A force of 4 lb. is required to stretch a spring 1/2 ft. beyond its natural length. How much work is required to stretch the spring from its natural length to 2 ft.

A. 8 ft–lbs. B. 12 ft–lbs. C. 16 ft–lbs. D. 24 ft–lbs. E. 32 ft–lbs.

force on a spring: $F = kx$ deviation from natural

spring constant

$$4 = k \cdot \frac{1}{2} \rightarrow k = 8$$

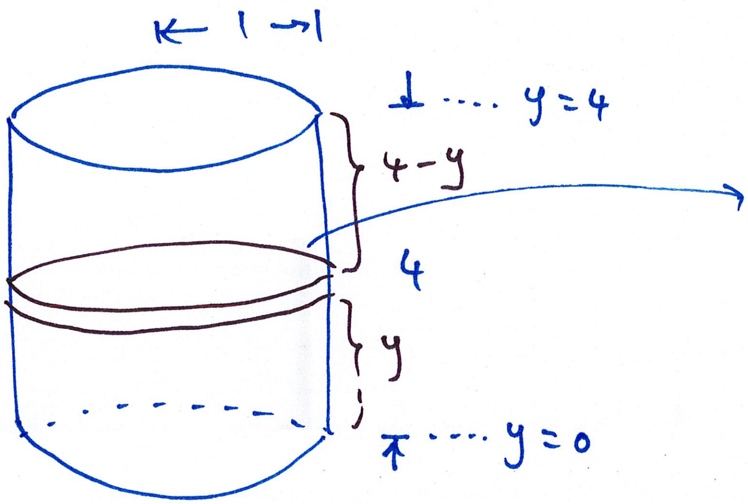
stretch to 2 ft

$$W = \int F\,dx = \int_0^{2} 8x\,dx = 4x^2 \Big|_0^2 = 16$$

stretch from natural

A cylindrical tank of height 4 feet and radius 1 foot is filled with water. How much work is required to pump all but 1 foot of water out of the tank. (Density = 62.5 lbs./ft^3)

A. $9\pi(62.5)$ ft-lbs. B. $3\pi(62.5)$ ft-lbs. C. $\frac{9\pi}{2}(62.5)$ ft-lbs. D. $18\pi(62.5)$ ft-lbs.
E. $6\pi(62.5)$ ft-lbs.



weight of slice: $\pi(1)^2 dy \cdot 62.5$

work of moving slice: $(62.5\pi)(4-y)\,dy$

$$\int_1^4 62.5\pi(4-y)\,dy = 62.5\pi\left(4y-\frac{1}{2}y^2\right)\Big|_1^4$$

$$= 62.5\pi\left[(16-8)-(4-\tfrac{1}{2})\right]$$

$$= 62.5\pi\left(\frac{9}{2}\right)$$

$$\int x(\ln x)^3 dx = \frac{x^2}{2}(\ln x)^3 - I, \text{ where } I =$$

A. $\frac{1}{4}\int (\ln x)^4 dx$ B. $\frac{1}{3}\int (\ln x)^2 dx$ C. $\frac{1}{3}\int (\ln x)^2 dx$ D. $\frac{3}{2}\int x^2(\ln x)^2 dx$ E. $\frac{3}{2}\int x(\ln x)^2 dx$

by parts: LIATE

$x, \ln x$

A L

choose $u = (\ln x)^3$ $dv = x\,dx$

$du = 3(\ln x)^2 \cdot \frac{1}{x} dx$ $v = \int x\,dx = \frac{1}{2}x^2$

then $uv - \int v\,du$

$$\frac{x^2}{2}(\ln x)^3 - \underbrace{\int \frac{1}{2}x^2 \cdot 3(\ln x)^2 \cdot \frac{1}{x}\,dx}_{I}$$

$$I = \frac{3}{2}\int x(\ln x)^2 dx$$

$$\int_0^{\pi/2} \sin^3 x dx =$$

A. 2/3 B. 4/3 C. 0 D. 1/4 E. 1/3

$$\int_0^{\pi/2} \sin^2 x \cdot \underbrace{\sin x \, dx}_{du}$$

$\downarrow$

$1 - \cos^2 x$, $u = \cos x$

$$= \int_0^{\pi/2} (1-\cos^2 x) \sin x \, dx = -\int_1^0 (1-u^2)\, du = -\left(u - \frac{1}{3}u^3\right)\Big|_1^0$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\left[(0) - \left(1 - \frac{1}{3}\right)\right]$$

$x = \pi/2 \rightarrow u = \cos \pi/2 = 0$

$$= \frac{2}{3}$$

$x = 0 \rightarrow u = \cos 0 = 1$

$$\int_0^{\pi/4} \sec^4 x \tan x \, dx =$$

A. 1 B. 1/3 C. 4/3 D. 3/4 E. 2/9

$$\int_0^{\pi/4} \sec^3 x \underbrace{(\sec x \tan x) dx}_{du \text{ if } u = \sec x} = \int_1^{\sqrt{2}} u^3 du = \frac{1}{4} u^4 \Big|_1^{\sqrt{2}} = \frac{1}{4}(4-1) = 3/4$$

$x = \pi/4 \rightarrow u = \sec \frac{\pi}{4} = \sqrt{2}$

$x = 0 \rightarrow u = \sec 0 = 1$

$$\int_0^{\pi/4} \sec^2 x \underbrace{\tan x}_{u} \underbrace{\sec^2 x \, dx}_{du \text{ if } u = \tan x} = \int_0^1 (1+u^2) u \, du$$

$\sec^2 x \downarrow$ $1 + \tan^2 x$ or $1 + u^2$

$$= \int_0^1 u + u^3 \, du = \frac{u^2}{2} + \frac{u^4}{4} \Big|_0^1$$

$$= \frac{3}{4}$$

In order to compute $\int \frac{dx}{(1+x^2)^{3/2}}$ we make the substitution $x = \tan\theta$. This gives an integral in θ whose value is

A. $\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$ B. $\ln(\sec^2\theta) + C$ C. $\frac{1}{2}\theta + \tan^{-1}\theta + C$ D. $\frac{1}{2}\sqrt{\cos\theta} + C$

E. $\sin\theta + C$

$x = \tan\theta \qquad dx = \sec^2\theta \, d\theta$

$1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$

$$\int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta = \int \frac{1}{\sec\theta}\, d\theta = \int \cos\theta \, d\theta = \sin\theta + C$$

$$\int \frac{dx}{\sqrt{9-4x^2}} =$$

A. $\sec^{-1}\left(\frac{3x}{2}\right)+C$ B. $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+C$ C. $\tan^{-1}\left(\frac{2x}{3}\right)+C$ D. $\frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)+C$

E. $\sqrt{9-4x^2}+\tan^{-1}\left(\frac{2x}{3}\right)+C$

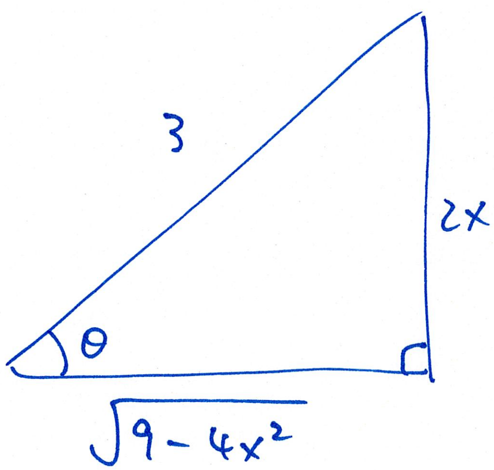
$\sqrt{9-4x^2} = \sqrt{9-(2x)^2}$

triangle with sides : 3, $2x$, $\sqrt{9-(2x)^2}$

hypotenuse : 3 because $3^2 = (2x)^2 + \left(\sqrt{9-(2x)^2}\right)^2$

adjacent : $\sqrt{9-(2x)^2}$

relate x and θ in the simplest way possible



$\sin\theta = \frac{2x}{3}$

$2x = 3\sin\theta$

$x = \frac{3}{2}\sin\theta$

$dx = \frac{3}{2}\cos\theta \, d\theta$

from triangle $\cos\theta = \frac{\sqrt{9-4x^2}}{3}$

$\sqrt{9-4x^2} = 3\cos\theta$

$$\int \frac{\frac{3}{2}\cos\theta}{3\cos\theta}\,d\theta = \int \frac{1}{2}\,d\theta = \frac{1}{2}\theta + C$$

from $x = \frac{3}{2}\sin\theta \rightarrow \sin\theta = \frac{2}{3}x \rightarrow \theta = \sin^{-1}\left(\frac{2}{3}x\right)$

$$\frac{1}{2}\sin^{-1}\left(\frac{2}{3}x\right) + C$$

$$\int \frac{x+1}{x^3-2x^2+x}\,dx =$$

A. $\ln|x| + \ln|x-1| + C$

B. $\ln|x| - \ln|x-1| + C$

C. $\ln|x| - \frac{2}{x-1} + C$

D. $\ln|x-1| - \frac{2}{x-1} + C$

E. $\ln|x| - \ln|x-1| - \frac{2}{x-1} + C$

the "ln" in answers suggest partial fraction expansion

$$\frac{x+1}{x^3-2x^2+x} = \frac{x+1}{(x)(x^2-2x+1)} = \frac{x+1}{(x)(x-1)(x-1)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{x+1}{(x)(x-1)(x-1)}$$

$A(x-1)^2 + B(x)(x-1) + C(x) = x+1$

choose $x=1 \rightarrow C=2$

choose $x=0 \rightarrow A=1$

choose $x=2 \rightarrow \underset{\uparrow_1}{A}(2-1)^2 + B(2)(2-1) + \underset{\uparrow_2}{C}(2) = 3$

$$1 + 2B + 4 = 3$$

$$2B = \cancel{0} \; -2$$

$$B = \cancel{0} \; -1$$

$$\int \left(\frac{1}{x} - \frac{1}{x-1} + \frac{2}{(x-1)^2} \right) dx$$

$$= \ln|x| - \ln|x-1| - \frac{2}{x-1} + C$$

A partial fraction decomposition of $\dfrac{x+2}{x^4+2x^2}$ has the form

A. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+2}$ B. $\dfrac{A}{x^2}+\dfrac{Bx+C}{x^2+2}$ C. $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^2+2}$ D. $\dfrac{A}{x^2}+\dfrac{B}{x^2+2}$

E. $\dfrac{A}{x}+\dfrac{B}{x^2+2}$

$$\frac{x+2}{x^4+2x^2} = \frac{x+2}{(x^2)(x^2+2)} = \frac{x+2}{\underbrace{(x)(x)}_{\text{linear factor repeated}}\underbrace{(x^2+2)}_{\text{irreducible quadratic}}} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+2}$$

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we can view x^2 as a quadratic itself

$$\frac{x+2}{(x^2)(x^2+2)} = \frac{Ax+B}{x^2}+\frac{Cx+D}{x^2+2}$$

$$= \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+2}$$

Indicate convergence or divergence for each of the following improper integrals:

$$\text{(I)} \int_2^{\infty} \frac{1}{(x-1)^2}\,dx \qquad \text{(II)} \int_0^{2} \frac{1}{(x-1)^2}\,dx \qquad \text{(III)} \int_0^{1} \frac{\ln x}{x}\,dx$$

C D

A. I converges, II and III diverge. B. II converges, I and III diverge. C. I and III converge, II diverges. D. I and II converge, III diverges. E. I, II and III diverge.

I : $\displaystyle\int_2^{\infty} \frac{1}{(x-1)^2}\,dx = \lim_{a\to\infty} \int_2^{a} \frac{1}{(x-1)^2}\,dx = \lim_{a\to\infty} -\frac{1}{x-1}\Big|_2^a$

$\displaystyle = \lim_{a\to\infty} \Big(\underbrace{-\frac{1}{a-1}}_{\text{to } 0} + \frac{1}{2-1} \Big) = 1$ converges

II : problem at $x=1$ $\displaystyle\int_0^{1} \frac{1}{(x-1)^2}\,dx + \int_1^{2} \frac{1}{(x-1)^2}\,dx$

$\displaystyle = \lim_{a\to 1^-} \int_0^{a} \frac{1}{(x-1)^2}\,dx + \lim_{b\to 1^+} \int_b^{2} \frac{1}{(x-1)^2}\,dx$

$\displaystyle = \lim_{a\to 1^-} -\frac{1}{x-1}\Big|_0^a + \lim_{b\to 1^+} -\frac{1}{x-1}\Big|_b^2$

$$= \underbrace{\lim_{a\to 1^-}\left(-\underbrace{\frac{1}{a-1}}_{\infty} - 1\right)}_{\infty} + \underbrace{\lim_{b\to 1^+}\left(-\cancel{\frac{1}{b-1}}\;1 + \underbrace{\frac{1}{b-1}}_{\infty}\right)}_{\infty} = \infty$$