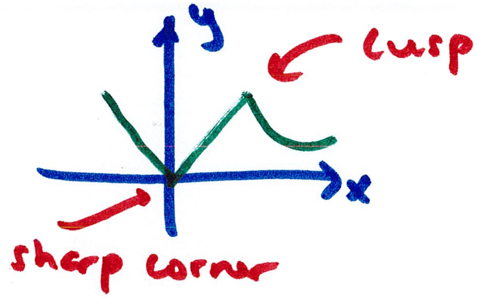
graph of $f'(x)$

$f(x)$ is <u>not differentiable</u> ($f'(x)$ DNE) if $f(x)$ is

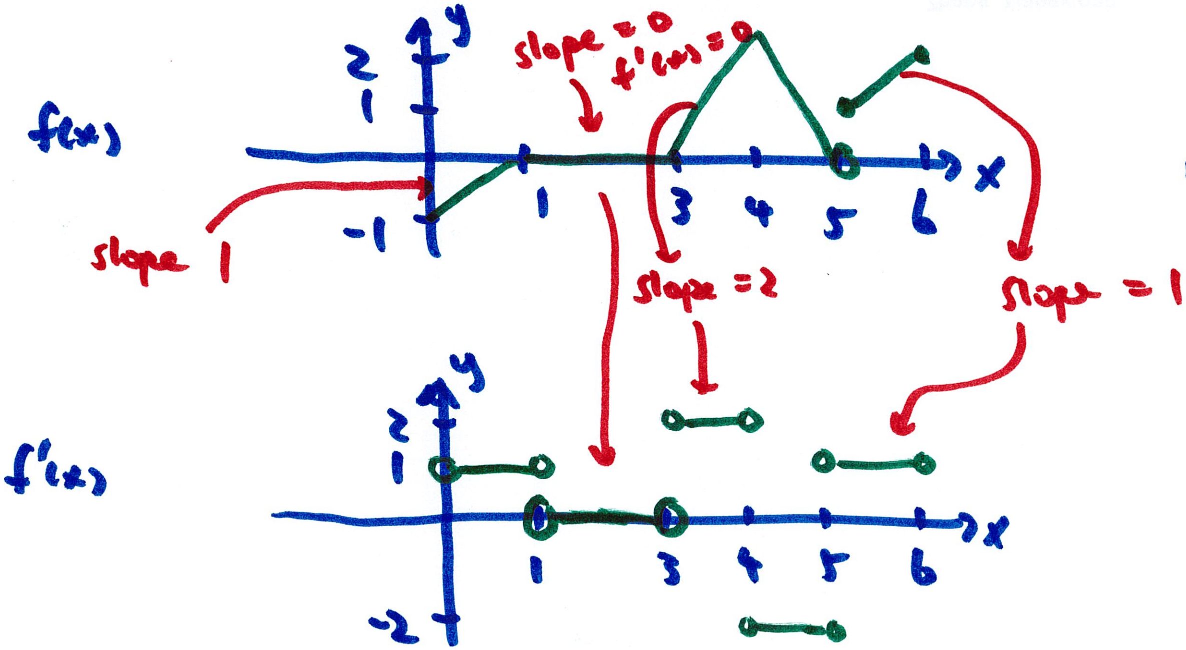
a) not continuous at x

b) has vertical tangent line (f' ∞ so DNE)

c) is not smooth (sharp corner or cusp)



<u>example</u> Sketch $f'(x)$ from graph of $f(x)$



discontinuous at $x = 0, 5, 6$

not smooth at $x = 1, 3, 4$

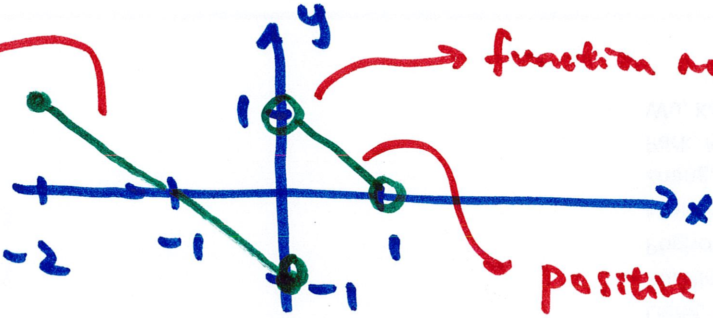
$f'(x)$ tells us the rate of change of $f(x)$

for example, on $0 < x < 1$, $f'(x) = 1 > 0 \rightarrow f(x)$ is <u>increasing</u> (positive rate)

on $4 < x < 5$, $f'(x) = -2 < 0 \rightarrow f(x)$ is <u>decreasing</u> (negative rate)

<u>example</u> Sketch graph of $f(x)$ from the given graph of $f'(x)$

$f'(x)$:



function not differentiable at $x=0$
possibly sharp corner or cusp

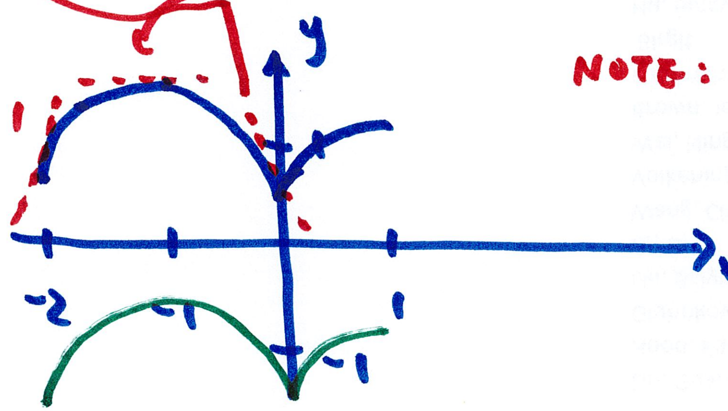
slope = 1
at $x=-2$,
then slope = 0 at $x=-1$ → horizontal tangent
then slope = -1 at $x=0$

positive slope then flatten out at $x=1$

NOTE: this $f(x)$ is not unique
many possible $f(x)$
to have the $f'(x)$ above

$f(x)$

slope = 1



↳ another possible $f(x)$