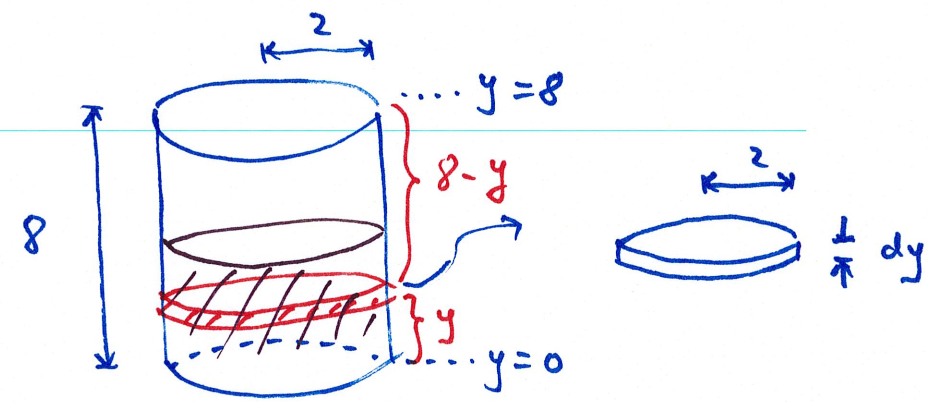
A cylindrical water tank has the height $8m$ and the radius $2m$. It is half full of water. Water has the density $\rho\ \frac{kg}{m^2}$ and the acceleration due to gravity is $g\ \frac{m}{s^2}$. The work done pumping all the water to the top of the tank is:



A. $32\pi\ \rho g$ J

B. $64\pi\ \rho g$ J

C. $128\pi\ \rho g$ J

D. $256\pi\ \rho g$ J

E. $96\pi\ \rho g$ J

slice mass $= \pi(2)^2 dy \cdot \rho = 4\pi\rho\, dy$

weight $= 4\pi\rho g\, dy$

work $= 4\pi\rho g\, dy \cdot (8-y) = 4\pi\rho g(8-y)dy$

accumulate from $y=0$ to $y=4$ ("half full")

$$\int_0^4 4\pi\rho g(8-y)dy = 4\pi\rho g \int_0^4 (8-y)dy$$

$$= 4\pi\rho g\left(8y - \tfrac{1}{2}y^2\right)\Big|_0^4 = 4\pi\rho g(32-8) = 96\pi\rho g$$

Compute the value of the integral $\int_1^e 2x \ln x \, dx$.

L

A

by parts

pick u → order: LIATE

pick $u = \ln x$ (L before A)

$du = \frac{1}{x}\,dx$

$dv = 2x\,dx \qquad v = x^2$

$uv - \int v\,du$

$$= x^2 \ln x \Big|_1^e - \int_1^e \underbrace{x^2 \cdot \frac{1}{x}}_{x}\,dx$$

$$= x^2 \ln x \Big|_1^e - \frac{1}{2}x^2 \Big|_1^e$$

$$= (e^2 \underbrace{\ln e}_{1} - 1 \cdot \cancel{\ln 1}^{\,0}) - \left(\frac{1}{2}e^2 - \frac{1}{2}\right)$$

$$= e^2 - \frac{1}{2}e^2 + \frac{1}{2} = \frac{1}{2}e^2 + \frac{1}{2} = \frac{1+e^2}{2}$$

A. $\frac{1}{2}$

B. $\frac{1-e^2}{2}$

C. $\frac{1+e^2}{2}$

D. $\frac{3e^2-1}{2}$

E. $\frac{e^2}{2}$

integration : directly

Substitution (u sub)

by parts (integrand is a product of two things)

trig subs (need sum/difference of squares)

Evaluate $\int_0^{\pi/2} \sin^3 x \cos^4 x dx$

A. $\frac{5}{12}$

B. $\frac{1}{7}$

C. $\frac{2}{35}$

D. $\frac{3}{28}$

E. $\frac{5}{16}$

sinx and cosx mixed

if $u = \sin x$, then $du = \cos x\, dx$

if $u = \cos x$, then $du = -\sin x\, dx$

then use $\underbrace{\sin^2 x + \cos^2 x = 1}_{\text{only useful if even power}}$ to get rid of the rest

here, if $u = \sin x$, then $du = \cos x\, dx$

$$\cos^4 x = \cos^3 x \underbrace{\cos x}_{du}$$

↳ odd power, $\sin^2 x + \cos^2 x = 1$ doesn't help

so only $u = \cos x$ as other choice, $\cos(\pi/2) = 0$

$du = -\sin x\, dx$

$$\int_0^{\pi/2} \sin^2 x \cos^4 x \sin x\, dx = \int_1^0 -(1-u^2)u^4\, du$$

$1 - \cos^2 x$ $= 1 - u^2$, u^4 , $-du$

$\cos(0) = 1$

$$= -\int_1^0 (u^4 - u^6)\,du = -\left(\frac{1}{5}u^5 - \frac{1}{7}u^7\right)\Big|_1^0$$

$$= \frac{1}{5} - \frac{1}{7} = \frac{2}{35}$$

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if $\sin x$ and $\cos x$ both have even power,

use $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$\cos^2 x = \dfrac{1+\cos 2x}{2}$

Compute

$$\int 7\sec^4 x\,dx$$

choices: $u = \sec x \qquad du = \sec x \tan x\,dx$

$u = \tan x \qquad du = \sec^2 x\,dx$

then $\tan^2 x + 1 = \sec^2 x$

A. $\frac{7}{3}\tan^3 x + C$

B. $-\frac{7}{3}\tan^3 x + C$

C. $7(\sec x + \tan x)^5 + C$

D. $\frac{7}{3}\tan x + 7\tan^3 x + C$

E. $7\tan x + \frac{7}{3}\tan^3 x + C$

here, if $u = \sec x$, then $du = \sec x \tan x\,dx$

missing

but if $u = \tan x$, then $du = \sec^2 x\,dx$

$$\int 7\sec^4 x\,dx = 7\int \underbrace{\sec^2 x}_{\underbrace{\tan^2 x + 1}_{u^2+1}} \cdot \underbrace{\sec^2 x\,dx}_{du}$$

$$= 7\int (u^2+1)\,du = 7\left(\frac{1}{3}u^3 + u\right) + C$$

$$= \frac{7}{3}\tan^3 x + 7\tan x + C$$

After a proper trigonometric substitution is used to transform $\int_1^4 \frac{dt}{t^2 - 2t + 10}$ into $\int_a^b f(\theta)\, d\theta$, what is the new upper integration limit b?

A. $\frac{\pi}{6}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. $\frac{\pi}{3}$

E. π

$$\int_1^4 \frac{dt}{\underbrace{t^2-2t+10}}$$

needs to be sum/difference of squares

need to complete the square

$$t^2-2t+10 = (t^2-2t+1)+10-1$$

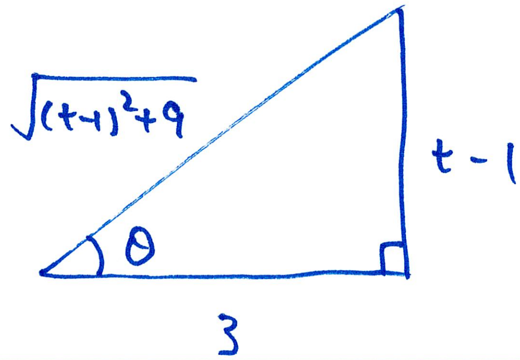
$$= (t-1)^2+9$$

$$\int_1^4 \frac{dt}{\left(\sqrt{(t-1)^2+9}\right)^2}$$

draw triangle w/ sides: $\sqrt{(t-1)^2+9}$, $t-1$, 3

hypotenuse : $\sqrt{(t-1)^2+9}$

adjacent : 3



relate θ and t as simply as possible

$$\tan\theta = \frac{t-1}{3}$$

$$3\tan\theta = t-1$$

$$t = 3\tan\theta + 1$$

old upper limit: $t = 4 \rightarrow \theta = ?$

$$4 = 3\tan\theta + 1$$

$$3 = 3\tan\theta$$

$$1 = \tan\theta \rightarrow \theta = \frac{\pi}{4}$$

Use the fact that

$$\int \frac{2x^4+15x^3+9x^2+11x+3}{x^5+x^4-x-1}\,dx = 5\ln|x-1| - 3\ln|x+1| - \frac{3}{x+1} + 2\tan^{-1}(x) + C$$

to find the partial fraction expansion of $\dfrac{2x^4+15x^3+9x^2+11x+3}{x^5+x^4-x-1}$

$$\int (\cdots) = \underbrace{5\ln|x-1| - 3\ln|x+1| - \frac{3}{x+1} + 2\tan^{-1}(x) + C}$$

A. $\dfrac{5}{x-1} + \dfrac{-3}{x+1} + \dfrac{3}{(x+1)^2} + \dfrac{2}{x^2+1}$

then $(\cdots) = \dfrac{d}{dx}\left(\;\cdots\; \right)$

B. $\dfrac{5}{x-1} + \dfrac{-3}{x+1} + \dfrac{-3x}{(x+1)^2} + \dfrac{2}{x^2+1}$

$$= 5\cdot\frac{1}{x-1} - 3\cdot\frac{1}{x+1} + \frac{3}{(x+1)^2} + 2\cdot\frac{1}{x^2+1}$$

C. $\dfrac{5}{x-1} + \dfrac{-3}{x+1} + \dfrac{-3}{(x+1)^2} + \dfrac{2x}{x^2+1}$

D. $\dfrac{-5}{x-1} + \dfrac{3}{x+1} + \dfrac{3}{(x+1)^2} + \dfrac{-2}{x^2+1}$

$$= \frac{5}{x-1} - \frac{3}{x+1} + \frac{3}{(x+1)^2} + \frac{2}{(x^2+1)}$$

E. $\dfrac{-5}{x-1} + \dfrac{3}{x+1} + \dfrac{3x}{(x+1)^2} + \dfrac{2x}{x^2+1}$

Each of the following statements is either true or false. Determine which are true and which are false.

(i) $\int_0^1 \ln(x)\,dx$ is divergent F

(ii) $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent for $p > 1$ T

(iii) $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is convergent F

A. (i) and (ii) are true; (iii) is false

B. (i) is true; (ii) and (iii) are false

C. (ii) is true; (i) and (iii) are false

D. (i) and (iii) are true; (ii) is false

E. (ii) and (iii) are true; (i) is false

(i) $\int_0^1 \ln(x)\,dx$

at $x=0$, $\ln(x) \to -\infty$

this integral is improper

$$\lim_{a\to 0^+} \int_a^1 \ln x\,dx \quad \text{by parts}$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$= \lim_{a\to 0^+} \left(x\ln x \Big|_a^1 - \int_a^1 x \cdot \frac{1}{x}\,dx \right)$$

$$= \lim_{a\to 0^+} \left(x\ln x \Big|_a^1 - x \Big|_a^1 \right)$$

$$= \lim_{a\to 0^+} \left(1\cdot\ln 1 - a\ln a - 1 + a \right)$$

$$= \lim_{a\to 0^+} \left(\underbrace{-a\ln a}_{?} - 1 + \cancel{a}^{\,0} \right) \quad \text{converge if} = \text{number}$$

$\to 0\cdot\infty = ?$

$$\lim_{a\to 0^+} a\ln a = \lim_{a\to 0^+} \frac{\ln a}{\frac{1}{a}} \to \frac{\infty}{\infty}$$

$$= \lim_{a\to 0^+} \frac{\frac{1}{a}}{-\frac{1}{a^2}} = \lim_{a\to 0^+} (-a) = 0$$

so, $\int_0^1 \ln x \, dx = -1$ converges

(ii) $\int_1^\infty \frac{1}{x^p} dx$ converges if $p > 1$

true

$$\lim_{a\to\infty} \int_1^a x^{-p} dx = \lim_{a\to\infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^a \qquad (p \neq 1)$$

$$= \lim_{a\to\infty} \frac{1}{-p+1} \left(a^{-p+1} - 1 \right)$$

$$= \lim_{a\to\infty} \frac{1}{-p+1} \left(\underbrace{\frac{a}{a^p}}_{\to 0 \text{ if } p>1} - 1 \right) = -1 \quad \text{if } p > 1$$