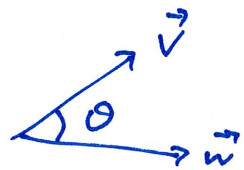
Find the angle between the vectors $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

A. $\cos^{-1}\left(\frac{8}{9}\right)$ B. $\cos^{-1}\left(\frac{5}{9}\right)$ C. $\cos^{-1}\left(\frac{2}{3}\right)$ D. $\cos^{-1}\left(\frac{7}{9}\right)$ E. $\cos^{-1}\left(\frac{1}{3}\right)$

$$\boxed{\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\theta}$$



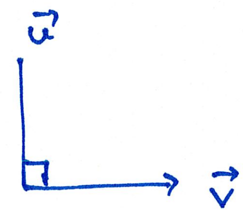
$$\langle 2, 2, 1 \rangle \cdot \langle 2, 2, -1 \rangle = \left(\sqrt{2^2+2^2+1^2}\right)\left(\sqrt{2^2+2^2+(-1)^2}\right) \cos\theta$$

$$(2 \cdot 2) + (2 \cdot 2) + (1 \cdot -1) = (3)(3) \cos\theta$$

$$\cos\theta = \frac{7}{9} \qquad \text{so,} \qquad \theta = \cos^{-1}\left(\frac{7}{9}\right)$$

Find a such that $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + a\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ are perpendicular.

A. 3 B. 2 C. 1 D. -1 E. -2



if perpendicular, $\theta = 90^\circ$

and $\cos\theta = 0$

so, $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta = 0$

$\vec{u} = \langle 2, -1, a \rangle$ $\vec{v} = \langle 1, 4, 2 \rangle$

$\vec{u} \cdot \vec{v} = 2 - 4 + 2a = 0$

$-2 + 2a = 0$ $a = 1$

If $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ is perpendicular to $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, and if $w_3 = 2$, then $w_1 =$

A. 4 B. 2 C. -2 D. -4 E. 1

cross product of two vectors is perpendicular to both

so, $\vec{w} = \vec{u} \times \vec{v}$ or $\vec{v} \times \vec{u}$

$\vec{u} = \langle 1, 1, -1 \rangle$ $\vec{v} = \langle 2, 1, 1 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= \vec{i}\,(1\cdot 1 - 1\cdot -1) - \vec{j}\,(1\cdot 1 - 2\cdot -1) + \vec{k}\,(1\cdot 1 - 2\cdot 1)$$

$$= 2\vec{i} - 3\vec{j} - \vec{k} = \langle 2, -3, -1 \rangle$$

we want negative of this multiplied by 2

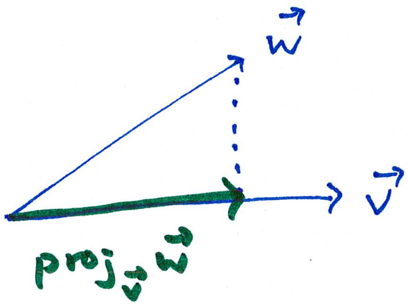
notice sign is wrong, so $\vec{v} \times \vec{u}$ is the cross product

and magnitude is off by factor of two

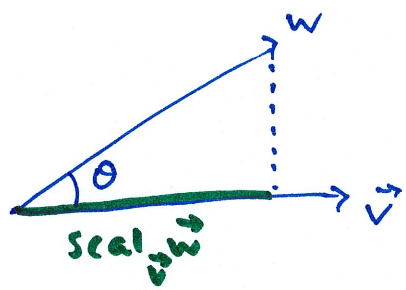
If $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{k}$, find $|\text{proj}_{\mathbf{v}}(\mathbf{w})|$.

A. $1/\sqrt{3}$ B. $\sqrt{3}$ C. $\sqrt{3}/5$ D. $2\sqrt{3}$ E. $\sqrt{3}/2$

$\vec{v} = \langle 1, 1, 1 \rangle \quad \vec{w} = \langle 2, 0, -1 \rangle$



$$|\text{proj}_{\vec{v}}\vec{w}| = \text{scal}_{\vec{v}}\vec{w}$$



$$\text{scal}_{\vec{v}}\vec{w} = |\vec{w}|\cos\theta$$

$$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}$$

$$\text{scal}_{\vec{v}}\vec{w} = \cancel{|\vec{w}|}\,\frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\cancel{|\vec{w}|}} = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|}$$

$$\vec{v}\cdot\vec{w} = 2 - 1 = 1$$

$$|\vec{v}| = \sqrt{3}$$

so, $\text{scal}_{\vec{v}}\vec{w} = \dfrac{1}{\sqrt{3}}$

$$\text{proj}_{\vec{v}}\vec{w} = \text{scal}_{\vec{v}}\vec{w}\,\frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{3}}\cdot\frac{\langle 1, 1, 1\rangle}{\sqrt{3}} = \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle$$

The radius of the sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z = 3$ is

A. $3 + \sqrt{13}$ B. $\sqrt{13}$ C. $\sqrt{65}$ D. $3 + \sqrt{56}$ E. $\sqrt{17}$

$$x^2 + 2x + y^2 + 4y + z^2 - 6z = +3$$

complete the square

$$x^2 + 2x + \left(\frac{2}{2}\right)^2 + y^2 + 4y + \left(\frac{4}{2}\right)^2 + z^2 - 6z + \left(\frac{-6}{2}\right)^2 = 3 + 1 + 4 + 9$$

$$x^2 + 2x + 1 + y^2 + 4y + 4 + z^2 - 6z + 9 = 17$$

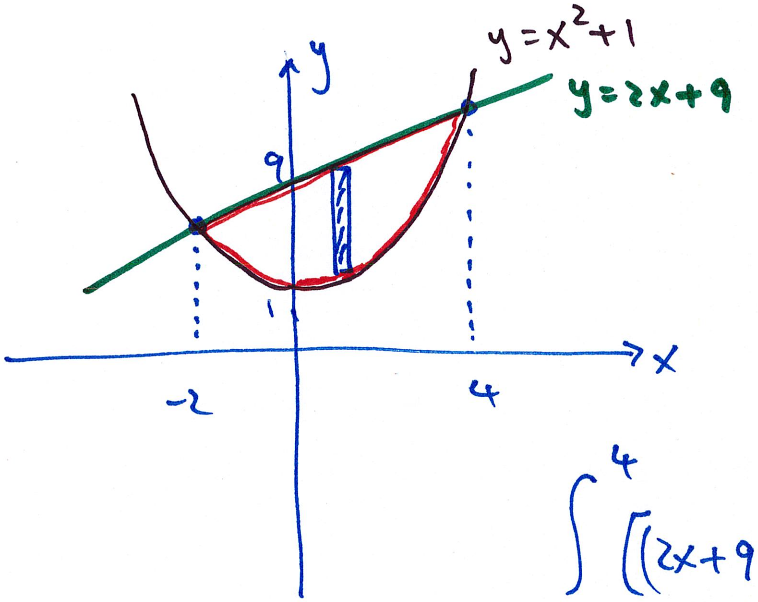
$$(x+1)^2 + (y+2)^2 + (z-3)^2 = 17$$

$\hookrightarrow$ $(\text{radius})^2$

center: $(-1, -2, 3)$ radius $\sqrt{17}$

The area of the region enclosed by the curves $y = x^2 + 1$ and $y = 2x + 9$ is given by

A. $\int_{-2}^{4} (x^2 + 1 - 2x - 9)\, dx$ B. $\int_{-2}^{4} (2x + 9 - x^2 - 1)\, dx$ C. $\int_{-2}^{2} (2x + 9 - x^2 - 1)\, dx$ ~~D.~~ $\int_{-4}^{2} (2x + 9 - x^2 - 1)\, dx$ E. $\int_{-4}^{2} (x^2 + 1 - 2x - 9)\, dx$



intersection :

$x^2+1 = 2x+9$

$x^2-2x-8=0$

$(x-4)(x+2)=0$

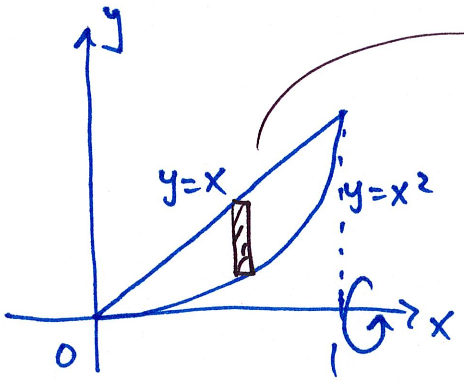
$x=-2, \; x=4$

$$\int_{-2}^{4} \left[(2x+9)-(x^2+1)\right] dx$$

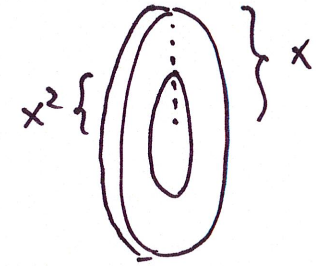
$$= \int_{-2}^{4} (2x+9-x^2-1)\, dx$$

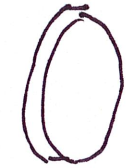
Let R be the region between the graphs of $y = x^2$ and $y = x$. Find the volume of the solid generated by revolving R about the x–axis.

A. $\frac{\pi}{6}$ B. $\frac{\pi}{12}$ C. $\frac{\pi}{4}$ D. $\frac{\pi}{15}$ E. $\frac{2\pi}{15}$



choice : disk/washer
shell





volume $= \pi (x)^2 dx$
$- \pi (x^2)^2 dx$

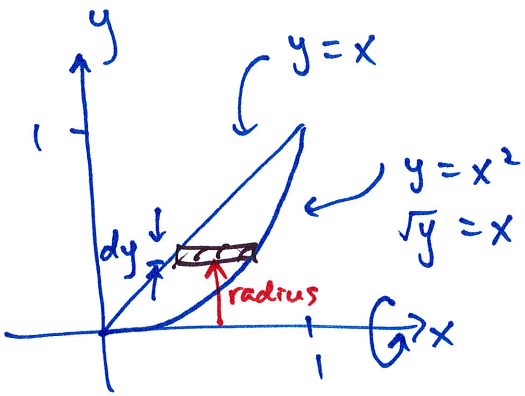
let's use disk / washer method since equations are in terms of x

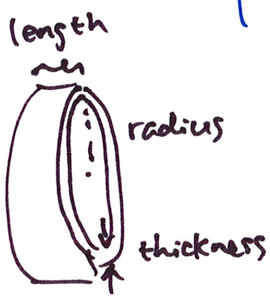
disk : rectangle perpendicular to axis of revolution

$$V = \int_0^1 [\underbrace{\pi (x)^2}_{\text{outer radius}} - \underbrace{\pi (x^2)^2}_{\text{inner radius}}] dx = \pi \int_0^1 (x^2 - x^4) dx = \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2}{15} \pi$$

Shell





shell : rectangle parallel to axis of revolution

$$V = \int_a^b 2\pi \,(\text{radius})\,(\text{height or length of rectangle})\,(\text{thickness})$$

radius: from axis of rev. to rectangle

here, radius = y

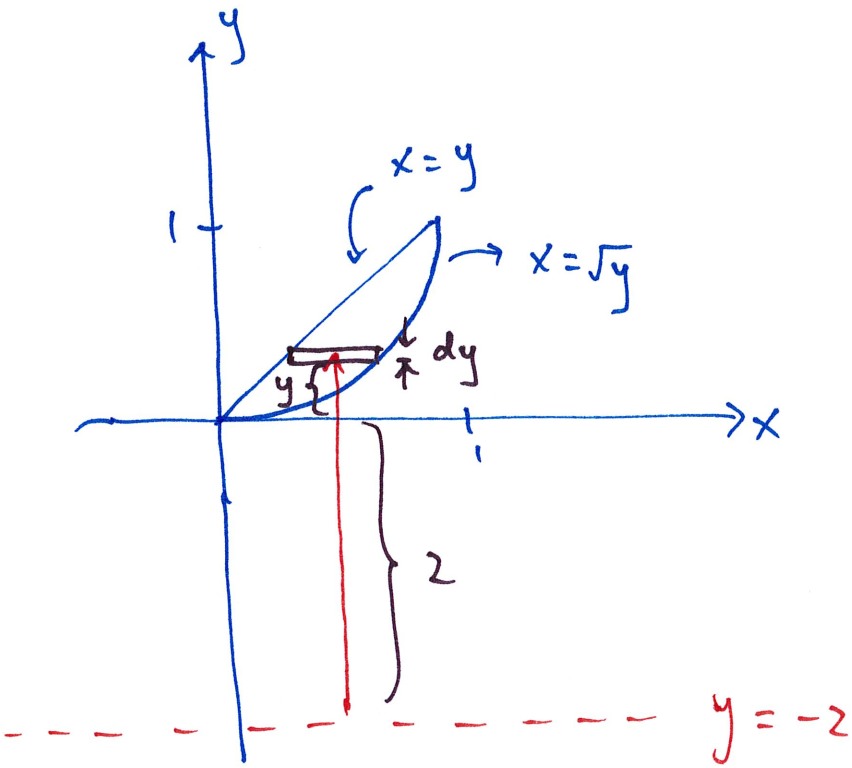
height/length of rectangle = right − left

$= \sqrt{y} - y$

thickness = dy

$$V = \int_0^1 2\pi (y)(\sqrt{y} - y)\, dy$$

what if we revolve around $y = -2$?



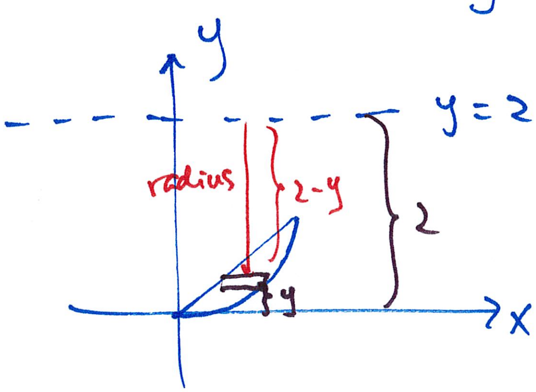
shell

radius = from $y = -2$ to rectangle

$= 2 + y$

$$V = \int_0^1 2\pi (2+y)(\sqrt{y} - y)\, dy$$

if revolved around $y = 2$



radius $= 2 - y$